On general Eulerian integral of certain products of two multivariable I-functions defined by Prasad and a class of polynomials

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ABSTRACT
The object of this paper is to establish an general Eulerian integral involving the product of two multivariable I-functions defined by Prasad [1], a class of multivariable polynomials and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular cases concerning the multivariable H-function and the Srivastava-Daoust polynomial

Keywords: Eulerian integral, multivariable I-function, Lauricella function of several variables, multivariable H-function, generalized hypergeometric function, Srivastava-Daoust polynomial

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1.Introduction

The object of this paper is to establish an general Eulerian integral involving the product of two multivariable I-functions defined by Prasad [1], a general class of multivariable polynomials and a generalized hypergeometric function with general argument which provide unification and extension of numerous results. The multivariable I-function is an extension of the multivariable H-function defined by Srivastava et al [7]. We will use the contracted form.

The multivariable I-function of r-variables is defined by Prasad [1] in term of multiple Mellin-Barnes type integral :

$$I(z'_1,\cdots,z'_s) = I^{0,n_2;0,n_3;\cdots;0,n_s:m^{(1)},n^{(1)};\cdots;m^{(s)},n^{(s)}}_{p_2,q_2,p_3,q_3;\cdots;p_s,q_s:p^{(1)},q^{(1)};\cdots;p^{(s)},q^{(s)}}\left(\begin{array}{c|c} z'_1 & (a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \vdots & \\ z'_s & (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (a_{sj};\alpha^{(1)}_{sj},\cdots,\alpha^{(s)}_{sj})_{1,p_s} : (a^{(1)}_j,\alpha^{(1)}_j)_{1,p^{(1)}};\cdots;(a^{(s)}_j,\alpha^{(s)}_j)_{1,p^{(s)}} \\ (b_{rj};\beta^{(1)}_{sj},\cdots,\beta^{(s)}_{sj})_{1,q_s} : (b^{(1)}_j,\beta^{(1)}_j)_{1,q^{(1)}};\cdots;(b^{(s)}_j,\beta^{(s)}_j)_{1,q^{(s)}} \end{array}\right) \quad (1.1)$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\zeta_i(t_i)z_i'^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \quad (1.2)$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz'_i| < \frac{1}{2}\Omega_i\pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)}\right) \tag{1.3}$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z'_1, \cdots, z'_s) = 0(\,|z'_1|^{\alpha_1}, \cdots, |z'_s|^{\alpha_r}\,), max(\,|z'_1|, \cdots, |z'_s|\,) \rightarrow 0$$

$$I(z'_1, \cdots, z'_s) \; = 0(\,|z'_1|^{\beta_1}, \cdots, |z'_s|^{\beta_r}\,), min(\,|z'_1|, \cdots, |z'_s|\,) \rightarrow \infty$$

where $k = 1, \cdots, r : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this section :

$$I(z''_1, \cdots, z''_u) = I^{0,n'_2;0,n'_3;\cdots;0,n'_u:m'^{(1)},n'^{(1)};\cdots;m'^{(u)},n'^{(u)}}_{p'_2,q'_2,p'_3,q'_3;\cdots;p'_u,q'_u:p'^{(1)},q'^{(1)};\cdots;p'^{(u)},q'^{(u)}} \left(\begin{array}{c|c} z"_1 & (\text{a'}_{2j};\alpha'^{(1)}_{2j},\alpha'^{(2)}_{2j})_{1,p'_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ z"_u & (\text{b'}_{2j};\beta'^{(1)}_{2j},\beta'^{(2)}_{2j})_{1,q'_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\text{a'}_{uj};\alpha'^{(1)}_{uj},\cdots,\alpha'_{uj}{}^{(u)})_{1,p'_u} : (a'^{(1)}_j,\alpha'^{(1)}_j)_{1,p'^{(1)}};\cdots;(a'_j{}^{(u)},\alpha'_j{}^{(u)})_{1,p'^{(u)}} \\ \\ (\text{b'}_{uj};\beta'^{(1)}_{uj},\cdots,\beta'_{uj}{}^{(u)})_{1,q'_u} : (b'^{(1)}_j,\beta'^{(1)}_j)_{1,q'^{(1)}};\cdots;(b'_j{}^{(u)},\beta'_j{}^{(u)})_{1,q'^{(u)}} \end{array}\right) \tag{1.4}$$

$$= \frac{1}{(2\pi\omega)^u}\int_{L''_1}\cdots\int_{L''_u}\psi(x_1,\cdots,x_u)\prod_{i=1}^{u}\xi_i(x_i)z_i''^{x_i}\mathrm{d}x_1\cdots\mathrm{d}x_u \tag{1.5}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

where $|argz''_i| < \frac{1}{2}\Omega''_i\pi$,

$$\Omega'_i = \sum_{k=1}^{n'^{(i)}} {\alpha'_k}^{(i)} - \sum_{k=n'^{(i)}+1}^{p'^{(i)}} {\alpha'_k}^{(i)} + \sum_{k=1}^{m'^{(i)}} {\beta'_k}^{(i)} - \sum_{k=m^{(i)}+1}^{q'^{(i)}} {\beta'_k}^{(i)} + \left(\sum_{k=1}^{n'_2} {\alpha'_{2k}}^{(i)} - \sum_{k=n_2+1}^{p'_2} {\alpha'_{2k}}^{(i)} \right)$$

$$+ \cdots + \left(\sum_{k=1}^{n'_u} {\alpha'_{uk}}^{(i)} - \sum_{k=n'_u+1}^{p'_u} {\alpha'_{uk}}^{(i)} \right) - \left(\sum_{k=1}^{q'_2} {\beta'_{2k}}^{(i)} + \sum_{k=1}^{q'_3} {\beta'_{3k}}^{(i)} + \cdots + \sum_{k=1}^{q'_u} {\beta'_{uk}}^{(i)} \right) \tag{1.6}$$

where $i = 1, \cdots, u$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z''_1, \cdots, z''_u) = 0(|z''_1|^{\alpha'_1}, \cdots, |z''_u|^{\alpha'_s}), max(|z''_1|, \cdots, |z''_u|) \to 0$$

$$I(z''_1, \cdots, z''_u) = 0(|z''_1|^{\beta'_1}, \cdots, |z''_u|^{\beta'_s}), min(|z''_1|, \cdots, |z''_u|) \to \infty$$

where $k = 1, \cdots, z : \alpha''_k = min[Re(b_j^{\prime(k)}/\beta_j^{\prime(k)})], j = 1, \cdots, m'_k$ and

$$\beta''_k = max[Re((a_j^{\prime(k)} - 1)/\alpha_j^{\prime(k)})], j = 1, \cdots, n'_k$$

Srivastava and Garg [4] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_u}[z_1, \cdots, z_u] = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} (-L)_{h_1R_1+\cdots+h_uR_u} B(E; R_1, \cdots, R_u) \frac{z_1^{R_1} \cdots z_u^{R_u}}{R_1! \cdots R_u!} \tag{1.7}$$

The coefficients are $B[E; R_1, \ldots, R_v]$ arbitrary constants, real or complex.

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [5 ,page 39 eq .30]

$$\frac{\prod_{j=1}^P \Gamma(A_j)}{\prod_{j=1}^Q \Gamma(B_j)} {}_PF_Q \left[(A_P); (B_Q); -(x_1 + \cdots + x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \frac{\prod_{j=1}^P \Gamma(A_j + s_1 + \cdots + s_r)}{\prod_{j=1}^Q \Gamma(B_j + s_1 + \cdots + s_r)} \Gamma(-s_1) \cdots \Gamma(-s_r) x_1^{s_1} \cdots x_r^{s_r} \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \cdots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \cdots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$

The Lauricella function $F_D^{(k)}$ is defined as

$$F_D^{(k)}\left[a,b_1,\cdots,b_k;c;x_1,\cdots,x_k\right]=\frac{\Gamma(c)}{\Gamma(a)\prod_{j=1}^{k}\Gamma(b_j)}\frac{1}{(2\pi\omega)^k}\int_{L_1}\cdots\int_{L_k}\frac{\Gamma\left(a+\sum_{j=1}^{k}\zeta_j\right)\Gamma(b_1+\zeta_1),\cdots,\Gamma(b_k+\zeta_k)}{\Gamma\left(c+\sum_{j=1}^{k}\zeta_j\right)}$$

$$\prod_{j=1}^{k}\Gamma(-\zeta_j)(-x_j)^{\zeta_i}\,\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_k \tag{2.2}$$

where $max\left[|arg(-x_1)|,\cdots,|arg(-x_k)|\right]<\pi, c\neq 0,-1,-2,\cdots$.

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t=(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha,-\sigma_1,\cdots,-\sigma_k;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right] \tag{2.3}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i\in\mathbb{C},(i=1,\cdots,k)$; $min(Re(\alpha),Re(\beta))>0$ and

$$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava et al ([6], page60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [3, page 454].

3. Eulerian integral

In this section , we note :

$$U=p_2,q_2;p_3,q_3;\cdots;p_{s-1},q_{s-1};p'_2,q'_2;p'_3,q'_3;\cdots;p'_{u-1},q'_{u-1};0,0;\cdots;0,0;0,0;\cdots;0,0 \tag{3.1}$$

$$V=0,n_2;0,n_3;\cdots;0,n_{s-1};0,n'_2;0,n'_3;\cdots;0,n'_{u-1};0,0;\cdots;0,0;0,0;\cdots;0,0 \tag{3.2}$$

$$X=m^{(1)},n^{(1)};\cdots;m^{(s)},n^{(s)};m'^{(1)},n'^{(1)};\cdots;m'^{(u)},n'^{(u)};1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{3.3}$$

$$Y=p^{(1)},q^{(1)};\cdots;p^{(s)},q^{(s)};p'^{(1)},q'^{(1)};\cdots;p'^{(u)},q'^{(u)};0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.4}$$

$$A=(a_{2k};\alpha_{2k}^{(1)},\alpha_{2k}^{(2)})_{1,p_2};\cdots;(a_{(s-1)k};\alpha_{(s-1)k}^{(1)},\alpha_{(s-1)k}^{(2)},\cdots,\alpha_{(s-1)k}^{(s-1)})_{1,p_{s-1}}:(a'_{2k};\alpha_{2k}'^{(1)},\alpha_{2k}'^{(2)})_{1,p'_2}$$

$$;\cdots;(a'_{(u-1)k};\alpha'^{(1)}_{(u-1)k},\alpha'^{(2)}_{(u-1)k},\cdots,\alpha'^{(u-1)}_{(u-1)k})_{1,p'_{u-1}} \tag{3.5}$$

$$B=(b_{2k};\beta^{(1)}_{2k},\beta^{(2)}_{2k})_{1,q_2};\cdots;(b_{(s-1)k};\beta^{(1)}_{(s-1)k},\beta^{(2)}_{(s-1)k},\cdots,\beta^{(s-1)}_{(s-1)k})_{1,q_{s-1}};(b'_{2k};\beta'^{(1)}_{2k},\beta'^{(2)}_{2k})_{1,q'_2};\cdots;$$

$$(b'_{(u-1)k};\beta'^{(1)}_{(u-1)k},\beta'^{(2)}_{(u-1)k},\cdots,\beta'^{(u-1)}_{(u-1)k})_{1,q'_{u-1}} \tag{3.6}$$

$$\mathfrak{A}=(a_{sk};\alpha^{(1)}_{sk},\alpha^{(2)}_{sk},\cdots,\alpha^{(s)}_{sk},0,\cdots,0,0\cdots,0,0,\cdots,0)_{1,p_s} \tag{3.7}$$

$$\mathfrak{A}'=(a'_{uk};0,\cdots,0,\alpha'^{(1)}_{uk},\alpha'^{(2)}_{uk},\cdots,\alpha'^{(u)}_{uk},0,\cdots,0,0,\cdots,0)_{1,p'_u} \tag{3.8}$$

$$\mathfrak{B}=(b_{sk};\beta^{(1)}_{sk},\beta^{(2)}_{sk},\cdots,\beta^{(s)}_{sk},0,\cdots,0,0,\cdots,0,0,\cdots,0)_{1,q_s} \tag{3.9}$$

$$\mathfrak{B}'=(b'_{uk};0,\cdots,0,\beta'^{(1)}_{uk},\beta'^{(2)}_{uk},\cdots,\beta'^{(u)}_{uk},0,\cdots,0,0,\cdots,0)_{1,q'_u} \tag{3.10}$$

$$A'=(a^{(1)}_k,\alpha^{(1)}_k)_{1,p^{(1)}};\cdots;(a^{(s)}_k,\alpha^{(s)}_k)_{1,p^{(s)}};(a'^{(1)}_k,\alpha'^{(1)}_k)_{1,p^{(1)}};\cdots;(a'^{(u)}_k,\alpha'^{(u)}_k)_{1,p'^{(u)}};$$

$$(1,0);\cdots;(1,0);(1,0);\cdots;(1,0) \tag{3.11}$$

$$B'=(b^{(1)}_k,\beta^{(1)}_k)_{1,q^{(1)}};\cdots;(b^{(s)}_k,\beta^{(s)}_k)_{1,q^{(s)}};(b'^{(1)}_k,\beta'^{(1)}_k)_{1,q'^{(1)}};\cdots;(b'^{(u)}_k,\beta'^{(u)}_k)_{1,q'^{(u)}};$$

$$(0,1);\cdots;(0,1);(0,1);\cdots;(0,1) \tag{3.12}$$

$$K_1=(1-\alpha-\sum_{i=1}^{r}R_i(\mu_i+\mu'_i);\mu_1,\cdots,\mu_s,\mu'_1,\cdots,\mu'_u,1,\cdots,1,\upsilon_1,\cdots,\upsilon_l) \tag{3.13}$$

$$K_2=(1-\beta-\sum_{i=1}^{r}R_i(\rho_i+\rho'_i);\rho_1,\cdots,\rho_s,\rho'_1,\cdots,\rho'_u,0,\cdots,0,\tau_1,\cdots,\tau_l) \tag{3.14}$$

$$K_P=[1-A_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1,\cdots,1]_{1,P} \tag{3.15}$$

$$K_j=[1+\sigma_j-\sum_{i=1}^{r}R_i(\lambda^{(j)}_i+\lambda'^{(j)}_i);\lambda^{(1)}_j,\cdots,\lambda^{(s)}_j,\lambda'^{(1)}_j\cdots,\lambda'^{(u)}_j,0\cdots,1,\cdots,0,\zeta'_j,\cdots,\zeta^{(l)}_j]_{1,k} \tag{3.16}$$

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$$L_1=(1-\alpha-\beta-\sum_{i=1}^{r}R_i(\mu_i+\mu'_i+\rho_i+\rho'_i);\mu_1+\rho_1,\cdots,\mu_s+\rho_s,\mu'_1+\rho'_1,\cdots,\mu'_u+\rho'_u,$$

$$1,\cdots,1,\upsilon_1+\tau_1,\cdots,\upsilon_l+\tau_l) \tag{3.17}$$

$$L_Q=[1-B_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1\cdots,1]_{1,Q} \tag{3.18}$$

$$L_j=[1+\sigma_j-\sum_{i=1}^{r}R_i(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(s)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(u)},0\cdots,0,\zeta_j',\cdots,\zeta_j^{(l)}]_{1,k} \tag{3.19}$$

$$B_r=\frac{(-L)_{h_1R_1+\cdots+h_rR_r}B(E;R_1,\cdots,R_r)}{R_1!\cdots R_r!} \tag{3.20}$$

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_r}\begin{pmatrix} \mathrm{x}_1(t-a)^{\mu_1+\mu_1'}(b-t)^{\rho_1+\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{x}_r(t-a)^{\mu_u+\mu_u'}(b-t)^{\rho_r+\rho_r'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}-\lambda_j'^{(r)}} \end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}} \end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z'}_1(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{z'}_u(t-a)^{\mu_u'}(b-t)^{\rho_u'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}} \end{pmatrix}$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=(b-a)^{\alpha+\beta-1}$$

$$\sum_{R_1,\cdots,R_r=0}^{h_1R_1+\cdots h_rR_r\leqslant L}(b-a)^{\sum_{i=1}^{r}R_i(\mu_i+\mu_i'+\rho_i+\rho_i')}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j-\sum_{i=1}^{r}R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}B_rz_1^{R_1}\cdots z_r^{R_r}$$

$$I_{U:p_s+p'_u+l+k+2,q_s+q'_u+l+k+1;Y}^{V;0,n_s+n'_u+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\mathrm{K}_1,K_2,K_P,K_j,\mathfrak{A},\mathfrak{A}';A' \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(u)}_j}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \mathrm{B}\ ;\mathrm{L}_1,L_j,L_Q,\mathfrak{B},\mathfrak{B}';B' \end{array}\right) \quad (3.21)$$

We obtain the I-function of $s+u+k+l$ variables.

Provided that

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\rho_i,\mu'_j,\rho'_j\lambda_v^{(i)};\lambda'^{(i)}_v\in\mathbb{R}^+$, $f_i,g_j,\tau_v,\sigma_j\in\mathbb{C}$ $(i=1,\cdots,s;j=1,\cdots;u;v=1,\cdots,k)$
$\zeta_j^{(i)}>0(i=1,\cdots,l;j=1,\cdots,k)$

(B) $a'_{ij},b'_{ik},\in\mathbb{C}\ (i=1,\cdots,s;j=1,\cdots,p'_i;k=1,\cdots,q'_i);a'^{(i)}_j,b'^{(k)}_j\in\mathbb{C}$

$(i=1,\cdots,s;j=1,\cdots,p'^{(i)};k=1,\cdots,q'^{(i)})$

$a''_{ij},b''_{ik},\in\mathbb{C}\ (i=1,\cdots,u';j=1,\cdots,p''_i;k=1,\cdots,q''_i);a''^{(i)}_j,b''^{(k)}_j,\in\mathbb{C}$

$(i=1,\cdots,u;j=1,\cdots,p''^{i)};k=1,\cdots,q''^{(i)})$

$\alpha'^{(k)}_{ij},\beta'^{(k)}_{ij}\in\mathbb{R}^+\ (i=1,\cdots,s,j=1,\cdots,p'_i,k=1,\cdots,s)\,;\alpha'^{(i)}_j,\beta'^{(i)}_i\in\mathbb{R}^+\,(i=1,\cdots,s;j=1,\cdots,p'_i)$

$\alpha''^{(k)}_{ij},\beta''^{(k)}_{ij}\in\mathbb{R}^+\ (i=1,\cdots,u,j=1,\cdots,p''_i,k=1,\cdots,u)\,;\alpha''^{(i)}_j,\beta''^{(i)}_i\in\mathbb{R}^+\,(i=1,\cdots,u;j=1,\cdots,p''_i)$

$$(C)\ \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$$

(D) $$Re\big[\alpha+\sum_{i=1}^{s}\mu_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}+\sum_{i=1}^{u}\mu'_i\min_{1\leqslant j\leqslant m'^{(i)}}\frac{b_j'^{(i)}}{\beta_j'^{(i)}}\big]>0$$

$$Re\big[\beta+\sum_{i=1}^{s}\rho_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}+\sum_{i=1}^{u}\rho'_i\min_{1\leqslant j\leqslant m'^{(i)}}\frac{b_j'^{(i)}}{\beta_j'^{(i)}}\big]>0$$

(E) $$\Omega_i=\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right)+\cdots+$$

$$\left(\sum_{k=1}^{n_s}\alpha_{sk}^{(i)}-\sum_{k=n_s+1}^{p_s}\alpha_{sk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_s}\beta_{sk}^{(i)}\right)-\mu_i-\rho_i$$

$$-\sum_{j=1}^{k}\lambda_j^{(i)}>0\quad(i=1,\cdots,s)$$

$$\Omega'_i=\sum_{k=1}^{n'^{(i)}}\alpha_k'^{(i)}-\sum_{k=n'^{(i)}+1}^{p'^{(i)}}\alpha_k'^{(i)}+\sum_{k=1}^{m'^{(i)}}\beta_k'^{(i)}-\sum_{k=m^{(i)}+1}^{q'^{(i)}}\beta_k'^{(i)}+\left(\sum_{k=1}^{n'_2}\alpha_{2k}'^{(i)}-\sum_{k=n_2+1}^{p'_2}\alpha_{2k}'^{(i)}\right)$$

$$+\cdots+\left(\sum_{k=1}^{n'_u}\alpha_{uk}'^{(i)}-\sum_{k=n'_u+1}^{p'_u}\alpha_{uk}'^{(i)}\right)-\left(\sum_{k=1}^{q'_2}\beta_{2k}'^{(i)}+\sum_{k=1}^{q'_3}\beta_{3k}'^{(i)}+\cdots+\sum_{k=1}^{q'_u}\beta_{uk}'^{(i)}\right)-\mu'_i-\rho'_i$$

$$-\sum_{j=1}^{k}\lambda_j'^{(i)}>0\quad(i=1,\cdots,u)$$

(F) $$\left|arg\left(z_i\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right|<\frac{1}{2}\Omega_i\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,s)$$

$$\left|arg\left(z'_i\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right|<\frac{1}{2}\Omega'_i\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,u)$$

(G) $P\leqslant Q+1$. The equality holds, when , in addition,

either $P>Q$ and $\left|z''_i\left(\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right)\right|^{\frac{1}{Q-P}}<1\qquad(a\leqslant t\leqslant b)$

or $P \leqslant Q$ and $\max_{1\leqslant i\leqslant k}\left[\left|\left(z''_i\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right)\right|\right] < 1 \quad (a \leqslant t \leqslant b)$

Proof

First expressing the class of multivariable polynomials defined by Srivastava et al [4] in serie with the help of (1.7), expressing the I-function of s-variables and u-variables defined by Prasad [1] by the Mellin-Barnes contour integral with the help of the equation (1.2) and (1.5) respectively, the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1). Now collect the power of (f_jt+g_j) with $j=1,\cdots,k$ and use the equations (2.1) and (2.2) and we obtain $k-$Mellin-Barnes contour integral. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and interpreting $(r+s+k+l)-$Mellin-barnes contour integral in multivariable I-function of Prasad, we obtain the desired result.

4. Multivariable H-function

If $A=B=U=V=0$, the multivariable I-function reduces to the multivariable H-function and we obtain

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_r}\begin{pmatrix} \mathrm{x}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{x}_r(t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_r+\rho'_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}-\lambda_j'^{(r)}} \end{pmatrix}$$

$$H\begin{pmatrix} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}} \end{pmatrix}$$

$$H\begin{pmatrix} \mathrm{z'}_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}} \end{pmatrix}$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t = (b-a)^{\alpha+\beta-1}$$

$$\sum_{R_1,\cdots,R_r=0}^{h_1R_1+\cdots h_rR_r\leqslant L} B_r z_1^{R_1}\cdots z_r^{R_r}(b-a)^{\sum_{i=1}^r R_i(\mu_i+\mu_i'+\rho_i+\rho_i')}\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}^k(af_j+g_j)^{\sigma_j-\sum_{i=1}^r R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}$$

$$H^{0,n_s+n_u'+l+k+2;X}_{p_s+p_u'+l+k+2,q_s+q_u'+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{K}_1,K_2,K_P,K_j,\mathfrak{A},\mathfrak{A}';A' \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_u'(b-a)^{\mu_u'+\rho_u'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(u)}}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z_1''(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_l''(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(l)}}} & \mathrm{L}_1,L_j,L_Q,\mathfrak{B},\mathfrak{B}';B' \end{array}\right) \quad (4.1)$$

under the same notations and conditions that (3.21) with $A=B=U=V=0$

b) If $B(L;R_1,\cdots,R_r)=\dfrac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta_j'+\cdots+R_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b_j')_{R_1\phi_j'}\cdots\prod_{j=1}^{B^{(r)}}(b_j^{(r)})_{R_r\phi_j^{(r)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{R_1\psi_j'+\cdots+R_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d_j')_{R_1\delta_j'}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{R_r\delta_j^{(r)}}}$ (4.2)

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_r}[z_1,\cdots,z_r]$ reduces to generalized Lauricella function defined by Srivastava et al [3]. We have the following result under the same notations and conditions that (3.21)

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$F^{1+\bar{A}:B';\cdots;B^{(r)}}_{\bar{C}:D';\cdots;D^{(r)}}\left(\begin{array}{c} \mathrm{x}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{x}_r(t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_r+\rho'_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}-\lambda_j'^{(r)}} \end{array}\right|$$

$$\left.\begin{array}{c}[(\text{-L});\mathrm{R}_1,\cdots,R_r][(a);\theta',\cdots,\theta^{(r)}]:[(b');\phi'];\cdots;[(b^{(r)});\phi^{(r)}] \\ [(\mathrm{c});\psi',\cdots,\psi^{(r)}]:[(d');\delta'];\cdots;[(d^{(r)});\delta^{(r)}]\end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}} \end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z'}_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z'}_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}} \end{array}\right)$$

$$ {}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t = (b-a)^{\alpha+\beta-1}$$

$$\sum_{R_1,\cdots,R_r=0}^{h_1R_1+\cdots h_rR_r\leqslant L}(b-a)^{\sum_{i=1}^{r}R_i(\mu_i+\mu'_i+\rho_i+\rho'_i)}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j-\sum_{i=1}^{r}R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}B'_r z_1^{R_1}\cdots z_r^{R_r}$$

$$I_{U:p_s+p'_u+l+k+2,q_s+q'_u+l+k+1;Y}^{V;0,n_s+n'_u+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A} ; \mathrm{K}_1, K_2, K_P, K_j, \mathfrak{A}, \mathfrak{A}'; A' \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^k(af_j+g_j)^{\lambda'^{(u)}_j}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(l)}}} & \mathrm{B} ; \mathrm{L}_1, L_j, L_Q, \mathfrak{B}, \mathfrak{B}'; B' \end{array}\right) \quad (4.3)$$

where $B'_r = \dfrac{(-L)_{h_1R_1+\cdots+h_rR_r}B(E;R_1,\cdots,R_r)}{R_1!\cdots R_r!}$, $B[E;R_1,\ldots,R_v]$ is defined by (4.2)

Remark:

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions and a class of multivariable polynomials defined by Srivastava et al [4].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable I-functions, defined by Prasad [1], a class of multivariable polynomials defined by Srivastava et al [4] and a generalized hypergeometric function with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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