On general Eulerian integral of certain products of two multivariable A-functions and a class of multivariable polynomials

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ABSTRACT
The object of this paper is to establish an general Eulerian integral involving the product of two multivariable A-functions defined by Gautam et al [1], a class of multivariable polynomials and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular cases concerning the multivariable H-function and the Srivastava-Daoust polynomial

Keywords: Eulerian integral, multivariable A-function, Lauricella function of several variables, multivariable H-function, generalized hypergeometric function, Srivastava-Daoust polynomial

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1.Introduction

The object of this paper is to establish an general Eulerian integral involving the product of two multivariable A-functions defined by Gautam et al [1], a general class of multivariable polynomials and a generalized hypergeometric function with general argument which provide unification and extension of numerous results. The multivariable A-function is an extension of the multivariable H-function defined by Srivastava et al [7]. We will use the contracted form. The A-function is defined and represented in the following manner.

$$A(z'_1,\cdots,z'_s) = A^{m',n':m'_1,n'_1;\cdots;m'_s,n'_s}_{p',q':p'_1,q'_1;\cdots;p'_s,q'_s}\left(\begin{array}{c|c} z'_1 & (a'_j;A'^{(1)}_j,\cdots,A'^{(s)}_j)_{1,p'}: \\ \cdot & \\ \cdot & \\ \cdot & \\ z'_s & (b'_j;B'^{(1)}_j,\cdots,B'^{(s)}_j)_{1,q'}: \end{array}\right.$$

$$\left.\begin{array}{c} (c'^{(1)}_j,C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(s)}_j,C'^{(s)}_j)_{1,p'_s} \\ \\ (d'^{(1)}_j,D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(s)}_j,D'^{(s)}_j)_{1,q'_s} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\zeta_i(t_i)z'^{t_i}_i\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.2}$$

where $\phi(t_1,\cdots,t_s),\zeta_i(t_i), i=1,\cdots,s$ are given by :

$$\phi(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{m'}\Gamma(b'_j-\sum_{i=1}^{s}B'^{(i)}_jt_i)\prod_{j=1}^{n'}\Gamma(1-a'_j+\sum_{i=1}^{s}A'^{(i)}_jt_j)}{\prod_{j=n'+1}^{p'}\Gamma(a'_j-\sum_{i=1}^{s}A^{(i)}_jt_j)\prod_{j=m'+1}^{q'}\Gamma(1-b'_j+\sum_{i=1}^{s}B'^{(i)}_jt_j)} \tag{1.3}$$

and

$$\zeta_i(t_i) = \frac{\prod_{j=1}^{n'_i}\Gamma(1-c_j'^{(i)}+C_j'^{(i)}t_i)\prod_{j=1}^{m'_i}\Gamma(d_j'^{(i)}-D_j'^{(i)}t_i)}{\prod_{j=n'_i+1}^{p'_i}\Gamma(c_j'^{(i)}-C_j'^{(i)}t_i)\prod_{j=m'_i+1}^{q'_i}\Gamma(1-d_j'^{(i)}+D_j'^{(i)}t_i)} \tag{1.4}$$

Here $m', n', p', m'_i, n'_i, p'_i, c'_i \in \mathbb{N}^*; i = 1, \cdots, s\,; a'_j, b'_j, c_j'^{(i)}, d_j'^{(i)}, A_j'^{(i)}, B_j'^{(i)}, C_j'^{(i)}, D_j'^{(i)} \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i)z'_k| < \frac{1}{2}\eta_k\pi, \xi^* = 0, \eta_i > 0 \tag{1.5}$$

$$\Omega_i = \prod_{j=1}^{p'}\{A_j'^{(i)}\}^{A_j'^{(i)}}\prod_{j=1}^{q'}\{B_j'^{(i)}\}^{-B_j'^{(i)}}\prod_{j=1}^{q'_i}\{D_j'^{(i)}\}^{D_j'^{(i)}}\prod_{j=1}^{p'_i}\{C_j'^{(i)}\}^{-C_j'^{(i)}}; i = 1, \cdots, s \tag{1.6}$$

$$\xi_i^* = Im(\sum_{j=1}^{p'}A_j'^{(i)} - \sum_{j=1}^{q'}B_j'^{(i)} + \sum_{j=1}^{q'_i}D_j'^{(i)} - \sum_{j=1}^{p'_i}C_j'^{(i)}); i = 1, \cdots, s \tag{1.7}$$

$$\eta_i = Re\left(\sum_{j=1}^{n'}A_j'^{(i)} - \sum_{j=n'+1}^{p'}A_j'^{(i)} + \sum_{j=1}^{m'}B_j'^{(i)} - \sum_{j=m'+1}^{q'}B_j'^{(i)} + \sum_{j=1}^{m'_i}D_j'^{(i)} - \sum_{j=m'_i+1}^{q'_i}D_j'^{(i)} + \sum_{j=1}^{n'_i}C_j'^{(i)} - \sum_{j=n'_i+1}^{p_i}C_j'^{(i)}\right)$$

$$i = 1, \cdots, s \tag{1.8}$$

Consider the second multivariable A-function.

$$A(z''_1, \cdots, z''_u) = A^{m'',n'':m''_1,n''_1;\cdots;m''_u,n''_u}_{p'',q'':p''_1,q''_1;\cdots;p''_u,q''_u}\left(\begin{array}{c|c} \text{z"}_1 & (\text{a"}_j; A_j''^{(1)}, \cdots, A_j''^{(u)})_{1,p''} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \text{z"}_u & (\text{b"}_j; B_j''^{(1)}, \cdots, B_j''^{(u)})_{1,q''} : \end{array}\right.$$

$$\left.\begin{array}{l} (\text{c"}_j^{(1)}, C_j''^{(1)})_{1,p''_1}; \cdots; (c_j''^{(u)}, C_j''^{(u)})_{1,p''_u} \\ \\ (\text{d"}_j^{(1)}, D_j''^{(1)})_{1,q''_1}; \cdots; (d_j''^{(u)}, D_j''^{(u)})_{1,q''_u} \end{array}\right) \tag{1.9}$$

$$= \frac{1}{(2\pi\omega)^u}\int_{L''_1}\cdots\int_{L''_u}\phi'(x_1, \cdots, x_u)\prod_{i=1}^{u}\theta'_i(x_i)z_i''^{x_i}\mathrm{d}x_1\cdots\mathrm{d}x_r \tag{1.10 }$$

where $\phi'(x_1, \cdots, x_u), \theta'_i(x_i), i = 1, \cdots, u$ are given by :

$$\phi'(x_1, \cdots, x_u) = \frac{\prod_{j=1}^{m''}\Gamma(b''_j - \sum_{i=1}^{u}B_j''^{(i)}x_i)\prod_{j=1}^{n''}\Gamma(1 - a''_j + \sum_{i=1}^{u}A_j''^{(i)}x_j)}{\prod_{j=n''+1}^{p''}\Gamma(a''_j - \sum_{i=1}^{u}A_j''^{(i)}x_j)\prod_{j=m''+1}^{q''}\Gamma\left(1 - b''_j + \sum_{i=1}^{u}B_j''^{(i)}x_j\right)} \tag{1.11}$$

and

$$\theta'_i(x_i) = \frac{\prod_{j=1}^{n''_i}\Gamma(1-c''^{(i)}_j+C''^{(i)}_j x_i)\prod_{j=1}^{m''_i}\Gamma(d''^{(i)}_j-D''^{(i)}_j x_i)}{\prod_{j=n''_i+1}^{p''_i}\Gamma(c''^{(i)}_j-C''^{(i)}_j x_i)\prod_{j=m''_i+1}^{q''_i}\Gamma(1-d''^{(i)}_j+D''^{(i)}_j x_i)} \tag{1.12}$$

Here $m'', n'', p'', m''_i, n''_i, p''_i, c''_i \in \mathbb{N}^*; i = 1, \cdots, u\,; a''_j, b''_j, c''^{(i)}_j, d''^{(i)}_j, A''^{(i)}_j, B''^{(i)}_j, C''^{(i)}_j, D''^{(i)}_j \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega'_i)z''_k| < \frac{1}{2}\eta'_k\pi, \xi'^* = 0, \eta'_i > 0 \tag{1.13}$$

$$\Omega'_i = \prod_{j=1}^{p''}\{A''^{(i)}_j\}^{A''^{(i)}_j}\prod_{j=1}^{q''}\{B''^{(i)}_j\}^{-B''^{(i)}_j}\prod_{j=1}^{q''_i}\{D''^{(i)}_j\}^{D''^{(i)}_j}\prod_{j=1}^{p''_i}\{C''^{(i)}_j\}^{-C''^{(i)}_j}; i = 1, \cdots, u \tag{1.14}$$

$$\xi'^*_i = Im(\sum_{j=1}^{p''}A''^{(i)}_j - \sum_{j=1}^{q''}B''^{(i)}_j + \sum_{j=1}^{q''_i}D''^{(i)}_j - \sum_{j=1}^{p''_i}C''^{(i)}_j); i = 1, \cdots, u \tag{1.15}$$

$$\eta'_i = Re\left(\sum_{j=1}^{n''}A''^{(i)}_j - \sum_{j=n''+1}^{p''}A''^{(i)}_j + \sum_{j=1}^{m''}B''^{(i)}_j - \sum_{j=m''+1}^{q''}B''^{(i)}_j + \sum_{j=1}^{m''_i}D''^{(i)}_j - \sum_{j=m''_i+1}^{q''_i}D''^{(i)}_j + \sum_{j=1}^{n''_i}C''^{(i)}_j - \sum_{j=n''_i+1}^{p''_i}C''^{(i)}_j\right)$$

$$i = 1, \cdots, u \tag{1.16}$$

Srivastava and Garg [4] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u] = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)\frac{z_1^{R_1}\cdots z_u^{R_u}}{R_1!\cdots R_u!} \tag{1.17}$$

The coefficients are $B[E; R_1, \ldots, R_v]$ arbitrary constants, real or complex.

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [5 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \cdots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \cdots, r$
The Lauricella function $F_D^{(k)}$ is defined as

$$F_D^{(k)}[a, b_1, \cdots, b_k; c; x_1, \cdots, x_k] = \frac{\Gamma(c)}{\Gamma(a)\prod_{j=1}^{k}\Gamma(b_j)}\frac{1}{(2\pi\omega)^k}\int_{L_1}\cdots\int_{L_k}\frac{\Gamma\left(a+\sum_{j=1}^{k}\zeta_j\right)\Gamma(b_1+\zeta_1),\cdots,\Gamma(b_k+\zeta_k)}{\Gamma\left(c+\sum_{j=1}^{k}\zeta_j\right)}$$

$$\prod_{j=1}^{k}\Gamma(-\zeta_j)(-x_j)^{\zeta_i}\,\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_k \tag{2.2}$$

where $max\left[|arg(-x_1)|,\cdots,|arg(-x_k)|\right]<\pi, c\neq 0,-1,-2,\cdots$.

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t=(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha,-\sigma_1,\cdots,-\sigma_k;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right] \tag{2.3}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i\in\mathbb{C},(i=1,\cdots,k)$; $min(Re(\alpha),Re(\beta))>0$ and

$$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava et al ([6], page60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [3, page 454].

3. Eulerian integral

Let

$$X=m'_1,n'_1;\cdots;m'_s,n'_s;m''_1,n''_1;\cdots;m''_u,n''_u;1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{3.1}$$

$$Y=p'_1,q'_1;\cdots;p'_s,q'_s;p''_1,q''_1;\cdots;p''_u,q''_u;0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.2}$$

$$A=(a'_j;A'^{(1)}_j,\cdots,A'^{(s)}_j,0,\cdots,0,0,\cdots,0,0,\cdots,0)_{1,p'} \tag{3.3}$$

$$B=(b'_j;B'^{(1)}_j,\cdots,B'^{(s)}_j,0\cdots,0,0\cdots,0,0,\cdots,0)_{1,q'} \tag{3.4}$$

$$A'=(a''_j;0,\cdots,0,A''^{(1)}_j,\cdots,A''^{(u)}_j,0,\cdots,0,0,\cdots,0)_{1,p''} \tag{3.5}$$

$$B'=(b''_j;0,\cdots,0,B''^{(1)}_j,\cdots,B''^{(u)}_j,0,\cdots,0,0,\cdots,0)_{1,q''} \tag{3.6}$$

$$\mathrm{C}=(\mathrm{c}'^{(1)}_j,C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(s)}_j,C'^{(s)}_j)_{1,p'_s};(c''^{(1)}_j,C''^{(1)}_j)_{1,p''_1};\cdots;(c''^{(u)}_j,C''^{(u)}_j)_{1,p''_u}$$

$$(1,0);\cdots;(1,0);(1,0);\cdots;(1,0) \tag{3.7}$$

$$D=(\mathrm{d'}_j^{(1)},D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(s)}_j,D'^{(s)}_j)_{1,q'_s};(\mathrm{d''}^{(1)}_j,D''^{(1)}_j)_{1,q''_1};\cdots;(d''^{(u)}_j,D''^{(u)}_j)_{1,q''_u};$$

$$(0,1);\cdots;(0,1);(0,1);\cdots;(0,1) \tag{3.8}$$

$$K_1=(1-\alpha-\sum_{i=1}^{r}R_i(\mu_i+\mu'_i);\mu_1,\cdots,\mu_s,\mu'_1,\cdots,\mu'_u,1,\cdots,1,\upsilon_1,\cdots,\upsilon_l) \tag{3.9}$$

$$K_2=(1-\beta-\sum_{i=1}^{r}R_i(\rho_i+\rho'_i);\rho_1,\cdots,\rho_s,\rho'_1,\cdots,\rho'_u,0,\cdots,0,\tau_1,\cdots,\tau_l) \tag{3.10}$$

$$K_P=[1-A_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1,\cdots,1]_{1,P} \tag{3.11}$$

$$K_j=[1+\sigma_j-\sum_{i=1}^{r}R_i(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(s)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(u)},0\cdots,1,\cdots,0,\zeta'_j,\cdots,\zeta_j^{(l)}]_{1,k} \tag{3.12}$$

j

$$L_1=(1-\alpha-\beta-\sum_{i=1}^{r}R_i(\mu_i+\mu'_i+\rho_i+\rho'_i);\mu_1+\rho_1,\cdots,\mu_s+\rho_s,\mu'_1+\rho'_1,\cdots,\mu'_u+\rho'_u,$$

$$1,\cdots,1,\upsilon_1+\tau_1,\cdots,\upsilon_l+\tau_l) \tag{3.13}$$

$$L_Q=[1-B_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1\cdots,1]_{1,Q} \tag{3.14}$$

$$L_j=[1+\sigma_j-\sum_{i=1}^{r}R_i(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(s)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(u)},0\cdots,0,\zeta'_j,\cdots,\zeta_j^{(l)}]_{1,k} \tag{3.15}$$

$$B_r=\frac{(-L)_{h_1R_1+\cdots+h_rR_r}B(E;R_1,\cdots,R_r)}{R_1!\cdots R_r!} \tag{3.16}$$

$$A=A,A';\mathfrak{B}=B,B' \tag{3.17}$$

We have the following result

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_r}\begin{pmatrix} \mathrm{x}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{x}_r(t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_r+\rho'_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}-\lambda_j'^{(r)}} \end{pmatrix}$$

$$A\begin{pmatrix} z_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}} \end{pmatrix} A\begin{pmatrix} z'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(1)}_j} \\ \cdot \\ \cdot \\ \cdot \\ z'_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(u)}_j} \end{pmatrix}$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=(b-a)^{\alpha+\beta-1}$$

$$\sum_{R_1,\cdots,R_r=0}^{h_1R_1+\cdots h_rR_r\leqslant L}(b-a)^{\sum_{i=1}^{r}R_i(\mu_i+\mu'_i+\rho_i+\rho'_i)}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j-\sum_{i=1}^{r}R_i(\lambda_j^{(i)}+\lambda'^{(i)}_j)}B_rz_1^{R_1}\cdots z_r^{R_r}$$

$$A^{m'+m'',n'+n''+l+k+2;X}_{p'+p''+l+k+2,q'+q''+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathfrak{A}\ ,\mathrm{K}_1,K_2,K_P,K_j:C \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(u)}_j}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \mathfrak{B}\ ,\mathrm{L}_1,L_j,L_Q,:D \end{array}\right) \quad (3.18)$$

We obtain the A-function of $s+u+k+l$ variables.

Provided that

(A) $m',n',p',m'_i,n'_i,p'_i,c'_i \in \mathbb{N}^*; i=1,\cdots,s$ **;** $a'_j,b'_j,c'^{(i)}_j,d'^{(i)}_j,A'^{(i)}_j,B'^{(i)}_j,C'^{(i)}_j,D'^{(i)}_j \in \mathbb{C}$

$m'',n'',p'',m''_i,n''_i,p''_i,c''_i \in \mathbb{N}^*; i=1,\cdots,u$ **;** $a''_j,b''_j,c''^{(i)}_j,d''^{(i)}_j,A''^{(i)}_j,B''^{(i)}_j,C''^{(i)}_j,D''^{(i)}_j \in \mathbb{C}$

(B) $m,n \in \mathbb{N}, \gamma_i,\tau_i,\gamma'_i,\tau'_i,c^{(i)}_j,c'^{(i)}_j,\lambda_l,\mu_l,a^{(l)}_j \in \mathbb{R}^+, \rho_j \in \mathbb{R}, p_i,q_i \in \mathbb{C}$**;** $\zeta^{(i)}_j > 0 (i=1,\cdots,l; j=1,\cdots,k)$

$$(C)\ \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\} < 1$$

(D) $$Re\big[\alpha+\sum_{i=1}^{s}\mu_i \min_{1\leqslant j\leqslant m'_i}\frac{d'^{(i)}_j}{D'^{(i)}_j}+\sum_{i=1}^{u}\mu'_i \min_{1\leqslant j\leqslant m''_i}\frac{d''^{(i)}_j}{D''^{(i)}_j}\big] > 0$$

$$Re\big[\beta+\sum_{i=1}^{s}\rho_i \min_{1\leqslant j\leqslant m'_i}\frac{d'^{(i)}_j}{D'^{(i)}_j}+\sum_{i=1}^{u}\rho'_i \min_{1\leqslant j\leqslant m''_i}\frac{d''^{(i)}_j}{D''^{(i)}_j}\big] > 0$$

(E) $$\xi^*_i = Im\left(\sum_{j=1}^{p'}A'^{(i)}_j-\sum_{j=1}^{q'}B'^{(i)}_j+\sum_{j=1}^{q'_i}D'^{(i)}_j-\sum_{j=1}^{p'_i}C'^{(i)}_j\right)=0; i=1,\cdots,s$$

$$\xi'^*_i = Im\left(\sum_{j=1}^{p''}A''^{(i)}_j-\sum_{j=1}^{q''}B''^{(i)}_j+\sum_{j=1}^{q''_i}D''^{(i)}_j-\sum_{j=1}^{p''_i}C''^{(i)}_j\right)=0; i=1,\cdots,u$$

(F) $|arg(\Omega_i)z_k| < \frac{1}{2}\eta_k\pi, \xi^*=0, \eta_i>0$

$$Re\left(\sum_{j=1}^{n}A'^{(i)}_j-\sum_{j=n+1}^{p}A'^{(i)}_j+\sum_{j=1}^{m}B'^{(i)}_j-\sum_{j=m+1}^{q}B'^{(i)}_j+\sum_{j=1}^{m_i}D'^{(i)}_j-\sum_{j=m_i+1}^{q_i}D'^{(i)}_j+\sum_{j=1}^{n_i}C'^{(i)}_j-\sum_{j=n_i+1}^{p_i}C'^{(i)}_j\right)$$

$$-\mu'_i-\rho'_i-\sum_{l=1}^{k}\lambda'^{(i)}_j > 0\,; i=1,\cdots,s$$

$$|arg(\Omega'_i)z'_k| < \frac{1}{2}\eta'_k\pi, \xi'^*=0, \eta'_i>0$$

$$Re\left(\sum_{j=1}^{n''}A''^{(i)}_j-\sum_{j=n''+1}^{p''}A''^{(i)}_j+\sum_{j=1}^{m''}B''^{(i)}_j-\sum_{j=m''+1}^{q''}B''^{(i)}_j+\sum_{j=1}^{m''_i}D''^{(i)}_j-\sum_{j=m''_i+1}^{q''_i}D''^{(i)}_j+\sum_{j=1}^{n''_i}C''^{(i)}_j-\sum_{j=n''_i+1}^{p''_i}C''^{(i)}_j\right)$$

$$-\mu'_i-\rho'_i-\sum_{l=1}^{k}\lambda'^{(i)}_j > 0\,; i=1,\cdots,u$$

(G) $$\left|arg\left(z_i\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda^{(i)}_j}\right)\right| < \frac{1}{2}\eta_i\pi \quad (a\leqslant t\leqslant b; i=1,\cdots,s)$$

$$\left|arg\left(z'_i\prod_{j=1}^k(f_jt+g_j)^{-\lambda'^{(i)}_j}\right)\right| < \frac{1}{2}\eta'_i\pi \quad (a\leqslant t\leqslant b; i=1,\cdots,u)$$

(H) $P\leqslant Q+1$. The equality holds, when , in addition,

either $P>Q$ and $\left|z''_i\left(\prod_{j=1}^k(f_jt+g_j)^{-\zeta^{(i)}_j}\right)\right|^{\frac{1}{Q-P}} <1 \qquad (a\leqslant t\leqslant b)$

or $P\leqslant Q$ and $\max_{1\leqslant i\leqslant k}\left[\left|\left(z''_i\prod_{j=1}^k(f_jt+g_j)^{-\zeta^{(i)}_j}\right)\right|\right]<1 \qquad (a\leqslant t\leqslant b)$

Proof

First expressing the the class of multivariable polynomials defined by Srivastava et al [4] in serie with the help of (1.17), expressing the A-function of s-variables and u-variables defined by Gautam et al [1] by the Mellin-Barnes contour integral with the help of the equation (1.2) and (1.10) respectively, the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1). Now collect the power of (f_jt+g_j) with $j=1,\cdots,k$ and use the equations (2.1) and (2.2) and we obtain $k-$Mellin-Barnes contour integral. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and interpreting $(r+s+k+l)-$Mellin-barnes contour integral in multivariable A-function defined by Gautam et al [1], we obtain the desired result.

4. Multivariable H-function

If $A'^{(i)}_j, B'^{(i)}_j, C'^{(i)}_j, D'^{(i)}_j \in \mathbb{R}$,$m'=0$ and $A''^{(i)}_j, B''^{(i)}_j, C''^{(i)}_j, D''^{(i)}_j \in \mathbb{R}$ and $m''=0$, the multivariable A-functions reduces to multivariable H-functions defined by Srivastava et al [7]. We obtain the following formula.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^k(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_r}\begin{pmatrix} \mathrm{x}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda^{(1)}_j-\lambda'^{(1)}_j}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{x}_r(t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_r+\rho'_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda^{(r)}_j-\lambda'^{(r)}_j} \end{pmatrix}$$

$$H\begin{pmatrix} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda^{(1)}_j}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^k(f_jt+g_j)^{-\lambda^{(s)}_j} \end{pmatrix}$$

$$H\begin{pmatrix} z'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(1)}_j} \\ . \\ . \\ . \\ z'_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(u)}_j} \end{pmatrix}$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=(b-a)^{\alpha+\beta-1}$$

$$\sum_{R_1,\cdots,R_r=0}^{h_1R_1+\cdots h_rR_r\leqslant L}(b-a)^{\sum_{i=1}^{r}R_i(\mu_i+\mu'_i+\rho_i+\rho'_i)}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j-\sum_{i=1}^{r}R_i(\lambda_j^{(i)}+\lambda'^{(i)}_j)}B_rz_1^{R_1}\cdots z_r^{R_r}$$

$$H^{0,n'+n''+l+k+2;X}_{p'+p''+l+k+2,q'+q''+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathfrak{A}\ ,\mathrm{K}_1,K_2,K_P,K_j:C \\ \cdots & . \\ \cdots & . \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(s)}}} & . \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(u)}_j}} & . \\ \frac{(b-a)f_1}{af_1+g_1} & . \\ \cdots & . \\ \cdots & . \\ \frac{(b-a)f_k}{af_k+g_k} & . \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \mathfrak{B}\ ,\mathrm{L}_1,L_j,L_Q,:D \end{array}\right) \quad (4.1)$$

under the same conditions and notations that (3.18) with $A'^{(i)}_j, B'^{(i)}_j, C'^{(i)}_j, D'^{(i)}_j \in \mathbb{R}$ $,m'=0$ and

$A_j''^{(i)}, B_j''^{(i)}, C_j''^{(i)}, D_j''^{(i)} \in \mathbb{R}$ and $m''=0$

b) If $B(L;R_1,\cdots,R_r) = \dfrac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(r)}}(b_j^{(r)})_{R_r\phi_j^{(r)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{R_1\psi'_j+\cdots+R_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{R_r\delta_j^{(r)}}}$ (4.2)

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_r}[z_1,\cdots,z_r]$ reduces to generalized Lauricella function defined by Srivastava et al [3]. We have

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$F_{\bar{C}:D';\cdots;D^{(r)}}^{1+\bar{A}:B';\cdots;B^{(r)}}\left(\begin{array}{c}\mathrm{x}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{x}_r(t-a)^{\mu_u+\mu'_u}(b-t)^{\rho_r+\rho'_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}-\lambda_j'^{(r)}}\end{array}\right|$$

$$\left.\begin{array}{c}[(\text{-L});\mathrm{R}_1,\cdots,R_r][(a);\theta',\cdots,\theta^{(r)}]:[(b');\phi'];\cdots;[(b^{(r)});\phi^{(r)}]\\ [(\mathrm{c});\psi',\cdots,\psi^{(r)}]:[(d');\delta'];\cdots;[(d^{(r)});\delta^{(r)}]\end{array}\right)$$

$$A\left(\begin{array}{c}\mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}}\end{array}\right)$$

$$A\left(\begin{array}{c}\mathrm{z'}_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z'}_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}}\end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=(b-a)^{\alpha+\beta-1}$$

$$\sum_{R_1,\cdots,R_r=0}^{h_1R_1+\cdots h_rR_r\leqslant L}(b-a)^{\sum_{i=1}^r R_i(\mu_i+\mu_i'+\rho_i+\rho_i')}\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}^k(af_j+g_j)^{\sigma_j-\sum_{i=1}^r R_i(\lambda_j^{(i)}+\lambda_j'^{(i)})}B_r'z_1^{R_1}\cdots z_r^{R_r}$$

$$A_{p'+p''+l+k+2,q'+q''+l+k+1;Y}^{m'+m'',n'+n''+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathfrak{A}\ ,\mathrm{K}_1,K_2,K_P,K_j:C \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_u'(b-a)^{\mu_u'+\rho_u'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(u)}}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z_1''(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\varsigma_j^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_l''(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\varsigma_j^{(l)}}} & \mathfrak{B}\ ,\mathrm{L}_1,L_j,L_Q,:D \end{array}\right) \quad (4.3)$$

under the same notations and conditions that (3.18)

where $B_r'=\dfrac{(-L)_{h_1R_1+\cdots+h_rR_r}B(E;R_1,\cdots,R_r)}{R_1!\cdots R_r!}$, $B[E;R_1,\ldots,R_v]$ is defined by (4.2)

Remark:

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable A-functions and a class of multivariable polynomials defined by Srivastava et al [4].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable A-functions, defined by Gautam et al [1], a class of multivariable polynomials defined by Srivastava et al [4] and a generalized hypergeometric function with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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