Selberg integral involving the S generalized Gauss's hypergeometric function, a class of polynomials the multivariable I-function and multivariable Aleph-functions

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ABSTRACT

In the present paper we evaluate the Selberg integral involving the S generalized Gauus's hypergeometric function, a multivariable Aleph-function, the multivariable I-function defined by Nambisan et al [3] and general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the I-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializating the parameters their in.

Keywords:Multivariable Aleph-function, general class of polynomials, Selberg integral, General sequence of functions, multivariable I-function, multivariable H-function

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1.Introduction and preliminaries.

The function Aleph of several variables generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [4] , itself is an a generalisation of G and H-functions of multiple variables defined by Srivastava et al [7]. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $$\aleph(z_1,\cdots,z_r)=\aleph^{0,N:M_1,N_1,\cdots,M_r,N_r}_{P_i,Q_i,\tau_i;R:P_{i^{(1)}},Q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;P_{i^{(r)}},Q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_r \end{array}\right|$$

$$[(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,N}],[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{N+1,P_i}]:$$
$$\cdots\cdots\cdots,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{M+1,Q_i}]:$$

$$\left.\begin{array}{l}[(c_j^{(1)});\gamma_j^{(1)})_{1,N_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}];\cdots;[(c_j^{(r)});\gamma_j^{(r)})_{1,N_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{N_r+1,P_i^{(r)}}]\\ [(d_j^{(1)});\delta_j^{(1)})_{1,M_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}];\cdots;[(d_j^{(r)});\delta_j^{(r)})_{1,M_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{M_r+1,Q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^r\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^N\Gamma(1-a_j+\sum_{k=1}^r\alpha_j^{(k)}s_k)}{\sum_{i=1}^R[\tau_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\sum_{k=1}^r\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{Q_i}\Gamma(1-b_{ji}+\sum_{k=1}^r\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $\theta_k(s_k) = \dfrac{\prod_{j=1}^{M_k}\Gamma(d_j^{(k)} - \delta_j^{(k)}s_k)\prod_{j=1}^{N_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]}$ (1.3)

Suppose , as usual , that the parameters

$b_j, j = 1,\cdots,Q; a_j, j = 1,\cdots,P;$

$c_{ji^{(k)}}^{(k)}, j = n_k+1,\cdots,P_{i^{(k)}}; c_j^{(k)}, j = 1,\cdots,N_k;$

$d_{ji^{(k)}}^{(k)}, j = M_k+1,\cdots,Q_{i^{(k)}}; d_j^{(k)}, j = 1,\cdots,M_k;$

with $k = 1\cdots,r, i = 1,\cdots,R$, $i^{(k)} = 1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N}\alpha_j^{(k)} + \tau_i\sum_{j=N+1}^{P_i}\alpha_{ji}^{(k)} + \sum_{j=1}^{N_k}\gamma_j^{(k)} + \tau_{i^{(k)}}\sum_{j=n_k+1}^{P_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} - \tau_i\sum_{j=1}^{Q_i}\beta_{ji}^{(k)} - \sum_{j=1}^{M_k}\delta_j^{(k)}$$

$$-\tau_{i^{(k)}}\sum_{j=M_k+1}^{Q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} \leqslant 0 \qquad (1.4)$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)}s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(k)}s_k)$ with $j = 1$ to N and $\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{N}\alpha_j^{(k)} - \tau_i\sum_{j=N+1}^{P_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{Q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{N_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=N_k+1}^{P_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{M_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=M_k+1}^{Q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1\cdots,r, i = 1,\cdots,R \text{ , } i^{(k)} = 1,\cdots,R^{(k)} \qquad (1.5)$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(y_1,\cdots,y_r) = 0(|y_1|^{\alpha_1}\dots|y_r|^{\alpha_r}), max(|y_1|\dots|y_r|) \rightarrow 0$

$\aleph(y_1,\cdots,y_r) = 0(|y_1|^{\beta_1}\dots|y_r|^{\beta_r}), min(|y_1|\dots|y_r|) \rightarrow \infty$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, M_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, N_k$$

Serie representation of Aleph-function of several variables is given by

$$\aleph(y_1, \cdots, y_r) = \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{M_1} \cdots \sum_{g_r=0}^{M_r} \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \psi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r})$$

$$\times \ \theta_1(\eta_{G_1,g_1}) \cdots \theta_r(\eta_{G_r,g_r}) y_1^{-\eta_{G_1,g_1}} \cdots y_r^{-\eta_{G_r,g_r}} \quad (1.6)$$

Where $\psi(., \cdots, .), \theta_i(.), i = 1, \cdots, r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1} = \frac{d_{g_1}^{(1)} + G_1}{\delta_{g_1}^{(1)}}, \cdots, \ \eta_{G_r,g_r} = \frac{d_{g_r}^{(r)} + G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i + p_i] \neq \delta_j^{(i)}[d_{g_i}^i + G_i]$ (1.7)

for $j \neq M_i, M_i = 1, \cdots \eta_{G_i,g_i}; P_i, N_i = 0, 1, 2, \cdots, ; y_i \neq 0, i = 1, \cdots, r$ (1.8)

In the document , we will note :

$$G(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}) = \phi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}) \theta_1(\eta_{G_1,g_1}) \cdots \theta_r(\eta_{G_r,g_r}) \quad (1.9)$$

where $\phi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}), \theta_1(\eta_{G_1,g_1}), \cdots, \theta_r(\eta_{G_r,g_r})$ are given respectively in (1.2) and (1.3)

We will note the Aleph-function of r variables $\aleph_{u:w}^{0,N:v}\left(\begin{array}{c} z_1 \\ \cdots \\ z_r \end{array}\right)$ (1.10)

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1, \cdots, z_s) = I_{p',q':p'_1,q'_1;\cdots;p'_s,q'_s}^{0,n':m'_1,n'_1;\cdots;m'_s,n'_s} \left(\begin{array}{c|c} z_1 & (\mathrm{a'}_j; \alpha_j'^{(1)}, \cdots, \alpha_j'^{(s)}; A'_j)_{1,p'} : \\ \cdot & \\ \cdot & \\ \cdot & \\ z_s & (\mathrm{b'}_j; \beta_j'^{(1)}, \cdots, \beta_j'^{(s)}; B'_j)_{1,q'} : \end{array}\right.$$

$$\left.\begin{array}{l} (\mathrm{c'}_j^{(1)}, \gamma_j'^{(1)}; C_j'^{(1)})_{1,p'_1}; \cdots; (c_j'^{(s)}, \gamma_j'^{(s)}; C_j'^{(r)})_{1,p'_r} \\ \\ (\mathrm{d'}_j^{(1)}, \delta_j'^{(1)}; D_j'^{(1)})_{1,q'_1}; \cdots; (d_j'^{(s)}, \delta_j'^{(s)}; D_j'^{(s)})_{1,q'_r} \end{array}\right) \quad (1.11)$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L'_1} \cdots \int_{L'_s} \phi(t_1, \cdots, t_s) \prod_{i=1}^{s} \theta_i(t_i) z_i^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.12}$$

where $\phi(t_1, \cdots, t_s), \theta_i(t_i), i = 1, \cdots, s$ are given by :

$$\phi(t_1, \cdots, t_s) = \frac{\prod_{j=1}^{n'} \Gamma^{A'_j}\left(1 - a'_j + \sum_{i=1}^{s} \alpha'^{(i)}_j t_j\right)}{\prod_{j=n'+1}^{p'} \Gamma^{A'_j}\left(a'_j - \sum_{i=1}^{s} \alpha'^{(i)}_j t_j\right) \prod_{j=m'+1}^{q'} \Gamma^{B'_j}\left(1 - b'_j + \sum_{i=1}^{s} \beta^{(i)}_j t_j\right)} \tag{1.13}$$

$$\theta_i(t_i) = \frac{\prod_{j=1}^{n'_i} \Gamma^{C'^{(i)}_j}\left(1 - c'^{(i)}_j + \gamma'^{(i)}_j t_i\right) \prod_{j=1}^{m'_i} \Gamma^{D'^{(i)}_j}\left(d'^{(i)}_j - \delta'^{(i)}_j t_i\right)}{\prod_{j=n'_i+1}^{p'_i} \Gamma^{C'^{(i)}_j}\left(c'^{(i)}_j - \gamma'^{(i)}_j t_i\right) \prod_{j=m'_i+1}^{q'_i} \Gamma^{D'^{(i)}_j}\left(1 - d'^{(i)}_j + \delta'^{(i)}_j t_i\right)} \tag{1.14}$$

For more details, see Nambisan et al [3].

Following the result of Braaksma [2] the I-function of r variables is analytic if

$$U_i = \sum_{j=1}^{p'} A'_j \alpha'^{(i)}_j - \sum_{j=1}^{q'} B'_j \beta'^{(i)}_j + \sum_{j=1}^{p'_i} C'^{(i)}_j \gamma'^{(i)}_j - \sum_{j=1}^{q'_i} D'^{(i)}_j \delta'^{(i)}_j \leqslant 0, i = 1, \cdots, s \tag{1.15}$$

The integral (2.1) converges absolutely if

where$|arg(z'_k)| < \frac{1}{2}\Delta_k \pi, k = 1, \cdots, s$

$$\Delta_k = -\sum_{j=n'+1}^{p'} A'_j \alpha'^{(k)}_j - \sum_{j=1}^{q'} B'_j \beta^{(k)}_j + \sum_{j=1}^{m'_k} D'^{(k)}_j \delta'^{(k)}_j - \sum_{j=m'_k+1}^{q'_k} D'^{(k)}_j \delta'^{(k)}_j + \sum_{j=1}^{n'_k} C'^{(k)}_j \gamma'^{(k)}_j - \sum_{j=n'_k+1}^{p'_k} C'^{(k)}_j \gamma'^{(k)}_j > 0 \tag{1.16}$$

We will note :

$$X = m'_1, n'_1; \cdots; m'_s, n'_s \tag{1.17}$$

$$Y = p'_1, q'_1; \cdots; p'_s, q'_s \tag{1.18}$$

$$A = (a'_j; \alpha'^{(1)}_j, \cdots, \alpha'^{(s)}_j; A'_j)_{1,p'} \tag{1.19}$$

$$B = (b'_j; \beta'^{(1)}_j, \cdots, \beta'^{(s)}_j; B'_j)_{1,q'} \tag{1.20}$$

$$\mathrm{C} = (\mathrm{c}'^{(1)}_j, \gamma'^{(1)}_j; C'^{(1)}_j)_{1,p'_1}; \cdots; (c'^{(s)}_j, \gamma'^{(s)}_j; C'^{(s)}_j)_{1,p'_s} \tag{1.21}$$

$$D = (\mathrm{d}'^{(1)}_j, \delta'^{(1)}_j; D'^{(1)}_j)_{1,q'_1}; \cdots; (d'^{(s)}_j, \delta'^{(s)}_j; D'^{(s)}_j)_{1,q_s} \tag{1.22}$$

The contracted form is :

$$I(z_1,\cdots,z_s) = I_{p',q';Y}^{0,n';X}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{C} \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s & \mathrm{B}:\mathrm{D} \end{array}\right) \tag{1.23}$$

The generalized polynomials defined by Srivastava [5], is given in the following manner :

$$S_{N_1',\cdots,N_t'}^{M_1',\cdots,M_t'}[y_1,\cdots,y_t] = \sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\frac{(-N_1')_{M_1'K_1}}{K_1!}\cdots\frac{(-N_t')_{M_t'K_t}}{K_t!}$$

$$A[N_1',K_1;\cdots;N_t',K_t]y_1^{K_1}\cdots y_t^{K_t} \tag{1.24}$$

Where $M_1',\cdots,M_s'$ are arbitrary positive integers and the coefficients $A[N_1',K_1;\cdots;N_t',K_t]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_K = \frac{(-N_1')_{M_1'K_1}}{K_1!}\cdots\frac{(-N_t')_{M_t'K_t}}{K_t!}A[N_1',K_1;\cdots;N_t',K_t] \tag{1.25}$$

In the document , we note :

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}) = \phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r}) \tag{1.26}$$

where $\phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}),\theta_1(\eta_{G_1,g_1}),\cdots,\theta_r(\eta_{G_r,g_r})$ are given respectively in (1.2) and (1.3)

2. S Generalized Gauss's hypergeometric function

The S generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(a,b,c;z)$ introduced and defined by Srivastava et al [6, page 350 , Eq.(1.12)] is represented in the following manner :

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b,c;z) = \sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \qquad (|z|<1) \tag{2.1}$$

provided that $(Re(p)\geqslant 0, minRe(\alpha,\beta,\tau,\mu)>0; Re(c)>Re(b)>0)$

where the S generalized Beta function $B_p^{(\alpha,\beta;\tau,\mu)}(x,y)$ was introduced and defined by Srivastava et al [6, page 350, Eq(1.13)]

$$B_p^{(\alpha,\beta;\tau,\mu)}(x,y) = \int_1^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^\tau(1-t)^\mu}\right)\mathrm{d}t \tag{2.2}$$

provided that $(Re(p)\geqslant 0, minRe(x,y,\alpha,\beta)>0; Min\{Re(\tau),Re(\mu\}>0)$

3. Required integral

We have the following result, see (Beals et al [1], page 54)

$$S(a,b,c)=\int_0^1\cdots\int_0^1\prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$=\prod_{j=0}^{n-1}\frac{\Gamma(a+jc)\Gamma(b+jc)\Gamma(1+(j+1)c)}{\Gamma(a+b+(n-1+j)c)\Gamma(1+c)} \tag{3.1}$$

with $Re(a)>0, Re(b)>0, Re(c)>Max\left\{-\frac{1}{n},-\frac{Re(a)}{n-1},-\frac{Re(b)}{n-1}\right\}$

$S(a,b,c)$ is the Selberg integral with three parameters

4. Main integral

Let $X_{u,v,w}=\prod_{i=1}^n x_i^u(1-x_i)^v\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2w}$

we have the following formula

Theorem

$$\int_0^1\cdots\int_0^1\prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\ \phi_{\lambda,\mu,v}(zX_{\alpha,\beta,\gamma},s',a)$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}\left(\begin{array}{c}\mathrm{y}_1X_{\alpha_1,\beta_1,\gamma_1}\\ \cdots\\ \mathrm{y}_tX_{\alpha_t,\beta_t,\gamma_t}\end{array}\right)\aleph_{u:w}^{0,N:v}\left(\begin{array}{c}\mathrm{z}_1X_{\delta_1,\psi_1,\phi_1}\\ \cdots\\ \mathrm{z}_rX_{\delta_r,\psi_r,\phi_r}\end{array}\right)I_{p',q';Y}^{0,n';X}\left(\begin{array}{c}\mathrm{Z}_1X_{\epsilon_1,\eta_1,\zeta_1}\\ \cdots\\ \mathrm{Z}_sX_{\epsilon_s,\eta_s,\zeta_s}\end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$\sum_{m=0}^{\infty}\ \sum_{G_1,\cdots,G_r=0}^{\infty}\ \sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\ \sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}a_K\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}(a)_m\frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+m,c-b)}{B(b,c-b)}\frac{z^m}{m!}$$

$$G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})\,y_1^{K_1}\cdots y_t^{K_t}z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}\ I_{p'+3n,q'+n+1;Y}^{0,n'+3n;X}\left(\begin{array}{c}\mathrm{Z}_1\\ \cdots\\ \cdots\\ \mathrm{Z}_s\end{array}\middle|\begin{array}{l}\mathrm{A},\\ \\ \\ \mathrm{B},\end{array}\right)$$

$$[\text{1-a-}\alpha m-\sum_{i=1}^t K_i\alpha_i-\sum_{i=1}^r\eta_{G_i,g_i}\delta_i-j(c+\gamma m+\sum_{i=1}^t\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i}),\epsilon_1+j\zeta_1,\cdots,\epsilon_s+j\zeta_s;1]_{0,n-1}$$
$$\cdots$$
$$(\text{-c-}\gamma m+\sum_{i=1}^t K_i\gamma_i-\sum_{i=1}^r\phi_i\eta_{G_i,g_i},\zeta_1,\cdots,\zeta_s;n),$$

$$[\text{1-b-}\beta m-\sum_{i=1}^t K_i\beta_i-\sum_{i=1}^r\eta_{G_i,g_i}\psi_i-j(c+\gamma m+\sum_{i=1}^t\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i}),\eta_1+j\zeta_1,\cdots,\eta_s+j\zeta_s;1]_{0,n-1}$$
$$\cdots$$
$$\mathrm{B}_1$$

$$-(j+1)(c+\gamma m+\sum_{i=1}^{t}\gamma_i K_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i}),j\zeta_1,\cdots,j\zeta_s);1]_{0,n-1}:C \\ \cdots \\ D \Bigg) \tag{4.1}$$

where $$B_1=[1-a-b-(\alpha+\beta)m-\sum_{i=1}^{t}K_i(\alpha_i+\beta_i)-\sum_{i=1}^{r}(\delta_i+\phi_i)\eta_{G_i,g_i}-(n-1+j)\times$$

$$(c+\gamma m+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});\epsilon_1+\eta_1+j\zeta_1,\cdots,\epsilon_s+\eta_s+j\zeta_s;1]_{0,n-1} \tag{4.2}$$

Provided that

a) $min\{\alpha,\beta,\gamma,\alpha_i,\beta_i,\gamma_i,\delta_j,\psi_j,\phi_j,\epsilon_k,\eta_k,\zeta_k\}>0,i=1,\cdots,t,j=1,\cdots,r,k=1,\cdots,s$,

b)$A=Re[a+\alpha m+\sum_{i=1}^{r}\delta_i\min_{1\leqslant j\leqslant M_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\epsilon_i\min_{1\leqslant j\leqslant m'-i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}]>0$

c) $B=Re[b+\beta m+\sum_{i=1}^{r}\psi_i\min_{1\leqslant j\leqslant M_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\eta_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}]>0$

d) $C=Re[c+\gamma m+\sum_{i=1}^{r}\phi_i\min_{1\leqslant j\leqslant M_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^{s}\zeta_i\min_{1\leqslant j\leqslant m'_i}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}]>Max\left\{-\frac{1}{n},-\frac{A}{n-1},-\frac{B}{n-1}\right\}$

e)$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.5) ; $i=1,\cdots,r$

f) $|argZ_k|<\frac{1}{2}\Delta_k\pi$, where Δ_k is defined by (1.16) ; $i=1,\cdots,s$

g) The multiple series occuring on the right-hand side of (4.1) is absolutely and uniformly convergent.

h) $(Re(p)\geqslant 0,minRe(\alpha,\beta,\tau,\mu)>0;Re(c)>Re(b)>0)$

Proof

First, expressing the generalized the sequence of functions $F_p^{(\alpha,\beta;\tau,\mu)}(a,b,c;.)$in serie with the help of equation (2.1), the Aleph-function of r variables in series with the help of equation (1.6), the general class of polynomial of several variables $S_{N_1',\cdots,N_t'}^{M_1',\cdots,M_t'}[y_1,\cdots,y_t]$ with the help of equation (1.24) and the I-function of s variables in defined by Nambisan et al [4] in Mellin-Barnes contour integral with the help of equation (1.12), changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) . Now evaluating the resulting Selberg integral with the help of equation (3.1). Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

5. Particular cases

1) If $A_j'=B_j'=C_j'^{(i)}=D_j'^{(i)}=1$, The multivariable I-functions defined by Nambisan reduces to multivariable H-function defined by Srivastava et al [7]. We have.

Corollary

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1} \prod_{1\leqslant j<k\leqslant n} |x_j - x_k|^{2c} \quad \phi_{\lambda,\mu,v}(zX_{\alpha,\beta,\gamma}, s', a)$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}\left(\begin{array}{c} \mathrm{y}_1 X_{\alpha_1,\beta_1,\gamma_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_t X_{\alpha_t,\beta_t,\gamma_t} \end{array}\right) \aleph_{u:w}^{0,N:v}\left(\begin{array}{c} \mathrm{z}_1 X_{\delta_1,\psi_1,\phi_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r X_{\delta_r,\psi_r,\phi_r} \end{array}\right) H_{p',q';Y}^{0,n';X}\left(\begin{array}{c} \mathrm{Z}_1 X_{\epsilon_1,\eta_1,\zeta_1} \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_s X_{\epsilon_s,\eta_s,\zeta_s} \end{array}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n =$$

$$\sum_{m=0}^{\infty} \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{M_1} \cdots \sum_{g_r=0}^{M_r} \sum_{K_1=0}^{[N_1'/M_1']} \cdots \sum_{K_t=0}^{[N_t'/M_t']} a_K \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} (a)_m \frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+m, c-b)}{B(b, c-b)} \frac{z^m}{m!}$$

$$G(\eta_{G_1,g_1}, \cdots \eta_{G_r,g_r})\, y_1^{K_1} \cdots y_t^{K_t} z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}} \; H_{p'+3n,q'+2n;Y}^{0,n'+3n;X}\left(\begin{array}{c|l} \mathrm{Z}_1 & \mathrm{A}, \\ \cdot\ \cdot\ \cdot & \\ \cdot\ \cdot\ \cdot & \\ \mathrm{Z}_s & \mathrm{B}, \end{array}\right.$$

$$[1\text{-a-}\alpha m - \sum_{i=1}^t K_i\alpha_i - \sum_{i=1}^r \eta_{G_i,g_i}\delta_i - j(c+\gamma m + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_{G_i,g_i}), \epsilon_1 + j\zeta_1, \cdots, \epsilon_s + j\zeta_s]_{0,n-1}$$
$$\cdot\ \cdot\ \cdot$$
$$(\text{-c-}\gamma m + \sum_{i=1}^t K_i\gamma_i - \sum_{i=1}^r \phi_i \eta_{G_i,g_i}, \zeta_1, \cdots, \zeta_s), \cdots,$$

$$[1\text{-b-}\beta m - \sum_{i=1}^t K_i\beta_i - \sum_{i=1}^r \eta_{G_i,g_i}\psi_i - j(c+\gamma s' + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_{G_i,g_i}); \eta_1 + j\zeta_1, \cdots, \eta_s + j\zeta_s]_{0,n-1}$$
$$\cdot\ \cdot\ \cdot$$
$$(\text{-c-}\gamma s' - \sum_{i=1}^t \gamma_i K_i - \sum_{i=1}^r \phi_i \eta_{G_i,g_i}; \zeta_1, \cdots, \zeta_s), B_1$$

$$\left.\begin{array}{c} \text{-(j+1)(c+}\gamma m + \sum_{i=1}^t \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_{G_i,g_i}), j\zeta_1, \cdots, j\zeta_s)]_{0,n-1} : C \\ \cdot\ \cdot\ \cdot \\ \mathrm{D} \end{array}\right) \tag{5.1}$$

under the same notations and conditions that (4.1) with $A_j' = B_j' = C_j'^{(i)} = D_j'^{(i)} = 1$

6. Conclusion

In this paper we have evaluated a modified Selberg integral involving the product of the multivariable Aleph-function , the multivariable I-function defined by Nambisan et al [3], the S generalized Gauss hypergeometric function and a general class of polynomials of several variables. The integral established in this paper is of very general nature as it contains multivariable I-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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