Some results on Invariant Submanifolds in an Indefinite Trans-Sasakian Manifold

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Abstract: The purpose of this paper is to study invariant submanifolds in a indefinite trans-Sasakian manifold. Necessary and sufficient conditions are given on an submanifold of a indefinite trans-Sasakian manifold to be invariant and invariant case is considered. In this case further properties and some

theorems are given related to an invariant submanifolds in a indefinite trans-Sasakian manifold.

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1. Introduction

Contact structure has most important applications in physics. Many authors gave their valuable and essential results on differential geometry. In 1976 K.Yano and M.Kon introduced invariant and anti-invariant submanifolds in [1]. J.A.Oubina [2] introduced the notion of a tran sasakian manifold of type (α, β).Trans sasakian manifold is an important kind of sasakian manifold such that $\alpha = 1$ and $\beta = 1$.

In [3]A.Bejancu and K.L.Duggal introduced the notion of ϵ-sasakian manifolds with indefinite metric. In [5]U.C.De and Avijit Sarkar introduced and studied the notion of ϵ- Kenmotsu manifolds with indefinite metric with an example.H.Bayram Karadag and Mehmet Atceken [4] obtained some results on invariant submanifolds of sasakian manifolds and further properties are obtained.

In 2010 S.S.Shukla and D.D.Singh [6]Studies ϵ-Trans sasakian manifolds. In this paper they have obtained some results on ϵ-Trans sasakian manifolds and Aysel Turgut Vanli and Ramazan Sari[7] obtained some results on invariant submanifolds of a trans sasakian manifolds.Conditions for Indefinite trans sasakian manifolds to be D-totally geodesics, $D^{\perp}$-totally geodesics, mixed totally geodesic is given by Arindam Bhattacharya and Bandana Das in [8]. Recently Dae Ho Jin [9]studied Indefinite tran sasakian manifold of quasi constant curvature with lightlike hypersurfaces.

In this paper necessary and sufficient conditions are given on an submanifold of an indefinite trans-Sasakian manifold to be invariant and further properties and some theorems are given related to an invariant submanifolds in a indefinite trans-Sasakian manifold.

2. **preliminaries**

Let M be an (2n+1)-dimensional indefinite almost contact metric manifold with indefinite almost contact metric structure (ϕ, ξ, η, g) then they satisfies

$$\phi^2 = -I + \eta \otimes \xi \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi\xi = 0$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.3}$$

$$\epsilon g(X, \xi) = \eta(X)$$

where X, Y are vector fields on M and $\epsilon = \pm 1$

An indefinite almost contact metric structure (ϕ, ξ, η, g) on M is called indefinite trans-Sasakian if

$$(2.4) \qquad (\nabla_X \phi)(Y) = \alpha[g(X,Y)\xi - \epsilon\eta(Y)X] + \beta[g(\phi X, Y)\xi - \epsilon\eta(Y)\phi X]$$

where α *and* β are non zero scalar funtions on $\bar{M}$of type (α, β). $\bar{\nabla}$ is a Levi-civita connection on $\bar{M}$. In particular, an indefinite trans-Sasakian manifold is normal.

From above formula, one easily obtains

$$(2.5) \qquad \nabla_X \xi = -\alpha\epsilon\phi X + \beta\epsilon X - \epsilon\eta(X)\xi = \epsilon[-\alpha\phi X + \phi^2 X],$$

$$(2.6) \qquad \nabla_X \eta(Y) = -\alpha g(\phi X, Y) + \beta[g(X,Y) - \epsilon\eta(X)\eta(Y)],$$

Further in an indefinite trans sasakian manifold,the following holds true,

$$(2.7) \qquad R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y]$$

$$+\epsilon[(Y\alpha)\phi X - (X\alpha)\phi X + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y]$$

Let M be an $(2m+1)$ dimensional $(n > m)$ manifold imbedded in $\overline{M}$. The induced metric g of M is given

$$g(X,Y) = \overline{g}(\overline{X}, \overline{Y})$$

for any vector fields X,Y on M.

Let $T_x(M)$ and $T_x(M)^{\perp}$ denote that tangent and normal bundles of M and $x \in M$. Let ∇_X denote the Riemannian connection on M determined by the induced metric g and R denote the Riemannian curvature tensor of M. Then Gauss-Weingarten formula is given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y), \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N(X) + D_X N \tag{2.9}$$

for any vector fields X,Y tangent to M and any vector field N normal to M, where D is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T_x(M)^{\perp}$. Both A and B are called the second fundamental forms of they satisfy

$$g(B(X,Y),N) = g(A_N(X,Y)).$$

3. Submanifolds in indefinite trans-Sasakian manifold

Let M be an (m+1)dimensional immersed submanifold of an almost contact metric manifold $(\bar{M}, \phi, \bar{\eta}, \xi, \bar{g})$. where $\bar{M}$ is$(2n+1)$-dimensional.

Let $i : M \to \bar{M}$ be an immersion; we denote by B the differential of i. The induced Riemannian metric g of $M$is given by $g = i * \bar{g}$.

$$TM = T_x M \oplus T_x M^{\perp},$$

where $T_x M$ the tangent space of M at $x \in M, T_x M^{\perp}$ the normal space of M in $\bar{M}$, respectively. Moreover, we denote by $[N_{1,} N_{2,} N_{3,} ....N_{t,}]$ $t = 2n - m$, an orthonormal basis of the normal space $T_x M^{\perp}$. Then

$$\phi BX = B\varphi X + \sum_{l=1}^{t} \nu_l(X) N_l \quad l = 1, 2, ...t. \tag{3.1}$$

For any $X \in T_x M$, where φ are induced (1,1) tensor and ν_l are induced 1-forms on M. Similarly,

$$\phi N_l = BU_l + \sum_{l=1}^{t} \lambda_{ls} N_s, \tag{3.2}$$

where,U_l are vector fields on M and λ_{ls} are functions on M. Furthermore,the vector field ξ can be expressed as follows:

$$\xi = BV + \sum_{l=1}^{t} \alpha_l N_l, \tag{3.3}$$

where, V is a vectors field on M,α_l are functions on M.Thus

$$g(\varphi X, Y) = \bar{g}(B\varphi X, BY) = \bar{g}(\phi BX - \sum_{l=1}^{t} \nu_l(X)N_l, BY),$$
$$= -\bar{g}(BX, \phi BY) = -\bar{g}(BX, B\varphi Y) = -g(X, \varphi Y).$$

Hence we get,

$$g(\varphi X, Y) = -g(X, \varphi Y), \tag{3.4}$$

for any $X, Y \in \Gamma(TM)$.Moreover,from (2.3),

$$\bar{g}(\phi BX, N_l) = -\bar{g}(BX, \phi N_l),$$

and

$$\bar{g}(\phi N_l, N_s) = -\bar{g}(N_l, \phi N_s),$$

we get the equations

$$\nu_s(X) = -g(X, U_s), \quad \lambda_{ls} = -\lambda_{sl}.$$

So λ_{ls} is skew-symmetric. The following Lemmas will be needed later. This Lemmas provided that for an immersed submanifold of a Sasakian manifold [4]. But this Lemmas true for an immersed submanifold of any almost contact metric manifold.

Lemma 3.1. *Let M be an immersed submanifold of an almost contact metric manifold M . Then we have*

$$\phi^2 = -I + \eta \otimes V - \sum_{l=1}^{t} \nu_l \otimes U_l. \tag{3.5}$$

$$\nu_p(\varphi X) + \sum_{l=1}^{t} \nu_l(X)\lambda_{lp} - \alpha_p \eta(X) = 0, \tag{3.6}$$

and

$$\varphi U_p - \sum_{l=1}^{t} \lambda_{lp} U_l - \alpha_p V = 0.$$

where η is an induced 1-form on M and $\eta(X) = \epsilon g(X, V)$

Lemma 3.2. *Let M be an immersed submanifold of an almost contact metric manifold $\bar{M}$. Then following equations :*

$$\varphi V + \sum_{l=1}^{t} \alpha_l U_l = 0, \tag{3.7}$$

$$\nu_k(V) + \sum_{l=1}^{t} \alpha_l \lambda_{lk} = 0, \tag{3.8}$$

and

$$\eta(V) = 1 - \sum_{l=1}^{t} \alpha_l^2.$$

4. Invariant Submanifolds of an indefinite trans-Sasakian manifold

Let M be an immersed submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. If $\phi(B(T_xM)) \subset T_xM$, for any point $x \in M$, then M is called an invariant sbmanifold of $\bar{M}$. In this case,we have

$$\phi BX = B\varphi X, \tag{4.1}$$

$$\phi N_l = \sum_{l=1}^{t} \lambda_{ls} N_s, \tag{4.2}$$

$$\xi = BV + \sum_{l=1}^{t} \alpha_l N_s. \tag{4.3}$$

Let ∇ be the Levi-civita connection of M with respect to the induced metric g. Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X \xi = \nabla_X \xi + h(X,Y), \tag{4.4}$$

$$\overline{\nabla}_X N = \nabla^{\perp}_X N - A_N X. \tag{4.5}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^{\perp}$. $\nabla^{\perp}$ is the connection in the normal bundle, h is the second fundamental form of M and A_N is the weigarten endomorhism associated with N.The second fundamental form h and the shape operator A related by,

$$g(h(X,Y),N) = g(A_N X, Y). \tag{4.6}$$

Lemma 4.3. *Let M be an invariant submanifold of a trans sasakian manifold $\bar{M}$ then we have*

$$\phi^2 = -I + \bar{\eta} \otimes V, \quad \alpha_l \bar{\eta} = 0, \quad l, k = 1, 2, 3, ...t \tag{4.7}$$

$$\phi V = 0, \sum_{l=1}^{t} \alpha_l \lambda_{lk} = 0.$$

Proof: For any $X \in \Gamma(T\bar{M})$,we have,

$$B\phi^2 X = \phi^2 BX$$

$$= -BX + \eta(BX)\xi,$$

$$= -BX + \eta(BX)BV + \eta(BX) \sum_{l=1}^{t} \alpha_l N_l$$

Then we get,

$$\phi^2 X = -X + \bar{\eta}(X)V, \quad \sum_{l=1}^{t} \bar{\eta}(X)\alpha_l N_l = 0,$$

or

$$\phi^2 \;=\; -I + \bar{\eta} \otimes V, \quad \alpha_l \bar{\eta} = 0.$$

furthermore, from $\phi\xi = 0$ we get

$$B\phi V + \sum_{l=1}^{t} \alpha_l \sum_{k=1}^{t} \lambda_{lk} N_k = 0.$$

Thus we have the following theorems.

Theorem 4.1. *Let M be an invariant submanifold of a indefinite trans sasakian manifold $\bar{M}$. Then ξ is tangent to M iff then the induced structure (ϕ, V, η, g) on M is a indefinite trans sasakian structure.*

Proof: ξ is tangent to M. $V \neq 0$ that is $\alpha_l = 0$, then from (3,3) we have

$$\xi = BV. \tag{4.8}$$

From (3.1) we have

$$\bar{g}(\phi X, Y) = \bar{g}(B\phi X, Y) + \sum_{l=1}^{t} \nu_l(X)\bar{g}(N_l, Y) = g(\phi X, Y), \tag{4.9}$$

Then,from (2.4)we get

$$(\overline{\nabla}_X \phi)Y = \alpha(\bar{g}(X, Y)\xi - \epsilon\bar{\eta}(Y)X) + \beta(\bar{g}(\phi X, Y)\xi - \epsilon\bar{\eta}(Y)\phi X),$$

BY using (4.8) and (2.3), we obtain,

$$(\overline{\nabla}_X \phi)Y = \alpha[\bar{g}(X, Y)BV - \epsilon^2\bar{g}(Y, \xi)X] + \beta[\bar{g}(\phi X, Y)BV - \epsilon^2\bar{g}(Y, \xi)\phi X],$$

$$(\overline{\nabla}_X \phi)Y = \alpha[\bar{g}(X, Y)BV - \epsilon^2\bar{g}(Y, BV)X] + \beta[\bar{g}(\phi X, Y)BV - \epsilon^2\bar{g}(Y, BV)\phi X].$$

from (4.9), we get

$$(\overline{\nabla}_X\phi)Y = \alpha[g(X,Y)V - \epsilon^2 g(Y,V)X] + \beta[g(\phi X,Y)V - \epsilon^2 g(Y,V)\phi X],$$

$$(\overline{\nabla}_X\phi)Y = \alpha[g(X,Y)V - \epsilon\eta(Y)X] + \beta[g(\phi X,Y)V - \epsilon\eta(Y)\phi X],$$

$$(\overline{\nabla}_X\phi)Y = (\nabla_X\phi)Y,$$

Hence by using (2.5) and (4.6),we have

$$\begin{aligned} \overline{\nabla}_X\xi &= \overline{\nabla}_X BV, \\ \epsilon(-\alpha\phi X - \beta\phi^2 X) &= \nabla_X V, \\ \epsilon(-\alpha\phi BX - \beta\phi^2 BX) &= \nabla_{BX} V. \end{aligned}$$

Hence by using (2.1) and (3.1)it follows that

$$\nabla_{BX}V = \epsilon[-\alpha(B\phi X + \sum_{l=1}^{t}\nu_l(X)N_l) + \beta(B\phi^2 X + \sum_{l=1}^{t}\nu_l(\phi X)N_l + \sum_{i=1}^{t}\nu_i(X)BU_i + \sum_{i=1}^{t}\nu_i(X)\sum_{l=1}^{t}\lambda_{ls}N_s)],$$

Thus,we have

$$\begin{aligned} \nabla_{BX}V &= \epsilon[-\alpha(B\phi X) - \beta(B\phi^2 X)], \\ \nabla_X V &= \epsilon[-\alpha(\phi X) + \beta(X - \eta(X)V)]. \end{aligned}$$

Then M is an indefinite trans sasakian manifolds with Indefinite Trans sasakian structure (ϕ, V, η, g).

Theorem 4.2. *Let$M$ be an immersed submanifolds of an indefinite transsasakian manifold $\bar{M}$.Then $M$is an invariant submanifold of an indefinite transsasakian manifold $\bar{M}$ iff the induced structure (ϕ, V, η, g) on M is an indefinite trans sasakian structure.*

Proof: Let M be an invariant submanifold of an indefinite transsasakian manifold $\bar{M}$ then, from (3.5)and (4.7)we get

$$\sum_{l=1}^{t} \nu_l(X)U_l = 0 \Rightarrow \nu_l(X) = 0,$$

By using (3.6) it follows that

$$\nu_p(\phi X) = \alpha_p \eta(X), \quad -g(\phi X, U_p) = \alpha_p \epsilon g(X, V),$$

Thus we have

$$g(X, \phi U_p) = \alpha_p \epsilon g(X, V),$$

That is

$$g(\phi U_p - \epsilon \alpha_p V, X)$$

Since g is non degenere, we have $\phi U_p = \epsilon \alpha_p V$. Thus we get,$\alpha_p = 0$ Then, from(3.3)it follows that $\xi = BV$, that is $\xi \in T_x M$.

References

[1] K.Yano and M.Kon, *Anti invariant submanifolds* Marcel Dekker Inc., New York,(1976).

[2] J.A.Oubina, *New classes of almost contact metric structures,* Publ.Math.Debrecen.,32,(1985),187-193.

[3] A.Bejancu and K.L.Duggal, *Real hypersurfaces of indefinite Kaehler manifolds,* Int.J.Math.Sci., 16(3) (1993) 545-556.

[4] H.Bayram Karadag and Mehmet Atceken , *Invariant submanifolds of sasakian manifolds,* Balkan.Journal of geometry and itsapplications.,12,1(2007), 68-75.

[5] U.C.De and Avijit Sarkar, *On ϵ Kenmotsu manifolds,* Hadronic Journal., 32(2009), 231-242.

[6] S.S.Shukla and D.D.Singh , *On ϵ-Trans sasakian manifolds,* Int.Journal.,32,(2009), 231-242.

[7] Aysel Turgut Vanli and Ramazan Sari , *On invariant submanifolds of a trans sasakian manifolds,* Differ. Geom. Dyn.Syst.,12,(2010),277-288.

[8] ArindamBhattacharya and Bandana Das , *Contact CR-Submanifolds of an Indefinite trans sasakian manifolds,* Int.J.Contemp.Math.Science.,Volume.6,no.26,(2011), 1271-1282.

[9] Dae Ho Jin , *Studied Indefinite trans sasakian manifold of quasi constant curvature with lightlike hypersurfaces,* Balkan Journal of Geometry and Its applications.,Volume 20, issue 1,(2015), 65-75.

[10] Blair.D.E., *Riemannian Geometry of Contact and Symplectic Manifold,* Boston,Berlin,(2002).