On A Type Of Conformal φ –Recurrent Trans-Sasakian Manifolds

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Abstract: *The object of the present paper is to study on a type of conformal φ –recurrent trans-Sasakian manifolds.*

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I. Introduction

The notion of locally φ –symmetric Sasakian manifold was introduced by T. Takahashi [11] in 1977. φ –recurrent Sasakian manifold was studied by the author [12]. Also J. A. Oubinain 1985 introduced a new class of almost contact metric structures which was a generalization of Sasakian [2], α –Sasakian [5], Kenmotsu [5], β –Kenmotsu [5] and cosympletic [5] manifolds, which was called trans-Sasakian manifold [6]. After him many authors ([3], [4], [5], [6], [8], [9], [10]) have studied various type of properties in trans-Sasakian manifold.

The paper is organized as follows. Section–**2** is concerned with preliminaries. Section–**3** is devoted to the study of conformal φ –recurrent trans-Sasakian manifold which satisfies the condition $\varphi \mathrm{grad}(\alpha) = (2n-1)\mathrm{grad}\beta$, and proved that such a manifold is an Einstein manifold.

It is shown that in a conformal φ –recurrent trans-Sasakian manifold (M^{2n+1}, g), $n \geq 1$, the characteristic vector field ξ and the vector field ρ associated to the 1 –form A are opposite directional.

II. Preliminaries

A $(2n+1)$ dimensional, $(n \geq 1)$ almost contact metric manifold M with almost contact metric structure (φ, ξ, η, g), where φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \varphi(\xi) = 0, \ \eta o \varphi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in \chi(M)$, is called trans-Sasakian manifold [1] if and only if

$$(\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi(X), \tag{2.4}$$

for some smooth functions α and β on M. From (2.4) it follows that

$$\nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi), \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y). \tag{2.6}$$

In [10], the authors obtained some results which shall be useful for next section. They are

$$R(X,Y)\xi = (\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y) + 2\alpha\beta(\eta(Y)\varphi X-\eta(X)\varphi Y)+(Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y, \tag{2.7}$$

$$R(\xi,X)\xi = (\alpha^2-\beta^2-\xi\beta)(\eta(X)\xi - X), \tag{2.8}$$

$$2\alpha\beta+\xi\alpha = 0, \tag{2.9}$$

$$S(X,\xi) = (2n(\alpha^2-\beta^2)-\xi\beta)\eta(X)-(2n-1)X\beta-(\varphi X)\alpha, \tag{2.10}$$

$$Q\xi = (2n(\alpha^2-\beta^2)-\xi\beta)\xi-(2n-1)\text{grad}\beta+\varphi(\text{grad}\alpha). \tag{2.11}$$

When $\varphi\text{grad}(\alpha) = (2n-1)\text{grad}\beta$, then (2.10) and (2.11) reduces to

$$S(X,\xi) = 2n(\alpha^2-\beta^2)\eta(X), \tag{2.12}$$

$$Q\xi = 2n(\alpha^2-\beta^2)\xi. \tag{2.13}$$

Again a trans-Sasakian manifold is said to be locally φ −symmetric [11] if

$$\varphi^2((\nabla_W R)(X,Y)Z) = 0, \tag{2.14}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Let us introduce conformal φ-recurrent manifold. A trans-Sasakian manifold is said to be conformal φ-recurrent manifold if there exists a non-zero 1-form A such that

$$\varphi^2((\nabla_W C)(X,Y)Z) = A(W)C(X,Y)Z, \tag{2.15}$$

for $X, Y, Z, W \in \chi(M)$, where the 1-form A is defined as

$$g(X,\rho) = A(X), \forall X \in \chi(M), \tag{2.16}$$

ρ being the vector field associated to the 1-form A and C is a conformal curvature tensor given by [2]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{(2n-1)}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] - \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.17}$$

where R is the curvature tensor, S is the Ricci-tensor and r is the scalar curvature. Also,

$$g(QX,Y) = S(X,Y), \tag{2.18}$$

Q being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S. The above results will be useful in the next section.

III. Conformal φ-recurrent Trans-Sasakian manifold

In this section we consider a trans-Sasakian manifold which is conformal φ-recurrent. Then by virtue of (2.1) and (2.15) we have

$$-(\nabla_W C)(X,Y)Z + \eta((\nabla_W C)(X,Y)Z)\xi = A(W)C)(X,Y)Z. \tag{3.1}$$

From (3.1) it follows that

$$-g((\nabla_W C)(X,Y)Z,U) + \eta((\nabla_W C)(X,Y)Z)\eta(U) = A(W)g(C)(X,Y)Z,U). \tag{3.2}$$

Let $\{e_i\}$, $i = 1,2,\ldots,2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = \{e_i\}$, in (3.2) and taking summation over i, $1 \le i \le 2n+1$, we get

$$\nabla_W S(Y,Z) = (2n-1)[-\frac{dr(W)}{2n}g(Y,Z) + \frac{dr(W)}{2n(2n-1)}[g(Y,Z) -\eta(Y)\eta(Z)] - A(W)[\frac{1}{2n-1}S(Y,Z) + \frac{r}{2n}g(Y,Z)]]. \tag{3.3}$$

Replacing Z by ξ and using (2.1), (2.3) and (2.12) we obtain

$$\nabla_W S(Y,\xi) = (2n-1)\left[-\frac{dr(W)}{2n} - A(W)[\frac{2n(\alpha^2-\beta^2)}{2n-1} + \frac{r}{2n}]\right]\eta(Y). \tag{3.4}$$

Now we know

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{3.5}$$

Using (2.5) and (2.12) in the above relation (3.5) we have

$$(\nabla_W S)(Y,\xi) = 2n(\alpha^2-\beta^2)[-\alpha g(\varphi W,Y) + \beta g(W,Y)] + \alpha S(Y,\varphi W) - \beta S(Y,W). \tag{3.6}$$

Using (3.6) in (3.4) we have

$$2n(\alpha^2-\beta^2)[-\alpha g(\varphi W,Y) + \beta g(W,Y)] + \alpha S(Y,\varphi W) - \beta S(Y,W) = (2n-1)\left[-\frac{dr(W)}{2n} - A(W)[\frac{2n(\alpha^2-\beta^2)}{2n-1} + \frac{r}{2n}]\right]\eta(Y). \tag{3.7}$$

Replacing Y and W by φY and φW respectively, we obtain

$$S(Y,W) = 2n(\alpha^2-\beta^2)g(Y,W) \tag{3.8}$$

and

$$S(\varphi Y,W) = 2n(\alpha^2-\beta^2)g(\varphi Y,W). \tag{3.9}$$

Hence we can state the following theorem:

Theorem 3.1. A conformal φ-recurrent trans-Sasakian manifold (M^{2n+1}, g) satisfying $\varphi\,\mathrm{grad}(\alpha) = (2n-1)\mathrm{grad}\beta$, is an Einstein manifold.

Now from (3.1) and (2.16) we have

$$\begin{aligned}(\nabla_W R)(X,Y)Z = {}&+\eta((\nabla_W R)(X,Y)Z)\xi - \frac{1}{(2n-1)}[(\nabla_W g)(Y,Z)\eta(QX) \\ &-(\nabla_W g)(X,Z)\eta(QY) - (\nabla_W S)(Y,Z)\eta(X) \\ &-(\nabla_W S)(X,Z)\eta(Y) + \frac{dr(W)}{2n(2n-1)}[g(Y,Z)\eta(X) \\ &-g(X,Z)\eta(Y)]\xi - A(W)[R(X,Y)Z \\ &-\frac{1}{(2n-1)}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] \\ &+\frac{dr(W)}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y] \\ &-\frac{1}{(2n-1)}[(\nabla_W g)(Y,Z)QX \\ &-(\nabla_W g)(X,Z)QY - (\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y.\end{aligned} \tag{3.10}$$

Using Bianchi's identity in (3.10) we obtain

$$\begin{aligned}&A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) \\ &= +\frac{r}{2n(2n-1)}A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &\qquad +\frac{r}{2n(2n-1)}A(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]\end{aligned}$$

$$+\frac{r}{2n(2n-1)}A(Y)[g(X,Z)\eta(W)-g(W,Z)\eta(X)]. \quad (3.11)$$

Putting $Y = Z = \{e_i\}$, where e_i be an orthonormal basis of the tangent space at any point of the manifold, in (3.11) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$A(W) = -\frac{1}{2n-1}[A(X)\eta(W) - A(W)\eta(X)]. \quad (3.12)$$

Putting again $X = \xi$ and using (2.1) and (2.3) we obtain

$$A(W) = -K\eta(\rho)\eta(W) \text{ where } (K = \frac{1}{2n-2}), \quad (3.13)$$

for any vector field W and ρ being the vector field associated to the 1-form A, defined as (2.16). Thus we can state the following theorem:

Theorem 2. In a conformal φ-recurrent trans-Sasakian manifold (M^{2n+1}, g), $n \geq 1$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are opposite directional and the 1-form A is given by

$$(2n-1)A(W) = -\eta(\rho)\eta(W) \forall W \in \chi(M).$$

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