Stability of Additive-Quadratic Functional Equation in Banach Space and Banach Algebra: Using Direct and Fixed Point Methods

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Abstract
In this paper, the authors proved Stability of new type of AQ functional equation of the form

$$g(x+y+z+w)+g(x-y-z+w)+g(x+y-z-w)+g(x-y+z-w)+g(-x+y+z-w)+g(-x+y-z+w)+$$
$$g(-x-y+z+w)=g(x-y)+g(x-z)+g(x-w)+g(y-z)+g(y-w)+g(z-w)+2\left[g(x)+g(-x)\right]$$
$$+2\left[g(y)+g(-y)\right]+2\left[g(z)+g(-z)\right]+2\left[g(w)+g(-w)\right]-\left[g(x)-g(-x)\right]+\left[g(z)-g(-z)\right]+2\left[g(w)-g(-w)\right]$$

in Banach space , Banach Algebra using direct and fixed point methods.

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1. INTRODUCTION AND PRELIMINARIES

The study of stability problems for functional equations is related to a question of Ulams[24] concerning the stability of group homomorphism's was affirmatively answered for Banach spaces by Hyers[9,10] for linear mappings and by Russians for linear mappings by considering an bounded Cauchy difference.

The articles of Rassias[17,18] have provided a lot of influence in the development of what we now call generalized Ulam-Hyers[9,10,24] stability of functional equations. The terminology generalized Ulam-Hyers[24] stability originates from historical backgrounds. The terminologies are also applied to the case of other functional equations.

Over the last seven decades, the above Ulam[24] problem was numerous authors who provided solutions a various forms of functional equations like's additive, quadratic, cubic, quartic, mixed type functional equations in-solving only these types of functional equations were discussed. We refer the interested readers for more information at such problems to the monographs. One of the most famous functional equations is the additive functional equation

$$f(x+y)=f(x)+f(y)$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honour of Cauchy .The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. The additive functional equation is the $f(x)=cx$ solution of the additive Functional equation .The quadratic function $f(x)=cx^2$ satisfies the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

In this paper, the authors investigate the generalized Ulam-Hyers stability of a quadratic functional equation

$$\begin{aligned}&g(x+y+z+w)+g(x-y-z+w)+g(x+y-z-w)+g(x-y+z-w)+g(-x+y+z-w)+g(-x+y-z+w)+\\&g(-x-y+z+w)=g(x-y)+g(x-z)+g(x-w)+g(y-z)+g(y-w)+g(z-w)+2\left[g(x)+g(-x)\right]\\&+2\left[g(y)+g(-y)\right]+2\left[g(z)+g(-z)\right]+2\left[g(w)+g(-w)\right]-\left[g(x)-g(-x)\right]+\left[g(z)-g(-z)\right]+2\left[g(w)-g(-w)\right]\end{aligned} \tag{1.1}$$

in Banach spaces using direct and fixed point methods

$$\begin{aligned}&D(X,Y,Z,W)=g(x+y+z+w)+g(x-y-z+w)+g(x+y-z-w)+g(x-y+z-w)+g(-x+y+z-w)+g(-x+y-z+w)\\&+g(-x-y+z+w)-g(x-y)-g(x-z)-g(x-w)-g(y-z)-g(y-w)-g(z-w)-2\left[g(x)+g(-x)\right]-2\left[g(y)+g(-y)\right]\\&-2\left[g(z)+g(-z)\right]-2\left[g(w)+g(-w)\right]+\left[g(x)-g(-x)\right]-\left[g(z)-g(-z)\right]-2\left[g(w)-g(-w)\right]\end{aligned}$$

for all $x, y, z \in X$

2. Stability Results for (1.1)

In this section, we present the generalized Ulam-Hyers stability of the functional equation (1.1) for odd case.

Theorem 2.1: Let $j \in \{-1,1\}$ and $\alpha : X^4 \to [0,\infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\alpha\left(2^{nj}x, 2^{nj}x, 2^{nj}x, 2^{nj}x\right)}{2^{kj}} \text{ converges in R and } \sum_{k=0}^{\infty} \frac{\alpha\left(2^{nj}x, 2^{nj}x, 2^{nj}x, 2^{nj}x\right)}{2^{kj}} = 0 \tag{2.1}$$

for all $x, y, z \in X$. Let $g_a : X \to Y$ be an even function satisfying the inequality

$$\| Dg_a(x,y,z,w) \| \le \alpha(x,y,z,w) \tag{2.2}$$

for all $x, y, z \in X$. There exists a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (1.1) and

$$\| g_a(x) - A(x) \| \le \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(2^{nj}x, 2^{nj}x, 0, 0\right)}{2^{nj}} \tag{2.3}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{n\to\infty} \frac{g_a\left(2^{nj}x\right)}{2^{nj}} \tag{2.4}$$

for all $x \in X$.

Proof: Assume that $j = 1$. Replacing (x, y, z, w) and $(x, x, 0, 0)$ in (2.2) of g_a , we get

$$\| g_a(2x) - 2g_a(x) \| \le \alpha(x, x, 0, 0) \tag{2.5}$$

for all $x \in X$. It follows from (2.5) that

$$\left\| \frac{g_a(2x)}{2} - g_a(x) \right\| \le \frac{\alpha}{2}(x, x, 0, 0) \tag{2.6}$$

for all $x \in X$. Replacing x by $2x$ in (2.6) and dividing by 2 , we obtain

$$\left\| \frac{g_a(2^2x)}{2^2} - \frac{g_a(2x)}{2} \right\| \le \frac{\alpha}{2} \frac{(2x, 2x, 0, 0)}{2} \tag{2.7}$$

for all $x \in X$. It follows from (2.5) and (2.6) that

$$\left\| \frac{g_a(2^2 x)}{2^2} - g_a(x) \right\| \leq \frac{1}{2}\left[\alpha(x,x,0,0) + \frac{\alpha}{2}(2x,2x,0,0) \right] \tag{2.8}$$

for all $x \in X$. Generalizing, we have

$$\left\| g_a(x) - \frac{g_a(2^n x)}{2^n} \right\| \leq \frac{1}{2}\sum_{k=0}^{n-1} \frac{\alpha\left(2^k x, 2^k x, 0, 0\right)}{2^k} \leq \frac{1}{2}\sum_{k=0}^{\infty} \frac{\alpha\left(2^k x, 2^k x, 0, 0\right)}{2^k} \tag{2.9}$$

for all $x \in X$. In order to prove convergence of the sequence

$$\left\{ \frac{g_a(2^k x)}{2^k} \right\},$$

Replace x by $2^m x$ and dividing 2^m (2.9), for any $\mathrm{m,n} > 0$, to deduce

$$\begin{aligned}
\left\| \frac{g_a(2^m x)}{2^m} - \frac{g_a(2^{n+m} x)}{2^{n+m}} \right\| &= \frac{1}{2^m}\left\| g_a(2^m x) - \frac{g_a(2^m .2^n x)}{2^n} \right\| \\
&\leq \frac{1}{2}\sum_{m=0}^{n-1} \frac{\alpha\left(2^{m+n} x, 2^{m+n} x, 0, 0\right)}{2^{n+m}} \\
&\leq \frac{1}{10}\sum_{k=0}^{\infty} \frac{\alpha\left(2^{m+n} x, 2^{m+n}, 0\right)}{2^{m+n}} \qquad (2.10)\\
&\to 0 \text{ as } m \to \infty
\end{aligned}$$

for all $x \in X$ .Hence the sequence $\left\{ \frac{g_a(2^m x)}{2^m} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X \to Y$ such that

$$\mathrm{A}(x) = \lim_{n\to\infty} \frac{g_a(2^n x)}{2^n}, \quad \forall\ x \in X .$$

Letting $m \to \infty$ in (2.9), we see that (2.3) holds for $x \in X$. To prove that A satisfies (1.1) replacing (x, y, z) by $(2^n x, 2^n x, 2^n x)$ and dividing 2^n in (2.2), we obtain

$$\frac{1}{2^n}\left\| Dg_a\left(2^n x, 2^n x, 2^n x, 2^n x\right) \right\| \leq \frac{1}{2^n}\alpha\left(2^n x, 2^n x, 2^n x, 2^n x\right)$$

for all $x, y, z, w \in X$. Letting $m \to \infty$ in the above inequality and using the definition of $\mathrm{A}(x)$, we see that

$$DA(x, y, z, w) = 0 .$$

Hence A satisfies (1.1) for all $x, y, z \in X$. To show that A is unique, let $B(x)$ be another quadratic mapping satisfying (1.1) and (2.3), then

$$\begin{aligned}
\| A(x) - B(x) \| = \frac{1}{2^n}\| A(2^n x) - B(2^n x) \| &\leq \frac{1}{2^n}\left\{\| A(2^n x) - g_a(2^n x) \| + \| g_a(2^n x) - B(2^n x) \|\right\} \\
&\leq \frac{1}{2}\sum_{m=0}^{\infty} \frac{\alpha\left(2^{n+m} x, 2^{n+m} x, 0, 0\right)}{2^{n+m}} \to 0 \text{ as } m \to \infty
\end{aligned}$$

for all $x \in X$. Hence A is unique. Now, replacing x by $\frac{x}{2}$ in (2.2), we get

$$\left\| g_a(x) - 2g_a\left(\frac{x}{2}\right) \right\| \leq \alpha\left(\frac{x}{2}, \frac{x}{2}, 0, 0\right) \tag{2.11}$$

for all $x \in X$. It follows from (1.12) that

for all $x \in X$. The rest of the proof is similar to that of $j = 1$. Hence for $j = -1$ also the theorem is true. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 2.1 concerning the stability of (1.1).

Corollary 2.2: Let η and p be a nonnegative real numbers. Let an odd function $f_a : X \to Y$ satisfying the inequality

$$\| Dg_a(x,y,z,w) \| \le \begin{cases} \eta \\ \eta\left(\| x \|^p + \| y \|^p + \| z \|^p + \| w \|^p\right), & p \ne 1; \\ \varepsilon\left(\| x \|^p \| y \|^p \| z \|^p \| w \|^p + \left\{\| x \|^{4p} + \| y \|^{4p} + \| z \|^{4p} + \| w \|^{4p}\right\}\right), & p \ne \frac{1}{4} \end{cases} \tag{2.13}$$

for all $x, y, z, w \in X$. Then there exists a unique quadratic function $A : X \to Y$ such that

$$\| g_a(x) - A(x) \| \le \begin{cases} \eta, \\ \dfrac{\eta \| x \|^p}{| 2 - 2^p |}, \\ \dfrac{\eta \| x \|^{4p}}{| 2 - 2^{4p} |}, \end{cases} \tag{2.14}$$

for all $x \in X$.

Proof: If we replace

$$\alpha(x,y,z,w) = \begin{cases} \eta; \\ \eta\left(\| x \|^w + \| y \|^w + \| z \|^w + \| w \|^w\right); \\ \eta\left(\| x \|^w \| y \|^w \| z \|^w \| w \|^w + \left\{\| x \|^{4w} + \| y \|^{4w} + \| z \|^{4w} + \| w \|^{4w}\right\}\right); \end{cases} \tag{2.15}$$

for all $x, y, z, w \in X$

3. Stability Results for (1.1)

In this section, we investigate the generalized Ulam- Hyers stability of the functional equation (1.1) for even case.

Theorem 3.1: Let $j \in \{-1,1\}$ and $\alpha : X^4 \to [0,\infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\alpha\left(2^{kj}x, 2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{4^{nj}} \text{ converges in R and } \sum_{k=0}^{\infty} \frac{\alpha\left(2^{kj}x, 2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{4^{kj}} = 0 \tag{3.1}$$

for all $x, y, z, w \in X$. Let $g_q : X \to Y$ be an even function satisfying the inequality

$$\left\| Dg_q(x,y,z,w) \right\| \le \alpha(x,y,z,w) \tag{3.2}$$

for all $x, y, z, w \in X$. There exists a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (1.1) and

$$\left\| g_q(x) - Q(x) \right\| \le \frac{1}{12} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(2^{nj}x, 2^{nj}x, 0, 0\right)}{4^{nj}} \tag{3.3}$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x)=\lim_{n\to\infty}\frac{g_q\left(2^{nj}x\right)}{4^{nj}} \tag{3.4}$$

for all $x\in X$.

Proof: Assume that $j=1$. Replacing (x,y,z,w) and $(x,x,0)$ in (3.2) of g_q, we get

$$\left\| 3g_q(2x)-12g_q(x) \right\| \le \alpha(x,x,0,0) \tag{3.5}$$

for all $x\in X$. It follows from (3.5) that

$$\left\| \frac{g_q(2x)}{4}-g_q(x) \right\| \le \frac{\alpha}{12}(x,x,0,0) \tag{3.6}$$

for all $x\in X$. Replacing x by $2x$ in (3.6) and dividing by 4, we obtain

$$\left\| \frac{g_q(2^2x)}{4^2}-\frac{g_q(2x)}{4} \right\| \le \frac{\alpha}{12}\frac{(2x,2x,0,0)}{4} \tag{3.7}$$

for all $x\in X$. It follows from (3.5) and (3.6) that

$$\left\| \frac{g_q(2^2x)}{4^2}-g_q(x) \right\| \le \frac{1}{20}\left[\alpha(x,x,0)+\frac{\alpha}{4}(2x,2x,0)\right] \tag{3.8}$$

for all $x\in X$. Generalizing, we have

$$\left\| g_q(x)-\frac{g_q(2^nx)}{4^n} \right\| \le \frac{1}{12}\sum_{k=0}^{n-1}\frac{\alpha\left(2^kx,2^kx,0,0\right)}{4^k} \le \frac{1}{20}\sum_{k=0}^{\infty}\frac{\alpha\left(2^kx,2^kx,0\right)}{4^k} \tag{3.9}$$

for all $x\in X$. In order to prove convergence of the sequence $\left\{\frac{g_q(2^kx)}{4^k}\right\}$.Replace x by 2^mx and dividing 4^m (3.9), for any $\mathrm{m,n}>0$, to deduce

$$\begin{aligned}\left\| \frac{g_q(2^mx)}{4^m}-\frac{g_q(2^{n+m}x)}{4^{n+m}} \right\| &= \frac{1}{4^m}\left\| g_q(2^mx)-\frac{g_q(2^m.2^nx)}{4^n} \right\| \\ &\le \frac{1}{12}\sum_{k=0}^{n-1}\frac{\alpha\left(2^{m+n}x,2^{m+n}x,0,0\right)}{4^{n+m}} \\ &\le \frac{1}{12}\sum_{k=0}^{\infty}\frac{\alpha\left(2^{m+n}x,2^{m+n},0,0\right)}{4^{m+n}} \\ &\to 0 \text{ as } m\to\infty \end{aligned} \tag{3.10}$$

for all $x\in X$ .Hence the sequence $\left\{\frac{g_q(2^nx)}{4^n}\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $\mathrm{Q}:X\to Y$ such that

$$Q(x)=\lim_{n\to\infty}\frac{g_q(2^nx)}{4^n}, \quad \forall\ x\in X.$$

Letting $m\to\infty$ in (3.9), we see that (3.3) holds for $x\in X$. To prove that A satisfies (1.1) replacing (x,y,z,w) by $(2^nx,2^nx,2^nx)$ and dividing 4^n in (3.2), we obtain

$$\frac{1}{4^n}\left\| Dg_q\left(2^nx,2^nx,2^nx,2^nx\right)\right\| \le \frac{1}{4^n}\alpha\left(2^nx,2^n\mathrm{x},2^nx,2^nx\right)$$

for all $x,y,z,w\in X$. Letting $m\to\infty$ in the above inequality and using the definition of $Q(x)$, we see that

$$DQ(x,y,z,w)=0.$$

Hence Q satisfies (1.1) for all $x, y, z, w \in X$. To show that Q is unique, let $B(x)$ be another quadratic mapping satisfying (1.1) and (3.3), then

$$\| Q(x)-B(x) \| = \frac{1}{4^n} \| Q(2^n x) - B(2^n x) \| \le \frac{1}{4^n}\left\{\| Q(2^n x) - g_q(2^n x) \| + \| g_q(2^n x) - B(2^n x) \|\right\}$$
$$\le \frac{1}{12}\sum_{k=0}^{\infty} \frac{\alpha\left(2^{n+m}x, 2^{n+m}x, 0, 0\right)}{4^{n+m}} \to 0 \text{ as } n\to\infty$$

for all $x \in X$. Hence Q is unique. Now, replacing x by $\frac{x}{2}$ in (3.2), we get

$$\left\| 3g_q(x) - 12 g_q\left(\frac{x}{2}\right)\right\| \le \alpha\left(\frac{x}{2}, \frac{x}{2}, 0, 0\right) \tag{3.11}$$

for all $x \in X$. It follows from (1.12) that

$$\left\| g_q(x) - 4 g_q\left(\frac{x}{2}\right)\right\| \le \frac{1}{3}\alpha\left(\frac{x}{2}, \frac{x}{2}, 0, 0\right) \tag{3.12}$$

for all $x \in X$. The rest of the proof is similar to that of $j=1$. Hence for $j=-1$ also the theorem is true. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.1).

Corollary 3.2: Let η and p be a nonnegative real numbers. Let an even function $g_q : X \to Y$ satisfying the inequality

$$\| Dg_q(x,y,z) \| \le \begin{cases} \eta \\ \eta\left(\| x\|^p + \| y\|^p + \| z\|^p + \| w\|^p\right), & p \ne 1; \\ \eta\left(\| x\|^p \| y\|^p \| z\|^p \| w\|^p + \left\{\| x\|^{4p} + \| y\|^{4p} + \| z\|^{4p} + \| w\|^{4p}\right\}\right), & p \ne \frac{1}{4} \end{cases} \tag{3.13}$$

for all $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\| g_q(x) - Q(x) \| \le \begin{cases} \eta, \\ \frac{\eta \| x\|^p}{|4-2^p|}, \\ \frac{\eta \| x\|^{4p}}{|4-2^{4p}|}, \end{cases} \tag{3.14}$$

for all $x \in X$.

Proof: If we replace

$$\alpha(x,y,z,w) = \begin{cases} \eta; \\ \eta\left(\| x\|^p + \| y\|^p + \| z\|^p + \| w\|^p\right); \\ \eta\left(\| x\|^p \| y\|^p \| z\|^p \| w\|^p + \left\{\| x\|^{4p} + \| y\|^{4p} + \| z\|^{4p} + \| w\|^{4p}\right\}\right); \end{cases} \tag{3.15}$$

for all $x, y, z, w \in X$

4. Stability Results for (1.1): Mixed Case

In this section, we establish the generalized Ulam-Hyers stability of the functional equation (1.1) for mixed case.

Theorem 4.1: Let $j \in \{-1,1\}$ and $\alpha : X^4 \to [0,\infty)$ be a function satisfying (2.1) and (3.1) for all $x,y,z,w \in X$. Let $g : X \to Y$ be a function satisfying the inequality

$$\| Dg(x,y,z,w) \| \le \alpha(x,y,z,w) \tag{4.1}$$

for all $x,y,z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (1.1) and

$$\| g(x) - A(x) - Q(x) \| \le \frac{1}{2}\left[\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(2^{nj}x,2^{nj}x,0,0)}{2^{nj}} + \frac{\alpha(-2^{nj}x,2^{nj}x,0,0)}{2^{nj}} \right) + \frac{1}{12}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(2^{nj}x,2^{nj}x,0,0)}{4^{nj}} + \frac{\alpha(-2^{nj}x,2^{nj}x,0,0)}{4^{nj}} \right) \right] \tag{4.2}$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ is defined in (2.4) and (3.4) respectively for all $x \in X$.

Proof: Let $g_o(x) = \dfrac{g_a(x) - g_a(-x)}{2}$ for all $x \in X$. Then $g_o(0) = 0$ and $g_o(-x) = -g_o(x)$ for all $x \in X$. Hence,

$$\| Dg_o(x,y,z,w) \| \le \frac{1}{2}\{\| Dg_o(x,y,z,w) \| + \| Dg_o(-x,-y,-z,w) \|\} \le \frac{\alpha(x,y,z,w)}{2} + \frac{\alpha(-x,-y,-z,w)}{2} \tag{4.3}$$

for all $x,y,z \in X$. By Theorem 2.1, we have

$$\| g_o(x) - A(x) \| \le \frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(2^{nj}x,2^{nj}x,0,0)}{2^{nj}} + \frac{\alpha(2^{nj}x,2^{nj}x,0,0)}{2^{nj}} \right) \tag{4.4}$$

for all $x \in X$. Also let, $g_e(x) = \dfrac{g_q(x) + g_q(-x)}{2}$ for all $x \in X$. Then $g_e(0) = 0$ and $g_e(-x) = g_e(x)$ for all $x \in X$. Hence,

$$\| Dg_e(x,y,z) \| \le \frac{1}{2}\{\| Dg_q(x,y,z,w) \| + \| Dg_q(-x,-y,-z,w) \|\} \le \frac{\alpha(x,y,z,w)}{2} + \frac{\alpha(-x,-y,-z,w)}{2} \tag{4.5}$$

for all $x,y,z,w \in X$. By Theorem 3.1, we have

$$\| g_e(x) - Q(x) \| \le \frac{1}{24}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(2^{nj}x,2^{nj}x,0,0)}{4^{nj}} + \frac{\alpha(2^{nj}x,2^{nj}x,0,0)}{4^{nj}} \right) \tag{4.6}$$

for all $x \in X$. Define

$$g(x) = g_e(x) + g_o(-x) \tag{4.7}$$

for all $x \in X$. It follows from (4.4), (4.6) and (4.7), we arrive

$$\begin{aligned} \| g(x) - A(x) - Q(x) \| &= \| g_e(x) + g_o(-x) - A(x) - Q(x) \| \\ &\le \| g_o(-x) - A(x) \| + \| g_e(x) - Q(x) \| \\ &\le \frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(2^{nj}x,2^{nj}x,0,0)}{2^{nj}} + \frac{\alpha(2^{nj}x,2^{nj}x,0,0)}{2^{nj}} \right) \end{aligned}$$

$$\leq \frac{1}{24}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(2^{nj}x,2^{nj}x,0,0)}{4^{nj}}+\frac{\alpha(2^{nj}x,2^{nj}x,0,0)}{4^{nj}}\right) \tag{4.8}$$

for all $x \in X$. Hence the theorem is proved.

Using Corollaries 2.2 and 3.2, we have the following corollary concerning the stability of (1.1).

Corollary 4.2: Let Υ and s be a nonnegative real numbers. Let a function $f : X \to Y$ satisfying the inequality

$$\| Dg(x,y,z,w) \| \leq \begin{cases} \Upsilon; & \\ \Upsilon\left(\| x \|^{s}+\| y \|^{s}+\| z \|^{s}+\|w\|^{s}\right); & s \neq 1,2; \\ \Upsilon\left(\| x \|^{s}\| y \|^{s}\| z \|^{s}\|w\|^{s}+\left\{\| x \|^{4s}+\| y \|^{4s}+\| z \|^{4s}+\|w\|^{4s}\right\}\right); & 4s \neq 1,2; \end{cases} \tag{4.9}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A: X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|g(x)-A(x)-Q(x)\| \leq \begin{cases} 2\Upsilon, \\ \Upsilon\| x \|^{s}\left[\frac{1}{\left|2-2^{s}\right|}+\frac{1}{\left|4-2^{s}\right|}\right], \\ \Upsilon\| x \|^{4s}\left[\frac{1}{\left|2-2^{4s}\right|}+\frac{1}{\left|4-2^{4s}\right|}\right], \end{cases} \tag{4.10}$$

for all $x \in X$.

5. Stability Results for (1.1): Using another Substitution-Direct Method

In this section, the generalized Ulam-Hyers stability of generalized quadratic functional equation (1.1) is investigated by using another substitution .Hence the details of the proof are omitted.

Theorem 5.1: Let $j \in \{-1,1\}$ and $\alpha : X^{4} \to [0,\infty)$ be a function such that

$$\sum_{k=0}^{\infty}\frac{\alpha\left(3^{kj}x,3^{kj}y,3^{kj}z,3^{kj}w\right)}{3^{kj}} \text{ converges in } \mathbb{R} \text{ and } \sum_{k=0}^{\infty}\frac{\alpha\left(3^{kj}x,3^{kj}y,3^{kj}z,3^{kj}w\right)}{3^{kj}}=0 \tag{5.1}$$

for all $x, y, z, w \in X$. Let $g_a : X \to Y$ be an even function satisfying the inequality

$$\| Dg_{a}(x,y,z,w) \| \leq \alpha(x,y,z,w) \tag{5.2}$$

for all $x, y, z, w \in X$. There exists a unique additive mapping $A: X \to Y$ which satisfies the functional equation (1.1) and

$$\| g_{a}(x)-A(x)\| \leq \frac{1}{3}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\alpha\left(3^{nj}x,3^{nj}x,3^{nj}x,0\right)}{3^{nj}} \tag{5.3}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x)=\lim_{k\to\infty}\frac{g_{a}\left(3^{kj}x\right)}{3^{kj}} \tag{5.4}$$

for all $x \in X$.

Corollary 5.2: Let Υ and p be a nonnegative real numbers. Let an even function $f_q : X \to Y$ satisfying the inequality

$$\| Dg_a(x,y,z,w) \| \leq \begin{cases} \Upsilon \\ \Upsilon\left(\| x \|^p + \| y \|^p + \| z \|^p + \|w\|^p\right), & p \neq 1; \\ \Upsilon\left(\| x \|^p \| y \|^p \| z \|^p \|w\|^p + \left\{\| x \|^{4p} + \| y \|^{4p} + \| z \|^{4p} + \|w\|^{4p}\right\}\right), & p \neq \frac{1}{4}; \end{cases} \tag{5.5}$$

for all $x, y, z, w \in X$. Then there exists a unique additive function $\mathrm{Q} : X \to Y$ such that

$$\| g_a(x) - A(x) \| \leq \begin{cases} \Upsilon, \\ \dfrac{\Upsilon \| x \|^P}{\left| 3 - 3^P \right|}, \\ \dfrac{\Upsilon \| x \|^{4P}}{\left| 3 - 3^{4P} \right|}, \end{cases} \tag{5.6}$$

for all $x \in X$.

Proof: If we replace

$$\alpha(x,y,z,w) \leq \begin{cases} \Upsilon \\ \Upsilon\left(\| x \|^p + \| y \|^p + \| z \|^p + \|w\|^p\right), & p \neq 1; \\ \Upsilon\left(\| x \|^p \| y \|^p \| z \|^p \|w\|^p + \left\{\| x \|^{4p} + \| y \|^{4p} + \| z \|^{4p} + \|w\|^{4p}\right\}\right), & p \neq \frac{1}{4}; \end{cases} \tag{3.7}$$

for all $x, y, z, w \in X$

Theorem 5.3: Let $j \in \{-1,1\}$ and $\alpha : X^4 \to [0,\infty)$ be a even function such that

$$\sum_{k=0}^{\infty} \frac{\alpha\left(3^{kj}x, 3^{kj}y, 3^{kj}z, 3^{kj}w\right)}{9^{kj}} \text{ converges in R and } \sum_{k=0}^{\infty} \frac{\alpha\left(3^{kj}x, 3^{kj}y, 3^{kj}z, 3^{kj}w\right)}{9^{kj}} = 0 \tag{5.8}$$

for all $x, y, z, w \in X$. Let $g_q : X \to Y$ be an even function satisfying the inequality

$$\left\| Dg_q(x,y,z,w) \right\| \leq \alpha(x,y,z,w) \tag{5.9}$$

for all $x, y, z, w \in X$. There exists a unique quadratic mapping $\mathrm{Q} : X \to Y$ which satisfies the functional equation (1.1) and

$$\left\| g_q(x) - Q(x) \right\| \leq \frac{1}{9} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(3^{nj}x, 3^{nj}x, 3^{nj}x\right)}{9^{nj}} \tag{5.10}$$

for all $x \in X$. The mapping $\mathrm{Q}(x)$ is defined by

$$\mathrm{Q}(x) = \lim_{k \to \infty} \frac{g_q\left(3^{kj}x\right)}{9^{kj}} \tag{5.11}$$

for all $x \in X$.

Corollary 5.4: Let Υ and p be a nonnegative real numbers. Let an even function $g_q : X \to Y$ satisfying the inequality

$$\| Dg_q(x,y,z,w) \| \leq \begin{cases} \Upsilon \\ \Upsilon\left(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p\right), & p \neq 2; \\ \Upsilon\left(\|x\|^p \|y\|^p \|z\|^p \|w\|^p + \left\{\|x\|^{4p} + \|y\|^{4p} + \|z\|^{4p} + \|w\|^{4p}\right\}\right), & p \neq \frac{2}{4}; \end{cases} \tag{5.12}$$

for all $x, y, z, w \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\| g_q(x) - Q(x) \| \leq \begin{cases} \Upsilon, \\ \dfrac{\Upsilon \|x\|^P}{|9 - 3^P|}, \\ \dfrac{\Upsilon \|x\|^{4P}}{|9 - 3^{4P}|}, \end{cases} \tag{5.13}$$

for all $x \in X$.

Theorem 5.5: Let $j \in \{-1, 1\}$ and $\alpha : X^4 \to [0, \infty)$ be a function satisfying (5.1) and (5.8) for all $x, y, z, w \in X$. Let $g : X \to Y$ be a function satisfying the inequality

$$\| Dg(x,y,z,w) \| \leq \alpha(x,y,z,w) \tag{5.14}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (1.1) and

$$\begin{aligned} \| g(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Bigg[& \frac{1}{3} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(3^{nj}x, 3^{nj}x, 3^{nj}x, 0)}{3^{nj}} + \frac{\alpha(3^{nj}x, 3^{nj}x, 3^{nj}x, 0)}{3^{nj}} \right) \\ & + \frac{1}{9} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(3^{nj}x, 3^{nj}x, 3^{nj}x, 0)}{9^{nj}} + \frac{\alpha(3^{nj}x, 3^{nj}x, 3^{nj}x, 0)}{9^{nj}} \right) \Bigg] \end{aligned} \tag{515}$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ is defined in (5.4) and (5.11) respectively for all $x \in X$.

Corollary 5.6: Let Υ and p be a nonnegative real numbers. Let a function $g : X \to Y$ satisfying the inequality

$$\| Dg(x,y,z,w) \| \leq \begin{cases} \Upsilon; \\ \Upsilon\left(\|x\|^p + \|y\|^p + \|z\|^p \|w\|^p\right); & p \neq 1, 2; \\ \Upsilon\left(\|x\|^p \|y\|^p \|z\|^p \|w\|^p + \left\{\|x\|^{4p} + \|y\|^{4p} + \|z\|^{4p} + \|w\|^{4p}\right\}\right); & 4p \neq 1, 2; \end{cases} \tag{5.16}$$

for all $x, y, z, w \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|g(x)-A(x)-Q(x)\| \leq \begin{cases} \Upsilon\left(\frac{1}{2}+\frac{1}{8}\right), \\ \Upsilon\|x\|^{p}\left[\frac{1}{\left|3-3^{p}\right|}+\frac{1}{\left|9-3^{p}\right|}\right], \\ \Upsilon\|x\|^{p}\left[\frac{1}{\left|3-3^{4p}\right|}+\frac{1}{\left|9-3^{4p}\right|}\right], \end{cases} \tag{5.17}$$

for all $x \in X$.

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