Inversion of Matrix by Elementary Column Transformation: Representation of Numerical Data by a Polynomial Curve

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Abstract:
There is necessity of a method/formula, for representing a given set of numerical data on a pair of variables by a the curve, in interpolation by the approach which consists of the representation of numerical data by a polynomial first and then to compute the value of the dependent variable from the polynomial corresponding to any given value of the independent variable. Due to this necessity, a method has been composed for representing a given set of numerical data on a pair of variables by a polynomial curve. The method has been composed with the help of elementary column transformation of matrix. This paper describes the method composed here with numerical example in order to show the application of the method to numerical data.

Keywords : *Numerical data, polynomial curve, matrix inversion, Gauss Jordan method.*

1. Introduction:

In the existing approach of interpolation {Hummel (1947), Erdos & Turan (1938) et al}, where a number of interpolation formulae are available {Bathe & Wilson (1976), Jan (1930), Hummel (1947) et al}, if it is wanted to interpolate the values of the dependent variable corresponding to a number of values of the independent variable by a suitable existing interpolation formula then it is required to apply the formula for each value separately and thus the numerical computation of the value of the dependent variable based on the given data are to be performed in each of the cases. In order to get rid of these repeated numerical computations from the given data, one can think of an approach which consists of the representation of the given numerical data by a suitable polynomial and then to compute the value of the dependent variable from the polynomial corresponding to any given value of the independent variable. However, a method is necessary for representing a given set of numerical data on a pair of variables by a suitable polynomial. Das & Chakrabarty (2016*a*, 2016*b*, 2016*c* & 2016*d*) derived four formulae for representing numerical data on a pair of variables by a polynomial curve. They derived the formulae from Lagranges Interpolation Formula, Newton's Divided Difference Interpolation Formula, Newton's Forward Interpolation Formula and Newton's Backward Interpolation respectively. In another study, Das & Chakrabarty (2016*e*) derived one method for representing numerical data on a pair of variables by a polynomial curve. The method derived is based on the inversion of a square matrix by Caley-Hamilton theorem on characteristic equation of matrix [Cayley *(1858, 1889) & Hamilton (1864a , 1864b , 1862)*]. In a separate study, Das & Chakrabarty (2016*f*) composed another method for the same purpose. The method composed there is based on the inversion of matrix by elementary row transformation of matrix. In this study, another method has been composed for the same purpose with the help elementary column transformation of matrix. This paper describes the method developed here with numerical example in order to show the application of the method to numerical data.

2. Representation of Numerical Data by Polynomial Curve:

Let y_1, y_2, ………… y_n be the values assumed by the function $y = f(x)$ corresponding to the values x_1, x_2, ………… x_n of the independent variable X. Now the problem is to interpolate the values of the function corresponding to some value of x which values of the function is not available. Now the interpolation is based on the mathematical fact that if n points x_0, x_1, ………… x_n are given then they can be suitably represented by a polynomial of degree n–1. In the present case thus the function $y = f(x)$ can be represented suitably by a polynomial in x of degree n. Suppose that the polynomial is

$y= f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$, $a_{n-1} \neq 0$

Since the n points lie on the polynomial curve describe above so we have

$$y_0 = a_0 + a_1 x_0 + a_2 x_0^2 + \ldots\ldots\ldots + a_{n-1} x_0^{n-1}$$
$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots\ldots\ldots\ldots + a_{n-1} x_1^{n-1}$$

$$y_2 = a_0 + a_1 x_2 + a_2 x_2^{\,2} + \dots\dots + a_{n-1} x_2^{\,n-1}$$
$$y_3 = a_0 + a_1 x_3 + a_2 x_3^{\,2} + \dots\dots + a_{n-1} x_3^{\,n-1}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y_n = a_0 + a_1 x_n + a_2 x_n^{\,2} + \dots\dots + a_{n-1} x_n^{\,n-1}$$

$$\rightarrow (2.1)$$

In order to obtain the polynomial we are to solve this n equations for the n unknown coefficients (parameters) $a_0, a_1, \dots\dots, a_n$.

Now solving these equations for the parameters, the polynomial can be obtained.

In order to solve the equations it is to be noted that the equations in (2.1) can be expressed as

$$AC = B \qquad \rightarrow (2.2)$$

where

$$A = \begin{pmatrix} 1 & x_0 & x_0^2 \dots\dots\dots\dots x_0^{n-1} \\ 1 & x_1 & x_1^2 \dots\dots\dots\dots x_1^{n-1} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ 1 & x_n & x_n^2 \dots\dots\dots\dots x_n^{n-1} \end{pmatrix},$$

$$C = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \end{pmatrix} \quad \& \quad B = \begin{pmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix} \rightarrow (2.3)$$

which gives $C = A^{-1}B \qquad \rightarrow (2.4)$

In order to find out C, it is required to find out A^{-1}.

To find out A^{-1} let us apply element are column transformation of matrix.

Following operations performed on a matrix are the elementary column transformations (operations):

(i) **Interchange of two columns**

The interchange of ith & jth column is denoted by C_{ij}

(ii) Multiplication of (each element of) a column by a non zero number k.

The multiplication of jth column by k is denoted by kC_i

(iii) Addition of k times the elements of a column to the corresponding elements of another column, k ≠ 0

The addition of k times the jth column to the ith column is denoted by C_i + kC_j

If a matrix B is obtained from a matrix A by one or more E operation then B is said to be equivalent to A and is denoted by A ~ B

The matrix obtained from a unit matrix **I** by applying it to any one of the E-operations (elementary operations) is called an elementary matrix.

To find A^{-1} can be summarized as follows:

The elementary column operations which reduce a square matrix A to the unit matrix, give the inverse matrix A^{-1}

Working rule:

To find the inverse of A by E-column operations, we write A and I side by side and the same operations are performed on both. As soon as A is reduced to I, I will reduce to A^{-1}.

3. An Example of Application of the Formula:

The following table shows the data on total population of India corresponding to the years:

Year	*Total Population*
1971	548159652
1981	683329097
1991	846302688
2001	1027015247

Taking 1971 as origin and changing scale by 1/10, one can obtain the following table for independent variable *x* (representing time) and *f*(*x*) (representing total population of India):

Year	x_i	$f(x_i)$
1971	0	548159652
1981	1	683329097
1991	2	846302688
2001	3	1027015247

Now here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$

$f(x_0) = 548159652$, $f(x_1) = 683329097$,

$f(x_2) = 846302688$, $f(x_3) = 1027015247$

Let $C = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}$

$$B = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

We know that AC = B ⇒ C = $A^{-1}B$ ……. (1)

Now, A = IA

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 4 \\ 1 & 3 & 9 & 18 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} A,$$

$$C_4 \to C_4 - C_3$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 4 \\ 1 & 3 & 6 & 18 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} A,$$

$$C_3 \to C_3 - C_2$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 6 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} A,$$

$$C_4 \to \frac{1}{2} C_4$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 3 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} A,$$

$$C_4 \to C_4 - C_3$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 0 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & \frac{3}{2} & 1 & -\frac{3}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} A,$$

$$C_2 \to C_2 - C_4$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} A,$$

$$C_3 \to \frac{1}{2} C_3$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & 1 \\ 0 & \frac{3}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} A,$$

$$C_3 \to C_3 - C_4$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 1 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} A,$$

$$C_2 \to C_2 - C_3$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -\frac{3}{2} & 1 \\ 0 & -\frac{5}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} A,$$

$$C_2 \to C_2 - C_3$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & -\frac{3}{2} & 1 \\ \frac{5}{2} & -\frac{5}{2} & 2 & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} A,$$

$$C_1 \to C_1 - C_2$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{3}{2} & 3 & -\frac{3}{2} & 1 \\ \frac{1}{2} & -\frac{5}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} A,$$

$$C_1 \to C_1 - C_3$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{3}{2} & 3 & -\frac{3}{2} & \frac{1}{3} \\ \frac{1}{2} & -\frac{5}{2} & 2 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix} A,$$

$$C_4 \rightarrow \frac{1}{3} C_4$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix} A,$$

$$C_1 \rightarrow C_1 - C_4$$

$$\Rightarrow I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix} A,$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix} A,$$

$\therefore (1) \Rightarrow C = A^{-1}B$

$$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-11}{6} & 3 & \frac{-3}{2} & \frac{1}{3} \\ 1 & \frac{-5}{2} & 2 & \frac{-1}{2} \\ \frac{-1}{6} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 548159652 \\ 683329097 \\ 846302688 \\ 1027015247 \end{pmatrix}$$

$$= \begin{pmatrix} 548159652 \\ 117912312.66 \\ 18934662 \\ -1677529.67 \end{pmatrix}$$

$\therefore a_0 = 548159652,\ a_1 = 117912312.66,$
$a_2 = 18934662,\ \ a_3 = -1677529.67$

$\therefore$ The polynomial representing the given numerical data is

$$y = 548159652 + 117912312.66x + 18934662x^2 - 1677529.67x^3$$

This polynomial yield the values of the function $y = f(x)$ as follows:

$y_0 = 548159652$,

$y_1 = 548159652 + 117912312.66 + 18934662 - 1677529.67$
$= 683329097$

$y_2 = 548159652 + 117912312.66 \times 2 + 18934662 \times 4 - 1677529.67 \times 8$
$= 846302688$

& $y_3 = 548159652 + 117912312.66 \times 3 + 18934662 \times 9 - 1677529.67 \times 27$
$= 1027015247$

These values of the function $y = f(x)$ are found to be identical with the respective observed values.

5. CONCLUSIONS:

The method developed here can be suitably applied in representing a set of numerical data on a pair of variables by a polynomial.

The polynomial that represents the given set of numerical data can be applied in interpolation at any position of the independent variable lying within its two extreme values.

The polynomial that represents the given set of numerical data can be applied in inverse interpolation also.

It has already been mentioned that in case of the interpolation by the existing formulae, if it is wanted to interpolate the values of the dependent variable corresponding to a number of values of the independent variable by a suitable existing interpolation formula then it is required to apply the formula for each value separately and thus the numerical computation of the value of the dependent variable based on the given data are to be performed in each of the cases. An advantage of the method composed here is that these repeated numerical computations from the given data are not necessary for representing a set of numerical data by a polynomial curve.

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