Equivalent Conditions on Conjugate Unitary Matrices

A. Govindarasu[#], S.Sassicala[#1]

[#]Department of Mathematics, A.V.C. College (Autonomous)
Mannampandal, Mayiladuthurai-609 305

Abstract — *Concept of conjugate unitary matrices are given. Some equivalent conditions on conjugate unitary matrices are obtained. Conditions related to secondary unitary matrices, hermitian, s-hermitian and involutary are also derived.*

AMS Classification — *15A09, 15A57.*

Keywords—*Unitary matrix, secondary transpose of a matrix, conjugate secondary transpose of a matrix, conjugate unitary matrix.*

I. Introduction

Anna Lee [1] has initiated the study of secondary symmetric matrices. Also she has shown that for a Complex matrix A, the usual transpose A^T and secondary transpose A^s are related as $A^s = VA^TV$, where 'V' is the permutation matrix with units in its secondary diagonal.

Also $\bar{A}^s$ denotes the conjugate secondary transpose of A. i.e., $\bar{A}^s = (c_{ij})$ where $c_{ij} = \overline{a_{n-j+1,n-i+1}}$ [2].

In this paper our intension is to prove some equivalent conditions on conjugate unitary matrices.

II. Preliminaries and Notations

Let C_{nxn} be the space of nxn complex matrices of order n. For $A \in C_{nxn}$. $A^T, \bar{A}, A^*, A^s, \bar{A}^s$ denote transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose of a matrix A respectively. Also V satisfies $V^T = \bar{V} = V^* = V$ and $V^2 = I$

A matrix $A \in C_{nxn}$ is called unitary if $AA^* = A^*A = I$ [7].

A matrix $A \in C_{nxn}$ is called conjugate normal if $AA^* = \overline{A^*A}$ [3].

A matrix $A \in C_{nxn}$ is called s-normal iff $AA^\theta = A^\theta A$ [6]

A matrix $A \in C_{nxn}$ is called secondary unitary (s-unitary) if $AA^\theta = A^\theta A = I$ [5].

A matrix $A \in C_{nxn}$ is called s-hermitian if $A = A^\theta = \bar{A}^S$.

A matrix $A \in C_{nxn}$ is called is called conjugate unitary if $AA^* = \overline{A^*A} = I$ [4].

III. Equivalent Conditions on Conjugate Unitary Matrices

Theorem 3.1 Any two of the following imply the other.

(i) A is conjugate unitary matrix (ii) A is hermitian (iii) A is involutary

Proof (i) and (ii) $\Longrightarrow$ (iii)

A is conjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$

Case (i) $AA^* = I$

$AA = I$ ($\because$ A is hermitian)

$A^2 = I$

Case (ii) $\overline{A^*A} = I$

Taking conjugate on both sides, we have

$\overline{\overline{A^*A}} = \bar{I}$

$A^*A = I$ ($\because \bar{I} = I$)

$AA = I$ ($\because$ A is hermitian)

$A^2 = I$

Therefore, in both cases implies A is involutary.

(ii) and (iii) $\Longrightarrow$ (i)

$A^2 = I$ ($\because$ A is involutary)

$AA = I$

$AA^* = I$ ($\because$ A is hermitian) (3.1.1)

Again take $A^2 = I$ ($\because$ A is involutary)

$AA = I$

$A^*A = I$ ($\because$ A is hermitian)

Taking conjugate on both sides, we have

$\overline{A^*A} = \bar{I}$

$\overline{A^*A} = I$ ($\because \bar{I} = I$) (3.1.2)

Therefore, from (3.1.1) and (3.1.2) we have

$AA^* = \overline{A^*A} = I$

$\therefore$ A is conjugate unitary matrix.

(iii) and (i) $\Longrightarrow$ (ii)

$AA^* = \overline{A^*A} = I$

Case (i) $AA^* = I$

Pre multiplying the above equation by A on both sides, we have

$AAA^* = AI$

$A^2A^* = A$

$I\ A^* = A$ ($\because$ A is involutary)

i.e., $A^* = A$

$\therefore$ A is hermitian

Case(ii) $\overline{A^*A} = I$

Taking conjugate on both sides, we have

$\overline{\overline{A^*A}} = \bar{I}$

$A^*A = I \quad (\because \bar{I} = I)$

Post multiplying the above equation by A on both sides, we have

$A^*AA = IA$

$A^*A^2 = A$

$A^*I = A \quad (\because A$ is involutary)

i.e., $A^* = A$

$\therefore$ A is hermitian

Therefore ,in both cases implies A is hermitian.

Theorem 3.2: Any two of the following imply the other.

(i) A is conjugate unitary matrix (ii) A is s-unitary

(iii) A commutes with V (i.e., $AV = VA$)

Proof: (i) and (ii) $\Longrightarrow$ (iii)

A is conjugate unitary matrix $\Rightarrow AA^* = \overline{A^*A} = I$

Case (i) $AA^* = I$

$A^* = A^{-1}$

$A^* = VA^*V \quad (\because A$ is s - unitary)

Taking conjugate transpose on both sides, we have

$(A^*)^* = (VA^*V)^*$

$A = V^*(A^*)^*V^*$

$= VAV \quad (\because V^* = V)$

Post multiplying the above equation by V on both sides, we have

$AV = VAVV$

$= VAV^2$

$= VA \quad (\because V^2 = I)$

Case (ii) $\overline{A^*A} = I$

Taking conjugate on both on both sides, we have

$\overline{\overline{A^*A}} = \bar{I}$

$A^*A = I \, (\because \bar{I} = I)$

$A^* = A^{-1}$

$A^* = VA^*V \quad (\because A$ is s - unitary)

Taking conjugate on both sides, we have

$(A^*)^* = (VA^*V)^*$

$A = V^*(A^*)^*V^*$

$= VAV \quad (\because V^* = V)$

Post multiply the above equation by V on both sides, we have

$AV = VAVV$

$= VAV^2$

$= VA \quad (\because V^2 = I)$

i.e., $AV = VA$

Therefore, in both cases implies A commutes with V

i.e., $AV = VA$

(ii) and (iii) $\Longrightarrow$ (i)

A is s – unitary $\Longrightarrow A^{-1} = VA^*V$

Taking conjugate transpose on both sides, we have

$(A^{-1})^* = (VA^*V)^*$

$(A^*)^{-1} = V^*(A^*)^*V^*$

$= VAV \quad (\because V^* = V)$

Post multiplying by A^* on both sides, we have

$(A^*)^{-1}A^* = VAVA^*$

$I = AVVA^* \quad (\because AV = VA)$

$= AV^2A^*$

$= AA^* \, (\because V^2 = I)$

i.e., $\overline{A^*A} = I$ (3.2.1)

$A^{-1} = VA^*V$

Taking conjugate transpose on both sides, we have

$(A^{-1})^* = (VA^*V)^*$

$(A^*)^{-1} = V^*(A^*)^*V^*$

$= VAV \quad (\because V^* = V)$

Pre multiplying by A^* on both sides, we have

$A^*(A^*)^{-1} = A^*VAV$

$I = A^*VVA \quad (\because AV = VA)$

$= A^*V^2A$

$I = A^*A \quad (\because V^2 = I)$

Taking conjugate on both sides, we have

$\bar{I} = \overline{A^*A}$

$I = \overline{A^*A} \quad (\because \bar{I} = I)$

$i.e.,\ \overline{A^*A} = I$ (3.2.2)

Therefore from equations (3.2.1) and (3.2.2) we have,

$AA^* = \overline{A^*A} = I$

(iii) and (i) $\Longrightarrow$(ii)

$AA^* = \overline{A^*A} = I$

Case (i) $AA^* = I$ (3.2.3)

Pre multiplying the above equation by V on both sides, we have

$VAA^* = VI$

$VAA^* = V$ (3.2.4)

Post multiplying equation (3.2.3) by V on both sides, we have

$AA^*V = IV$

$AA^*V = V$ (3.2.5)

Therefore, from (3.2.4) and (3.2.5), we have

$VAA^* = AA^*V = V$

Case (ii) $\overline{A^*A} = I$

Taking conjugate on both sides, we have

$\overline{\overline{A^*A}} = \bar{I}$

$A^*A = I \quad (\because \bar{I} = I)$ (3.2.6)

Pre multiplying equation (3.2.6) by V on both sides, we have

$VA^*A = VI$

$VA^*A = V$ (3.2.7)

Post multiplying equation (3.2.6) by V on both sides, we have

$A^*AV = IV$

$A^*AV = V$ (3.2.8)

Therefore from equations (3.2.7) and (3.2.8), we have

$VA^*A = A^*AV = V$

Therefore, in both cases, implies A is s-unitary matrix.

Theorem 3.3: Any two of the following imply the other.

(i) A is conjugate unitary matrix (ii) A is s–hermitian (iii) $VA^*$is involutary

Proof: (i) and (ii) ⟹ (iii)

A is conjugate unitary matrix ⟹ $AA^* = \overline{A^*A} = I$

Case (i) $AA^* = I$

$VA^*VA^* = I \quad (\because A = VA^*V)$

$(VA^*)^2 = I$

Case (ii) $\overline{A^*A} = I$

Taking conjugate on both sides, we have

$\overline{\overline{A^*A}} = \bar{I}$

$A^*A = I \quad (\because \bar{I} = I)$

Post multiplying by A^{-1} on both sides, we have

$A^*AA^{-1} = IA^{-1}$

$A^* = A^{-1}$

Pre multiplying by A on both sides, we have

$AA^* = AA^{-1}$

$AA^* = I$

$VA^*VA^* = I \quad (\because A$ is s- hermitian i.e., $A = VA^*V)$

$(VA^*)^2 = I$

Therefore, in both cases implies $(VA^*)^2 = I$

i.e., $VA^*$is involutary

(ii) and (iii) ⟹ (i)

$A = VA^*V$

$IA = VA^*V$

$(VA^*)^2A = VA^*V \quad (\because VA^*$ is involutary)

$VA^*VA^*A = VA^*V$

$(VA^*V)A^*A = VA^*V$

$AA^*A = A$

Pre multiplying by A^{-1} on both sides, we have

$A^{-1}AA^*A = A^{-1}A$

$A^*A = I$

Taking conjugate on both sides, we have

$\overline{A^*A} = \bar{I}$

$\overline{A^*A} = I \quad (\because \bar{I} = I) \qquad (3.3.1)$

$A = VA^*V$

$AI = VA^*V$

$A(VA^*)^2 = VA^*V \quad (\because VA^*$is involutary)

$AVA^*VA^* = VA^*V$

$A(VA^*V)A^* = VA^*V$

$AAA^* = A$

Pre multiplying by A^{-1} on both sides, we have

$A^{-1}AA^*A = A^{-1}A$

$IA^*A = I$

i.e., $A^*A = I \qquad (3.3.2)$

Therefore from equations (3.3.1) and (3.3.2) we have

$AA^* = \overline{A^*A} = I$

∴ A is conjugate unitary matrix.

(iii) and (i) ⟹ (ii)

A is conjugate unitary matrix ⟹ $AA^* = \overline{A^*A} = I$

Case (i) $AA^* = I$

$AA^* = (VA^*)^2 \quad (\because VA^*$ is involutary)

$AA^* = VA^*VA^*$

Post multiplying by $(A^*)^{-1}$ on both sides, we have

$AA^*(A^*)^{-1} = (VA^*V)A^*(A^*)^{-1}$

$A = VA^*V$

Case (ii) $\overline{A^*A} = I$

Taking conjugate on both sides, we have

$\overline{\overline{A^*A}} = \bar{I}$

$A^*A = I \quad (\because \bar{I} = I)$

Pre multiplying by A on both sides, we have

$A^*A = I$

$AA^*A = A$

Post multiplying by A^{-1} on both sides, we have

$AA^*AA^{-1} = AA^{-1}$

$AA^*I = I$

$AA^* = (VA^*)^2 \quad (\because VA^*$ is involutary)

$AA^* = VA^*VA^*$

Post multiplying by $(A^*)^{-1}$ on both sides, we have

$AA^*(A^*)^{-1} = VA^*VA^*(A^*)^{-1}$

$A = VA^*V$

Therefore, in both cases implies $A = VA^*V$

i.e., A is s- hermitian.

Theorem3.4: Any two of the following imply the other.

(i) A is conjugate unitary matrix (ii) A is skew hermitian (iii) (iA) is Involutary

Proof : (i) and (ii) ⟹(iii)

$AA^* = \overline{A^*A} = I$

Case (i) $AA^* = I$

$A(-A) = I \quad (\because A^* = -A)$

$-A^2 = I$

$i^2A^2 = I$

$(iA)^2 = I$

Case (ii) $\overline{A^*A} = I$

Taking conjugate on both sides, we have

$\overline{\overline{A^*A}} = \bar{I}$

$A^*A = I \quad (\because \bar{I} = I)$

$-AA = I \quad (\because A^* = -A)$

$-A^2 = I$

$i^2A^2 = I$

$(iA)^2 = I$

Therefore, in both cases implies (iA) is involutary

(ii)and (iii) ⟹ (i)

$A^* = -A$

$IA^* = -A$

$(iA)^2A^* = -A$

$i^2A^2A^* = -A$

$-AAA^* = -A$

i.e., $AAA^* = A$

Pre multiplying by A^{-1} on both sides, we have

$A^{-1}AAA^* = A^{-1}A$

$AA^* = I \qquad (3.4.1)$

Again take $A^* = -A$

$A^*I = -A$

$A^*(iA)^2 = -A$

$A^*i^2A^2 = -A$

$-A^*AA = -A$

$A^*AA = A$

Post multiplying by A^{-1} on both sides, we have

$A^*AAA^{-1} = AA^{-1}$

$A^*A = I$

Taking conjugate on both sides, we have,

$\overline{\overline{A^*A}} = \bar{I}$

$\overline{A^*A} = I \quad (\because \bar{I} = I)$ (3.4.2)

Therefore, from (3.4.1) and (3.4.2) we have

$AA^* = \overline{A^*A} = I$

(iii) and (ii) $\Longrightarrow$ (i)

Take $\overline{A^*A} = I$

Taking conjugate on both sides, we have

$\overline{\overline{A^*A}} = \bar{I}$

$A^*A = I \quad (\because \bar{I} = I)$

$A^*A = (iA)^2$

$A^*A = -A^2$

$A^*A = -AA$

Post multiplying by A^{-1} on both sides, we have

$A^*AA^{-1} = -AAA^{-1}$

$A^* = -A$ (3.4.3)

Now we take $AA^* = I$

$AA^* = (iA)^2$

$AA^* = -AA$

Pre multiplying by A^{-1} on both sides, we have

$A^{-1}AA^* = -A^{-1}AA$

$A^* = -A$ (3.4.4)

Therefore, from (3.4.3) and (3.4.4) we have $A^* = -A$

$\Longrightarrow$ A is skew-hermitian

Theorem3.5: If $VA^* = AV$ then any two of the following imply the other.

(i) A is s-unitary (ii) A is conjugate unitary (iii) A is hermitian

Proof: Given that $VA^* = AV$

(i) and (ii) $\Longrightarrow$ (iii)

$A^{-1} = VA^*V \quad (\because A \text{ is s- unitary})$

$IA^{-1} = VA^*V$

$A^*AA^{-1} = VA^*V \ (\because A \text{ is conjugate unitary})$

$A^* = AVV \quad (\because VA^* = AV)$

$A^* = AV^2$

$A^* = A \ (\because V^2 = I)$

i.e., A is hermitian

(ii) and (iii) $\Longrightarrow$(i)

Case (i) $AA^* = I$ (3.5.1)

Pre multiplying by V on both sides, we have

$VAA^* = VI$

$VAA^* = V$ (3.5.2)

Post multiplying equation (3.5.1) by V on both sides, we have

$A^*V = IV$

$AA^*V = V$ (3.5.3)

Therefore, from equations (3.5.2) and (3.5.3) we have

$VAA^* = AA^*V = V$

Case (ii) $\overline{A^*A} = I$

Taking conjugate on both sides, we have

$\overline{\overline{A^*A}} = \bar{I}$

$A^*A = I \quad (\because \bar{I} = I)$

$AA = I \quad (\because A \text{ is hermitian})$

$AA^* = I \quad (\because A \text{ is hermitian})$ (3.5.4)

Pre multiplying by V on both sides, we have,

$VAA^* = V$ (3.5.5)

Post multiplying equation (3.5.4) by V on both sides, we have,

$AA^*V = I$ (3.5.6)

Therefore, from equations (3.5.5) and (3.5.6), we have,

$VAA^* = AA^*V = V$

Therefore, in both cases, implies A is s-unitary.

(iii) and (i) $\Longrightarrow$ (ii)

$A^{-1} = VA^*V \ (\because A \text{ is s – unitary})$

Pre multiplying by A on both sides, we have

$AA^{-1} = AVA^*V$

$I = AVAV \quad (\because A \text{ is hermitian})$

$I = AVVA^*$

$I = AV^2A^*$

$I = AA^* \quad (\because V^2 = I)$

i.e., $AA^* = I$ (3.5.7)

$A^{-1} = VA^*V$

$A^{-1} = VAV \quad (\because A \text{ is hermitian})$

Pre multiplying by A on both sides, we have

$AA^{-1} = AVAV$

$I = A^*VAV \quad (\because A \text{ is hermitian})$

$I = A^*VVA^*$

$I = A^*V^2A^*$

$I = A^*A^* \quad (\because V^2 = I)$

$I = A^*A \quad (\because A \text{ is hermitian})$

i.e., $A^*A = I$

Taking conjugate on both sides, we have,

$\overline{A^*A} = \bar{I}$

$\overline{A^*A} = I \ (\because \bar{I} = I)$ (3.5.8)

Therefore, from equations (3.5.7) and (3.5.8), we have

$AA^* = \overline{A^*A} = I$

i.e., A is conjugate unitary matrix.

Theorem 3.6: Any two of the following imply the other.

(i)A is conjugate unitary matrix (ii) A is hermitian (iii) A^T is involutary

Proof: (i) and (ii) $\Longrightarrow$ (iii)

A is conjugate unitary matrix $\Longrightarrow$

$AA^* = \overline{A^*A} = I$

Case (i) $AA^* = I$

Take transpose on both sides, we have

$(AA^*)^T = I^T$

$(A^*)^T A^T = I \quad (\because I^T = I)$

$A^T A^T = I$

$(A^T)^2 = I$

$i.e.,$ A^T is involutary

Case (ii) $\overline{A^*A} = I$

Taking conjugate on both sides, we have

$\overline{\overline{A^*A}} = \bar{I}$

$A^*A = I \quad (\because \bar{I} = I)$

$AA = I \quad (\because A$ is hermitian $)$

$A^2 = I$

Taking transpose on both sides, we have

$(A^2)^T = I^T$

$(A^T)^2 = I \quad (\because I^T = I)$

Therefore, in both cases we have A^T is involutary.

(ii) and (iii) $\Longrightarrow$ (i)

$A^* = A$

Pre multiplying by A on both sides, we have

$AA^* = AA$

$AA^* = A^2$

Taking transpose on both sides, we have

$(AA^*)^T = (A^2)^T$

$(AA^*)^T = (A^T)^2$

$(AA^*)^T = I \quad (\because A^T$ is involutary $)$

(iii) and (ii) $\Longrightarrow$ (i)

$(A^T)^2 = I$

$A^T A^T = I$

Taking transpose on both sides, we have

$(A^T)^T (A^T)^T = I^T$

$AA = I \quad (\because I^T = \mathrm{I})$

Pre multiplying by A^{-1} on both sides, we have

$A^{-1}AA = A^{-1}I$

$A = A^{-1}$

$A = A^* \quad (\because AA^* = I \Longrightarrow A^* = A^{-1})$

$\Longrightarrow A$ is hermitian

Again taking transpose on both sides, we have

$((AA^*)^T)^T = I^T$

$AA^* = I \quad (\because I^T = I) \qquad (3.6.1)$

Again take $A^* = A$

Post multiplying by A on both sides, we have

$A^*A = AA$

$A^*A = A^2$

Taking transpose on both sides, we have

$(A^*A)^T = (A^2)^T$

$(A^*A)^T = (A^T)^2$

$(A^*A)^T = I \quad (\because A^T$ is involutary $)$

Again taking transpose on both sides, we have

$((A^*A)^T)^T = I^T$

$A^*A = I \quad (\because I^T = I)$

Taking conjugate on both sides, we have

$\overline{A^*A} = \bar{I}$

$\overline{A^*A} = I \quad (\because \bar{I} = I) \qquad (3.6.2)$

Therefore, from equations (3.6.1) and (3.6.2), we have

$AA^* = \overline{A^*A} = I$

i.e., A is conjugate unitary matrix.

IV. CONCLUSION

Some equivalent condition on conjugate unitary matrices are obtained. Conditions related to secondary unitary matrices, unitary matrices, hermitian, s-hermitian and involutary matrix are also derived.

REFERENCES

[1]. Anna Lee, Secondary symmetric, secondary skew symmetric, secondary orthogonal matrices, *Period. Math. Hungary*, Vol. 7, pp. 63-70, 1976.

[2]. Anna Lee, On s-symmetric, s-skew symmetric and s-orthogonal matrices, *Period. Math. Hungary*, Vol. 7, pp. 71-76, 1976.

[3]. A. Faßbender and Kh.D.Ikramov, Conjugate-normal matrices: a survey, *Lin. Alg. Appl.*, Vol. 429, pp.1425-1441, 2008.

[4]. A. Govindarasu and S. Sassicala, Characterizations of conjugate unitary matrices, *Journal of ultra scientist of physical science*, Vol. 29, No. 1, pp. 33–39, 2017.

[5]. S. Krishnamoorthy and A. Govindarasu, On Secondary Unitary Matrices, *International Journal of computational science and Mathematics*, Vol.2, No.3, pp. 247-253, 2010.

[6]. S. Krishnamoorthy and R. Vijayakumar, Some equivalent conditions on s-normal matrices, *International Journal of Comtemporary Mathematical Sciences*, Vol.4, No.29, pp.1449 – 1454, 2009.

[7]. A.K. Sharma, *Text book of matrix theory*, DPH Mathematics series, 2004