Characterizations for solutions of certain classes of Non-linear Diophantine equations

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ABSTRACT: *The main aim of this paper is to introduce a method, to solve certain class of non-homogenous non-Linear Diophantine equations and investigate various properties using the well-known Euler's theorem and the theory of congruence. Some of the interesting special cases of our main results have been discussed.*

KEY WORDS: *Diophantine Equations, Non-linear Diophantine equations, Euler phi-function, Euler's theorem.*

1. INTRODUCTION AND PRILIMINARY OBSERVATIONS

This paper focuses on the non-linear Diophantine equations of the form:

$$x^n + D = \lambda y^m, \qquad (1.1)$$

for positive integers $x, y, \lambda = -1$, D<0, and n, m are even natural numbers.
The case when $D > 0$, λ =1, n=2; Arif and Abu Muriefah [1] conjectured that the only solutions of (1.1) are given by (x, y) = (2^k, 2^{2k+1}) and (x , y) = (11. 2^{k-1}, 5. $2^{2(k-1)/3}$), with the latter solution existing only when (k, n) = (3M+1, 3) for some integer M ≥0. For another result concerning the case D =2^a refer to [9]. Arif and Abu Muriefah [2] proved that if D = 3^{2k+1} then (1.1) has exactly one infinite family of solutions. The case D = 32k has been solved by Luca [10] under the additional hypothesis that x and y are co prime. This was extended by Tao [17] to arbitrary positive integers x, y and D = 3^{2k}. In [11] Luca solved completely equation (1.1) if D = $2^a 3^b$ and gcd(x; y) = 1. Abu Muriefah [3] established that equation (1.1) with D = 5^{2k} may have a solution only if 5 divides x and p does not divide k for any odd prime p dividing n. For related results concerning the case D = 5^m one may consult the paper of Tao [18]. The case D = $2^a 3^b 5^c 7^d$ with gcd(x, y) = 1, where a, b, c, d are non-negative integers was studied by Pink [16] and for a ≥1 and gcd(x, y) = 1 all the solutions of (1.1) were listed. The complete solution of (1.1) in the cases when D = 7^{2k} and D = $2^a 5^b$ were given by Luca and Togbe [12], [13]. In [4] Abu Muriefah, Luca and Togbe determined all solutions of (1.1) with D = $2^a 3^b$ and gcd(x, y) = 1. Further, the case D =$2^a 3^b$ 11^c has been considered by Cangul, Demirci, Inam, Luca and Soydan [7], where all the solutions of (1.1) were given with gcd(x, y) = 1. Pink and Rabai [19]
gave all solutions of equation (1.1) if D =$5^a 17^b$ and gcd(x,y) = 1. Godinho, Marques and Togbe [8] solved completely (1.1) the case D = $2^a 5^b 17^c$ under the assumption gcd(x, y) = 1. Soydan, Ulas and Zhu [20] solved completely (1.1) with D = $2^a 19^b$ and gcd(x, y) = 1. Xiaowei [21] gave solutions of equation (1.1) with D =p^{2k} for a prime number p and gcd(x, y) = 1.

Moreover; when the case $D < 0$ and λ = -1, for even natural numbers n and m, Yann Bugeaud [5] gave solution for (1.1) when the case $D = -p^m$, prime number p and state their main result in [5, Theorem 1 and Theorem 2], depending only on the value of p $modulo$ 4. They showed in these theorem that If p ≡3 mod 4, then $x^2 - \mathrm{p}^{\mathrm{m}} = -y^n$ have only finitely many solutions (x, y, m, n). Moreover, those solutions satisfy n ≤ 5.6 · $10^5 \mathrm{p}^2 \log^2 \mathrm{p}$. If p ≡1 mod 4, then $x^2 - \mathrm{p}^{\mathrm{m}} = -y^n$ have only finitely many solutions (x, y, m, n) with even m or odd y. those solutions satisfy n ≤ 5.6 · $10^5 \mathrm{p}^2 \log^2 \mathrm{p}$.

The main tools which have been used for this are the well known Euler phi-function, Euler's theorem and the theory of congruence.

Euler phi-function for each positive integer n is defined as φ(n) is the number of positive integers less than or equal to n that are co primes to n and $\varphi(1) = 1$ and, when $n > 1$, Leo Moser [14].

In Euler phi-function for the power of a prime number p for $n \in N$ is proved that $\varphi(p^n)$=p^{n-1}(p-1),Lindsay N. Childs[15,Page180]. This is generalized for any $n \in N$ as $\varphi(n)$ =n $\prod_{i=1}^{n}(1-\frac{1}{p_i})$ for distinct prime factors $p_1, p_2, p_3, \dots, p_n$ of n, Melvyn B. Nathanson[6, page 58]. Moreover, it is proved that $\varphi(n)$ is even positive integer which is less than n for n>2.

In Leo Moser [5, page44] the Euler's Theorem that for a positive integer m and $(a, m) = 1$ is proved that $a^{\varphi(m)} \equiv 1(\mathrm{mod} m)$.

2. Main Results

In this paper, when the case $D < 0, \lambda = \pm 1$, conjectures of certain classes of Non-linear Diophantine equations as type of Equation (1.1) and their proofs have been given as follows:

$x^m + D = \lambda y^n$ $with$ $D < 0, \lambda = \pm 1, m$ and n Even positive integers. …………………..(2.1.)

To see this, we put two possible values of λ.

Case 1: when $m = 2$ and $\lambda = 1, D < 0$

This automatically reveals different solutions for different D values by Arif and Abu Muriefah [1] and [2], Luca [10], Tao [17], Luca [11] , Abu Muriefah [3], Tao [18]; Luca and Togbe [12], [13]; Abu Muriefah, Luca and Togbe [4];Cangul, Demirci, Inam, Luca and Soydan [7], Pink and Rabai [19]; Godinho, Marques and Togbe [8]; Soydan, Xiaowei [21].

However, this paper is mainly lies over the case where m and n be any even natural numbers, for some $D < 0$ and $\lambda = -1$ as follows:

Case2: m and n are even positive integers , $\lambda = -1, D = -K < 0$.

In this paper, to characterize Case2, the following theorem to introduce certain classes of Non-linear Diophantine equations with a new characterization and their possible solutions have resulted in. In this result, the solutions have been determined by depending on the degree of the equations.

Define: $Pim_{\varphi}(m)$ as the pre- image of m over the Euler phi-function φ.

Theorem2.1: The non-linear Diophantine of the form $x^m + D = -y^n$ has solutions of triplets(x, y, D) if

i. $Gcd(x, y) = 1$
ii. $Pim_{\varphi}(m)$=y, $Pim_{\varphi}(n) = x$
iii. with $D = -k$ such that k is expressed as $k \equiv 1 mod xy$.

$Proof$: By Euler's theorem, $a^{\varphi(b)} \equiv 1(mod b)$ and $b^{\varphi(a)} \equiv 1(mod\ a)$, with $\gcd(a, b) = 1$,Leo Moser [1, page44].

And also $\varphi(a) = a\prod_{i=1}^{n}(1-\frac{1}{p_i})$ for distinct prime factors $p_1, p_2, p_3, \dots, p_n$ of a, similarly for $\varphi(b)$, Melvyn B. Nathanson[6, page 58].

From this we have that $a^{\varphi(b)} - 1 \equiv 0(mod b)$ and $b^{\varphi(a)} - 1 \equiv 0(mod\ a)$.

Let R =$a^{\varphi(b)} + b^{\varphi(a)}$.$R - 1 = a^{\varphi(b)} + (b^{\varphi(a)} - 1) \equiv 0 moda$, and $R \equiv 1(mod\ b)$.

Moreover; $R - 1 = (a^{\varphi(b)} - 1) + b^{\varphi(a)} \equiv 0 modb$, and $R \equiv 1(mod\ b)$.Therefore,$R \equiv 1(mod\ ab)$.

$$\Rightarrow a^{\varphi(b)} + b^{\varphi(a)} \equiv 1(modab)$$

This congruence can be re-expressed as

$a^{\varphi(b)} + b^{\varphi(a)} = D, where\ D = abr + 1$, for some $r\epsilon\ Z$…………….. (**2.2**)

Then, Equation 2.2 shows that $a^m + b^n = D$, where $D = abr + 1$, for some $r\epsilon\ Z$ has solutions of the form (x, y) =(a, b) in such a way that gcd(a, b)=1. Moreover; m, n are even natural numbers , Melvyn B. Nathanson[21].

Thus, (x, y) =(a, b) can satisfy the non-linear Diophantine equation of the form

$x^m + D = -y^n$, with some $D = -k$ such that $k \equiv 1 mod xy$ and for even natural numbers m, n. ………………………………………………………………………..……………….… (**2.3**)

Remark: The computation of these two integers $x = a$ and $y = b$ is based on the fixed positive evenintegers m, n given in the Diophantine eqaution(2.3) as $b = Pim_{\varphi}(m)$=y, $a = Pim_{\varphi}(n) = x$. In general (x, y) = (a, b)=($Pim_{\varphi}(m)$,$Pim_{\varphi}(n)$),with gcd(a,b)=1.

Thus from the given even natural numbers $m\ and\ n$, degree of terms of the Diophantine equation, computing $Pim_{\varphi}(m)$ and $Pim_{\varphi}(n)$ is mandatory.

Algorithm to determine $Pim_{\varphi}(n)[Anteneh's\ algorithim]$:

Consider any positive integer r such that $Pim_{\varphi}(n) = r$, where the prime factorization for r is given by, r=$\prod_{j=1}^{i} p_j{}^{k_j}$, for disinict primes p_j.

Step 1: Consider those even divisors θ of n such that $\theta = \prod_{j=1}^{i}(p_j - 1)$ for some distinict primes p_j.

Step 2: Consider the possible prime numbers $p_1,\ p_2, \dots, p_i$ one by one such that θ in step 1 above are expressed as $\prod_{j=1}^{i}(p_j - 1)$.

Step 3: Consider $k_i's$ corresponding to those $p_i' s$ obtained in step 2 satisfying $\frac{r}{\theta} = \prod_{j=1}^{i} p_j{}^{k_j - 1}$.

Step 4: However, if there is no p_i obtained in step 3, then consider p_i and p_j paire wise and determine the corresponding k_i and k_j, for $i \neq j$ satisfying the property $\frac{r}{\theta} = \prod_{j=1}^{i} p_j{}^{k_j - 1}$.

Step5. If result is not obtained in step 4, then consider distinct $p_i,\ p_j, p_t$and determine the corresponding $k_i, k_j\ and\ k_t$ satisfying the property $\frac{r}{\theta} = \prod_{j=1}^{i} p_j{}^{k_j - 1}$.

Step 6: If result is not obtained in the above steps; continuing the process by considering all the possible distinct primes $p_1, p_2, \dots, p_i$ at the same time and identify the corresponding $k_1, k_2, \dots, k_i$satisfying the property $\frac{r}{\theta} = \prod_{j=1}^{i} p_j{}^{k_j - 1}$.

Step 7: Using p_i's and k_i's,$\forall i$ which are obtained in the above steps determine $Pim_{\varphi}(n) = r = \prod_{j=1}^{i} p_j{}^{k_j}$.

Special cases

Corrolary2.2. The possible non-trivial solutions for the Diophantine equation $x^2 - D = -y^4$ are expressed as (5,3,106),(5,4,281),(5,6,1321),(8,3,145),(10,3,181)

Proof: Suppose $a = Pim_{\varphi}(2),\ b = Pim_{\varphi}(4)$

Step1: The possible even divisors n= 2 are θ=1,2.

Step2: i).For θ=1=$\prod_{j=1}^{i}(p_j - 1)$, for satisfying this equality there is no prime numbers except $p_1 = 2$.So $\frac{a}{\theta}$=$\frac{2}{1}$=2=$\prod_{j=1}^{i} p_j{}^{k_j - 1}$.Hence, the possible k_1 corresponding to p_1=2 is obtained as $p_1{}^{k_1 - 1} = 2^{k_1 - 1} = 2$. Therefore; $k_1 = 2$. Finally, $n = \prod_{j=1}^{i} p_j{}^{k_j}$=4.

ii). For θ=2=$\prod_{j=1}^{i}(p_j-1)$, the prime numbers corresponding to this θ are:

a. $p_1=3$.So by the property $\prod_{j=1}^{i}{p_j}^{k_j-1}=\frac{a}{\theta}$ we have that 3^{k_1-1}=1. As a result; $k_1=1$. Finally, $n=\prod_{j=1}^{i}{p_j}^{k_j}$=3.

b. $p_1=2$ and$p_1=3$ Pairwise. So by the property $\prod_{j=1}^{i}{p_j}^{k_j-1}=\frac{a}{\theta}$ we have that $2^{k_1-1}.3^{k_1-1}$=1. As a result;$k_1=1$ and $k_2=1$. Consequently; $n=\prod_{j=1}^{i}{p_j}^{k_j}$=6.
Hence; the possible values of $a=Pim_{\varphi}(2)$ are 3,4,6.

To determine $b=Pim_{\varphi}(4)$ follow the following steps

Step1: The possible even divisors $\varphi(n)$= 4 are θ=1, 2, 4.

Step2: i). For θ=1=$\prod_{j=1}^{i}(p_j-1)$, for satisfying this equality there is no prime numbers except $p_1=2$. So $\frac{b}{\theta}$=$\frac{4}{1}$=4=$\prod_{j=1}^{i}{p_j}^{k_j-1}$.Hence, the possible k_1 corresponding to p_1=2 is obtained as ${p_1}^{k_1-1}=2^{k_1-1}=4$. Therefore; $k_1=3$. Finally, $n=\prod_{j=1}^{i}{p_j}^{k_j}$=$2^{k_1}$ =8.

ii). For θ=2=$\prod_{j=1}^{i}(p_j-1)$, the prime numbers corresponding to this θ are:

a. $p_1=3$.So by the property $\prod_{j=1}^{i}{p_j}^{k_j-1}=\frac{b}{\theta}$ we have that 3^{k_1-1}=2. From this it is impossible to get an integer k_1 satisfying this equality.

b. $p_1=2$ and $p_2=3$ Pairwise. So by the property $\prod_{j=1}^{i}{p_j}^{k_j-1}=\frac{b}{\theta}$ we have that $2^{k_1-1}.3^{k_2-1}$=2. As a result;$k_1=2$ and $k_2=1$.Finally, $n=\prod_{j=1}^{i}{p_j}^{k_j}$=$2^{k_1}.3^{k_2}$=12.

iii). For θ=4=$\prod_{j=1}^{i}(p_j-1)$, the prime numbers corresponding to this θ are:

a. $p_1=5$.So by the property $\prod_{j=1}^{i}{p_j}^{k_j-1}=\frac{b}{\theta}$ we have that 5^{k_1-1}=1. From this k_1 =1. Therefore; $n=\prod_{j=1}^{i}{p_j}^{k_j}$=$5^{k_1}$=5.

b. $p_1=2$ and$p_2=5$ Pairwise. So by the property $\prod_{j=1}^{i}{p_j}^{k_j-1}=\frac{b}{\theta}$ we have that $2^{k_1-1}.5^{k_2-1}$=1. As a result;$k_1=1$ and $k_2=1$.Finally, $n=\prod_{j=1}^{i}{p_j}^{k_j}$=$2^{k_1}.5^{k_2}$ =10.

Hence; the values of $b=Pim_{\varphi}(4)$ are 5,8,10,12.

Now, (a, b)= (5,3,106),(5,4,281),(5,6,1321),(8,3,145),(10,3,181)are all pairs of co-prime positive integers. From these values, the possible D is expressed as D≡ $1\mathrm{mod}(ab)$ satisfying the given Diophantine equation. Consequently, the possible non-trivial positive solutions will be (5,3,106),(5,4,281),(5,6,1321),(8,3,145),(10,3,181).

Corollary 2.3: The possible non-trivial solutions for the Diophantine equation $x^2-D=-y^4$ are expressed as (7,5,18026), (9,5,22186), (14,5,54041), (18,5,120601),(9,8,268705).

Corollary 2.4: The non-linear Diophantine of the form $x^n+D=-y^n$ has solutions of triplets (x,y,D)if

i. $Gcd(x,y)=1$
ii. $x=Pim_{\varphi}(n)=y$
iii. with $D=-k$ such that k $\equiv 1modxy$.

Proof: This is the direct Consequence of Theorem 1.1 above by considering $m=n$.

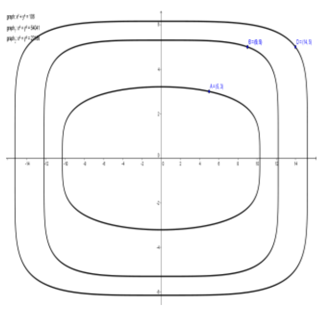

Figure 1

Note: In figure 1 above, using Geogebra soft ware, for the graphs of the given specific higher degree polynomial it is indicated and assured that some integers are roots for the equations (i.e. Existence of Non-trivial integer solution for certain classes of Non-linear Diophantine equations is checked).

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