Eulerian integral associated with product of three multivariable Aleph-functions and a class of polynomials

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ABSTRACT
The present paper is evaluated a new Eulerian integral associated with the product of three multivariable Aleph-functions , a generalized Lauricella function ,a class of multivariable polynomials with general arguments . We will study the cases concerning the multivariable I-function defined by Sharma et al [2] and Srivastava-Daoust polynomial [3].

Keywords: Eulerian integral, multivariable I-function, generalized Lauricella function of several variables, multivariable Aleph-function, generalized hypergeometric function, class of polynomials, Srivastava-Daoust polynomial

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1. Introduction

In this paper, we consider a general class of Eulerian integral concerning the product of three Multivariable Aleph-functions and a general class of multivariable polynomials defined by Srivastava et al [6].
The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [2] , itself is an a generalisation of G and H-functions of several variables defined by Srivastava et al [6]. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1''',\cdots,z_v''') = \aleph^{0,N:M_1,N_1,\cdots,M_v,N_v}_{P_i,Q_i,\tau_i;R:P_{i^{(1)}},Q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;P_{i^{(r)}},Q_{i^{(v)}};\tau_{i^{(v)}};R^{(v)}} \left(\begin{array}{c} \mathrm{z}'''_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}'''_v \end{array} \right|$

$$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(v)})_{1,\mathfrak{n}}] \ ,[\tau_i'(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(v)})_{\mathfrak{n}+1,p_i}] :$$
$$\cdots\cdots\cdots \ ,[\tau_i'(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(v)})_{m+1,q_i}] :$$

$$\left. \begin{array}{l} [(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,n_1}],[\tau'_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots; \ [(\mathrm{c}_j^{(v)});\gamma_j^{(v)})_{1,n_v}],[\tau'_{i^{(v)}}(c_{ji^{(v)}}^{(v)};\gamma_{ji^{(v)}}^{(v)})_{n_v+1,p_i^{(v)}}] \\ [(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,m_1}],[\tau'_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots; \ [(\mathrm{d}_j^{(v)});\delta_j^{(v)})_{1,m_v}],[\tau'_{i^{(v)}}(d_{ji^{(v)}}^{(v)};\delta_{ji^{(v)}}^{(v)})_{m_v+1,q_i^{(v)}}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^v}\int_{L_1}\cdots\int_{L_v}\psi_1(s_1,\cdots,s_v)\prod_{k=1}^{v}\xi_k(s_k)z_k'''^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi_1(s_1,\cdots,s_v) = \frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{k=1}^{v}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i'\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\sum_{k=1}^{v}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{Q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{v}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

$$\text{and } \xi_k(s_k) = \frac{\prod_{j=1}^{M_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{N_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau'_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, P; b_j, j = 1, \cdots, Q;$

$c_j^{(k)}, j = 1, \cdots, N_k; c_{ji^{(k)}}^{(k)}, j = N_k + 1, \cdots, P_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, M_k; d_{ji^{(k)}}^{(k)}, j = M_k + 1, \cdots, Q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} + \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} + \tau'_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{M_k} \delta_j^{(k)}$$

$$-\tau'_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to M_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k'''| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} - \tau'_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau'_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} - \tau'_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau'_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1, \cdots, r, i = 1, \cdots, R \text{ , } i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1''', \cdots, z_v''') = 0(|z_1'''|^{\alpha_1}, \cdots, |z_r'''|^{\alpha_r}), max(|z_1'''|, \cdots, |z_v'''|) \rightarrow 0$

$\aleph(z_1''', \cdots, z_v''') = 0(|z_1'''|^{\beta_1}, \cdots, |z_v'''|^{\beta_r}), min(|z_1'''|, \cdots, |z_v'''|) \rightarrow \infty$

where $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

Serie representation of Aleph-function of $u-$variables is given by

$$\aleph(z_1''', \cdots, z_v''') = \sum_{G_1,\cdots,G_v=0}^{\infty} \sum_{g_1=0}^{M_1} \cdots \sum_{g_v=0}^{M_v} \frac{(-)^{G_1+\cdots+G_v}}{\delta_{g_1} G_1! \cdots \delta_{g_v} G_v!} \psi_1(\eta_{G_1,g_1}, \cdots, \eta_{G_v,g_v})$$

$$\times\ \xi_1(\eta_{G_1,g_1}) \cdots \xi_v(\eta_{G_v,g_v}) z_1^{-\eta_{G_1,g_1}} \cdots z_v^{-\eta_{G_v,g_v}} \tag{1.6}$$

Where $\psi(.,\cdots,.)$, $\theta_i(.)$,$i = 1, \cdots, r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1} = \frac{d_{g_1}^{(1)} + G_1}{\delta_{g_1}^{(1)}}, \cdots,\ \eta_{G_v,g_v} = \frac{d_{g_v}^{(v)} + G_v}{\delta_{g_v}^{(v)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i + p_i] \neq \delta_j^{(i)}[d_{g_i}^i + G_i]$ (1.7)

for $j \neq M_i, M_i = 1, \cdots \eta_{G_i,g_i}; P_i, N_i = 0, 1, 2, \cdots, ; y_i \neq 0, i = 1, \cdots, v$

We have : $\aleph(z_1, \cdots, z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \right.$

$[(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,\mathfrak{n}}]$ $,[\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]$:
.... $,[\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}]$:

$[(c_j^{(1)}), \gamma_j^{(1)})_{1,n_1}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}]; \cdots ;\ ; [(c_j^{(r)}), \gamma_j^{(r)})_{1,n_r}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]$
$[(d'^{(1)}_j), \delta'^{(1)}_j)_{1,m_1}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}]; \cdots ;; [(d'^{(r)}_j), \delta'^{(r)}_j)_{1,m_r}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \Big)$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1'} \cdots \int_{L_r'} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.8}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{\mathfrak{n}} \Gamma(1 - a_j + \sum_{k=1}^r \alpha_j^{(k)} s_k)}{\sum_{i=1}^{R} [\tau_i \prod_{j=\mathfrak{n}+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^r \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^r \beta_{ji}^{(k)} s_k)]} \tag{1.9}$$

and $$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma(d'^{(k)}_j - \delta'^{(k)}_j s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} [\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_{i^{(k)}}} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k)]} \tag{1.10}$$

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta'^{(k)}_j$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.11}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d'^{(k)}_j - \delta'^{(k)}_j s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta'^{(k)}_j - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \cdots, r, i = 1, \cdots, R \text{ , } i^{(k)} = 1, \cdots, R^{(k)} \tag{1.12}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$

where $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \text{ ; } V = m_1, n_1; \cdots; m_r, n_r \tag{1.13}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.14}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.15}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.16}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.17}$$

$$D = \{(d_j'^{(1)}; \delta_j'^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j'^{(r)}; \delta_j'^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.18}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} z_1 & \mathrm{A} : \mathrm{C} \\ \cdot & \cdots \\ \cdot & \\ \cdot & \\ z_r & \mathrm{B} : \mathrm{D} \end{array}\right) \tag{1.19}$$

Consider the Aleph-function of s variables

$$\aleph(z_1, \cdots, z_s) = \aleph^{0,n':m_1',n_1',\cdots,m_s',n_s'}_{p_i',q_i',\iota_i;r':p_{i^{(1)}}',q_{i^{(1)}}',\iota_{i^{(1)}};r^{(1)};\cdots;p_{i^{(s)}}',q_{i^{(s)}}';\iota_{i^{(s)}};r^{(s)}}\left(\begin{array}{c|} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_s \end{array}\right.$$

$$\begin{array}{ll} [(\mathrm{u}_j; \mu_j^{(1)}, \cdots, \mu_j^{(r')})_{1,n'}] & ,[\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots, \mu_{ji}^{(r')})_{n'+1,p_i'}] : \\ \ldots\ldots\ldots\ldots & ,[\iota_i(v_{ji}; \upsilon_{ji}^{(1)}, \cdots, \upsilon_{ji}^{(r')})_{m'+1,q_i'}] : \end{array}$$

$$\left.\begin{array}{l} [(\mathrm{a}_j^{(1)}); \alpha_j^{(1)})_{1,n_1'}], [\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{n_1'+1,p_i'^{(1)}}]; \cdots; [(\mathrm{a}_j^{(s)}); \alpha_j^{(s)})_{1,n_s'}], [\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)}; \alpha_{ji^{(s)}}^{(s)})_{n_s'+1,P_i^{(s)}}] \\ [(\mathrm{b}_j^{(1)}); \beta_j^{(1)})_{1,m_1'}], [\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{m_1'+1,q_i'^{(1)}}]; \cdots; [(\mathrm{b}_j^{(s)}); \beta_j^{(s)})_{1,m_s'}], [\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)}; \beta_{ji^{(s)}}^{(s)})_{m_s'+1,Q_i^{(s)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1''}\cdots\int_{L_s''}\zeta(t_1, \cdots, t_s)\prod_{k=1}^{s}\phi_k(t_k) z_k^{t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.20}$$

with $\omega = \sqrt{-1}$

$$\zeta(t_1, \cdots, t_s) = \frac{\prod_{j=1}^{n'}\Gamma(1 - u_j + \sum_{k=1}^{s}\mu_j^{(k)} t_k)}{\sum_{i=1}^{r'}[\iota_i \prod_{j=n'+1}^{P_i}\Gamma(u_{ji} - \sum_{k=1}^{s}\mu_{ji}^{(k)} t_k)\prod_{j=1}^{q_i'}\Gamma(1 - v_{ji} + \sum_{k=1}^{s}\upsilon_{ji}^{(k)} t_k)]} \tag{1.21}$$

and $$\phi_k(t_k) = \frac{\prod_{j=1}^{m_k'}\Gamma(b_j^{(k)} - \beta_j^{(k)} t_k)\prod_{j=1}^{n_k'}\Gamma(1 - a_j^{(k)} + \alpha_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=m_k'+1}^{Q_{i^{(k)}}}\Gamma(1 - b_{ji^{(k)}}^{(k)} + \beta_{ji^{(k)}}^{(k)} t_k)\prod_{j=n_k'+1}^{P_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)} - \alpha_{ji^{(k)}}^{(k)} s_k)]} \tag{1.22}$$

Suppose , as usual , that the parameters

$u_j, j = 1, \cdots, p'; v_j, j = 1, \cdots, q';$

$a_j^{(k)}, j = 1, \cdots, n'_k; a_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p'_{i^{(k)}};$

$b_{ji^{(k)}}^{(k)}, j = m'_k + 1, \cdots, q'_{i^{(k)}}; b_j^{(k)}, j = 1, \cdots, m'_k;$

with $k = 1 \cdots, s, i = 1, \cdots, r', i^{(k)} = 1, \cdots, r^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i'^{(k)} = \sum_{j=1}^{n'} \mu_j^{(k)} + \iota_i \sum_{j=n'+1}^{p'_i} \mu_{ji}^{(k)} + \sum_{j=1}^{n'_k} \alpha_j^{(k)} + \iota_{i^{(k)}} \sum_{j=n'_k+1}^{p'_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)} - \iota_i \sum_{j=1}^{q'_i} \upsilon_{ji}^{(k)} - \sum_{j=1}^{m'_k} \beta_j^{(k)}$$

$$-\iota_{i^{(k)}} \sum_{j=m'_k+1}^{q'_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} \leqslant 0 \qquad (1.23)$$

The reals numbers τ_i are positives for $i = 1, \cdots, s$, $\iota_{i^{(k)}}$ are positives for $i^{(k)} = 1 \cdots r^{(k)}$

The contour L_k is in the t_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(b_j^{(k)} - \beta_j^{(k)} t_k)$ with $j = 1$ to m'_k are separated from those of $\Gamma(1 - u_j + \sum_{i=1}^{s} \mu_j^{(k)} t_k)$with $j = 1$ to N and $\Gamma(1 - a_j^{(k)} + \alpha_j^{(k)} t_k)$ with $j = 1$ to n'_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} B_i^{(k)} \pi$, where

$$B_i^{(k)} = \sum_{j=1}^{n'} \mu_j^{(k)} - \iota_i \sum_{j=n'+1}^{p'_i} \mu_{ji}^{(k)} - \iota_i \sum_{j=1}^{q'_i} \upsilon_{ji}^{(k)} + \sum_{j=1}^{n'_k} \alpha_j^{(k)} - \iota_{i^{(k)}} \sum_{j=n'_k+1}^{p'_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m'_k} \beta_j^{(k)} - \iota_{i^{(k)}} \sum_{j=m'_k+1}^{q'_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1, \cdots, s, i = 1, \cdots, r, i^{(k)} = 1, \cdots, r^{(k)} \qquad (1.24)$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\alpha'_1}, \cdots, |z_s|^{\alpha'_s}), max(|z_1|, \cdots, |z_s|) \rightarrow 0$

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\beta'_1}, \cdots, |z_s|^{\beta'_s}), min(|z_1|, \cdots, |z_s|) \rightarrow \infty$

where $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)} / \beta_j^{(k)})], j = 1, \cdots, m'_k$ and

$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n'_k$

We will use these following notations in this paper

$$U' = p'_i, q'_i, \iota_i; r'\,;\; V' = m'_1, n'_1; \cdots ; m'_s, n'_s \tag{1.25}$$

$$W' = p'_{i^{(1)}}, q'_{i^{(1)}}, \iota_{i^{(1)}}; r^{(1)}, \cdots, p'_{i^{(r)}}, q'_{i^{(r)}}, \iota_{i^{(s)}}; r^{(s)} \tag{1.26}$$

$$A' = \{(u_j; \mu_j^{(1)}, \cdots, \mu_j^{(s)})_{1,n'}\}, \{\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots, \mu_{ji}^{(s)})_{n'+1,p'_i}\} \tag{1.27}$$

$$B' = \{\iota_i(v_{ji}; v_{ji}^{(1)}, \cdots, v_{ji}^{(s)})_{m'+1,q'_i}\} \tag{1.28}$$

$$C' = (a_j^{(1)}; \alpha_j^{(1)})_{1,n'_1}, \iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{n'_1+1,p'_{i^{(1)}}}, \cdots, (a_j^{(s)}; \alpha_j^{(s)})_{1,n'_s}, \iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)}; \alpha_{ji^{(s)}}^{(s)})_{n'_s+1,p'_{i^{(s)}}} \tag{1.29}$$

$$D' = (b_j^{(1)}; \beta_j^{(1)})_{1,m'_1}, \iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{m'_1+1,q'_{i^{(1)}}}, \cdots, (b_j^{(s)}; \beta_j^{(s)})_{1,m'_s}, \iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)}; \beta_{ji^{(s)}}^{(s)})_{m'_s+1,q'_{i^{(s)}}} \tag{1.30}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_s) = \aleph_{U':W'}^{0,n':V'}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A'} : \mathrm{C'} \\ \cdot & \cdots \\ \cdot & \\ \mathrm{z}_s & \mathrm{B'} : \mathrm{D'} \end{array}\right) \tag{1.31}$$

Srivastava and Garg [4] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_u}[z_1, \cdots, z_u] = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} (-L)_{h_1R_1+\cdots+h_uR_u} B(E; R_1, \cdots, R_u) \frac{z_1^{R_1} \cdots z_u^{R_u}}{R_1! \cdots R_u!} \tag{1.32}$$

The coefficients are $B[E; R_1, \ldots, R_v]$ arbitrary constants, real or complex.

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [5 ,page 39 eq .30]

$$\frac{\prod_{j=1}^P \Gamma(A_j)}{\prod_{j=1}^Q \Gamma(B_j)}\, {}_PF_Q\left[(A_P); (B_Q); -(x_1 + \cdots + x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \frac{\prod_{j=1}^P \Gamma(A_j + s_1 + \cdots + s_r)}{\prod_{j=1}^Q \Gamma(B_j + s_1 + \cdots + s_r)} \Gamma(-s_1) \cdots \Gamma(-s_r) x_1^{s_1} \cdots x_r^{s_r} \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \cdots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \cdots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \cdots, r$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdot\cdot\cdot\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right) \tag{2.2}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i,\tau_j,h_j\in\mathbb{C},\lambda_j\in\mathbb{R}^+(i=1,\cdots,k;j=1,\cdots,l)$

$$min(Re(\alpha),Re(\beta))>0,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1,\ \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,$$

and $F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[3,page 454] and [5] given by :

$$F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdot\cdot\cdot\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\prod_{j=1}^{l}\Gamma(\lambda_j)\prod_{j=1}^{k}\Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}}\int_{L_1}\cdots\int_{L_{l+k}}\frac{\Gamma\left(\alpha+\sum_{j=1}^{l}h_js_j+\sum_{j=1}^{k}s_{l+j}\right)}{\Gamma\left(\alpha+\beta+\sum_{j=1}^{l}h_js_j+\sum_{j=1}^{k}s_{l+j}\right)}\prod_{j=1}^{l}\Gamma(\lambda_j+s_j)\prod_{j=1}^{k}\Gamma(-\sigma_j+s_{l+j})$$

$$\prod_{j=1}^{l+k}\Gamma(-s_j)z_1^{s_1}\cdots z_l^{s_l}z_{l+1}^{s_{l+1}}\cdots,z_{l+k}^{s_{l+k}}\,\mathrm{d}s_1\cdots\mathrm{d}s_{l+k} \tag{2.3}$$

Here the contour $L_j's$ are defined by $L_j=L_{w\zeta_j\infty}(Re(\zeta_j)=v_j'')$ starting at the point $v_j''-\omega\infty$ and terminating at the point $v_j''+\omega\infty$ with $v_j''\in\mathbb{R}(j=1,\cdots,l)$ and each of the remaining contour $L_{l+1},\cdots,L_{l+k}$ run from $-\omega\infty$ to $\omega\infty$

(2.2) can be easily established by expanding $\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [1, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [3, page 454].

3. Eulerian integral

In this section , we note :

$$\theta_i = \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}, \zeta_j^{(i)} > 0 (i=1,\cdots,r) \, ; \theta'_i = \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'^{(i)}}, \zeta_j'^{(i)} > 0 (i=1,\cdots,s)$$

$$\theta''_i = \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j''^{(i)}}, \zeta_j''^{(i)} > 0 (i=1,\cdots,u)$$

$$\theta'''_i = \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'''^{(i)}}, \zeta_j'''^{(i)} > 0 (i=1,\cdots,v) \tag{3.1}$$

$$K_2 = (1-\beta-\sum_{i=1}^{u} R_i b_i - \sum_{i=1}^{v} \eta_{G_i,g_i} b'_i ; \rho_1,\cdots,\rho_r,\rho'_1,\cdots,\rho'_s,0,\cdots,0,0\cdots,0) \tag{3.3}$$

$$K_j = [1-\lambda_j - \sum_{i=1}^{u} R_i \zeta_j''^{(i)} - \sum_{i=1}^{v} \eta_{G_i,g_i} \zeta_j'''^{(i)} ; \zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},$$

$$0,\cdots,\underset{\text{j}}{1},\cdots,0,0\cdots,0]_{1,l} \tag{3.4}$$

$$K'_j = [1+\sigma_j - \sum_{i=1}^{u} R_i \lambda_j''^{(i)} - \sum_{i=1}^{v} \eta_{G_i,g_i} \lambda_j'''^{(i)} ; \lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},$$

$$0,\cdots,0,0\cdots,\underset{\text{j}}{1},\cdots,0]_{1,k} \tag{3.5}$$

$$L_1 = (1-\alpha-\beta-\sum_{i=1}^{u} R_i(a_i+b_i) - \sum_{i=1}^{v} (a'_i+b'_i)\eta_{G_i,g_i} ; \mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu'_1+\rho'_1,\cdots,\mu'_r+\rho'_r,$$

$$h_1,\cdots,h_l,1,\cdots,1) \tag{3.6}$$

$$L_j = [1-\lambda_j - \sum_{i=1}^{u} R_i \zeta_j''^{(i)} - \sum_{i=1}^{s} \zeta_j'''^{(i)} \eta_{G_i,g_i} ; \zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,0,0\cdots,0]_{1,l} \tag{3.7}$$

$$L'_j = [1+\sigma_j - \sum_{i=1}^{u} R_i \lambda_j''^{(i)} - \sum_{i=1}^{v} \lambda_j'''^{(i)} \eta_{G_i,g_i} ; \lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0,\cdots,0]_{1,k} \tag{3.8}$$

$$P_1 = (b-a)^{\alpha+\beta-1} \left\{ \prod_{j=1}^{h} (af_j+g_j)^{\sigma_j} \right\} \tag{3.9}$$

$$B_{u,v} = (b-a)^{\sum_{i=1}^{v}(a'_i+b'_i)\eta_{G_i,g_i} + \sum_{i=1}^{u}(a_i+b_i)R_i} \left\{ \prod_{j=1}^{h} (af_j+g_j)^{-\sum_{i=1}^{v}\lambda_i'''\eta_{g_i,h_i} - \sum_{i=1}^{u}\lambda_i'' R_i} \right\} G_v \tag{3.10}$$

where $G_v = \psi(\eta_{G_1,g_1},\cdots,\eta_{G_v,g_v}) \times \xi_1(\eta_{G_1,g_1})\cdots\xi_v(\eta_{G_v,g_v})$

$\psi_1,\xi_i, i=1,\cdots,v$ are defined respectively by (1.2) and (1.3)

$$B_u = \frac{(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!} \tag{3.11}$$

$$V_1 = V;V';1,0;\cdots;1,0;1,0;\cdots;1,0 \;;W_1 = W;W';0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.12}$$

$$C_1 = C;C';(1,0),\cdots,(1,0);(1,0),\cdots,(1,0)\,;D_1 = D;D';(0,1),\cdots,(0,1);(0,1),\cdots,(0,1) \tag{3.13}$$

We have the general Eulerian integral

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c} \mathrm{z}_1''\theta_1''(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_u''\theta_u''(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j''^{(u)}} \end{array}\right)$$

$$\aleph\left(\begin{array}{c} \mathrm{z}_1'''\theta_1'''(t-a)^{a_1'}(b-t)^{b_1'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_v'''\theta_v'''(t-a)^{a_v'}(b-t)^{b_v'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'''^{(v)}} \end{array}\right)$$

$$\aleph\left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$\aleph\left(\begin{array}{c} \mathrm{z}_1'\theta_1'(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_s'\theta_s'(t-a)^{\mu_s'}(b-t)^{\rho_s'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t$$

$$= P_1\sum_{h_1=1}^{M_1}\cdots\sum_{h_v=1}^{M_v}\sum_{k_1=0}^{\infty}\cdots\sum_{k_v=0}^{\infty}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u}z''^{R_k}B_uB_{u,v}$$

$$\aleph^{0,n+n'+l+k+2:V_1}_{U;U';l+k+2,l+k+1:W_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{A}\ ;\ \mathrm{K}_1, K_2, K_j, K'_j : C_1 \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(s)}_j}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \\ \cdots & \mathrm{B}\ ;\mathrm{B'};\ \mathrm{L}_1, L_j, L'_j : D_1 \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (3.14)$$

We obtain the Aleph-function of $r+s+k+l$ variables. The quantities $A, A', B, B', C, C', C_1, D_1, V_1$ and W_1 are defined above.

(A) $a, b \in \mathbb{R}(a<b); \mu_i, \mu'_u, \rho_i, \rho'_u, \lambda_j^{(i)}, \lambda_j'^{(u)}, h_v \in \mathbb{R}^+, f_i, g_j, \tau_v, \sigma_j, \lambda_v \in \mathbb{C}\ (i=1,\cdots,r; j=1,\cdots;k;$

$u=1,\cdots,s; v=1,\cdots,l), a_i, b_i, \lambda_j''^{(i)}, \zeta_j''^{(i)} \in \mathbb{R}^+, (i=1,\cdots,u; j=1,\cdots,k)$

$a'_i, b'_i, \lambda_j'''^{(i)}, \zeta_j'''^{(i)} \in \mathbb{R}^+, (i=1,\cdots,v; j=1,\cdots,k)$

(B) See the section 1

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1$

(D) $Re\left[\alpha+\sum_{j=1}^{v}a'_j \min_{1\leqslant k\leqslant M_i}\frac{d_k^{(j)}}{\delta_k^{(j)}}+\sum_{j=1}^{r}\mu_j \min_{1\leqslant k\leqslant m_i}\frac{d_k'^{(j)}}{\delta_k'^{(j)}}+\sum_{j=1}^{s}\mu'_i \min_{1\leqslant k\leqslant m'_i}\frac{b_k^{(j)}}{\beta_k^{(j)}}\right]>0$

$$Re\left[\beta+\sum_{j=1}^{v}b'_j \min_{1\leqslant k\leqslant M_i}\frac{d_k^{(j)}}{\delta_k^{(j)}}+\sum_{j=1}^{r}\rho_j \min_{1\leqslant k\leqslant m_i}\frac{d_k'^{(j)}}{\delta_k'^{(j)}}+\sum_{j=1}^{s}\rho'_j \min_{1\leqslant k\leqslant m'_i}\frac{b_k^{(j)}}{\beta_k^{(j)}}\right]>0$$

(E) $Re\left(\alpha+\sum_{i=1}^{v}\eta_{G_i,g_i}a'_i+\sum_{i=1}^{u}R_ia_i+\sum_{i=1}^{r}\mu_is_i+\sum_{i=1}^{s}t_i\mu'_i\right)>0$

$$Re\left(\beta+\sum_{i=1}^{v}\eta_{G_i,g_i}b'_i+\sum_{i=1}^{u}R_ib_i+\sum_{i=1}^{r}v_is_i+\sum_{i=1}^{s}t_i\rho'_i\right)>0$$

$$Re\left(\lambda_j+\sum_{i=1}^{v}\eta_{G_i,g_i}\lambda_j'''^{(i)}+\sum_{i=1}^{u}R_i\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\zeta_j^{(i)}+\sum_{i=1}^{s}t_i\zeta_j'^{(i)}\right)>0(j=1,\cdots,l);$$

$$Re\left(-\sigma_j+\sum_{i=1}^{v}\eta_{G_i,g_i}\lambda'''^{(i)}+\sum_{i=1}^{u}R_i\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\lambda_j^{(i)}+\sum_{i=1}^{s}t_i\lambda_j'^{(i)}\right)>0(j=1,\cdots,k);$$

(F) $U_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}+\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}+\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}-\sum_{j=1}^{m_k}\delta_j'^{(k)}$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}\leqslant 0$$

$$U_i'^{(k)}=\sum_{j=1}^{n'}\mu_j^{(k)}+\iota_i\sum_{j=n'+1}^{p'_i}\mu_{ji}^{(k)}+\sum_{j=1}^{n'_k}\alpha_j^{(k)}+\iota_{i^{(k)}}\sum_{j=n'_k+1}^{p'_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)}-\iota_i\sum_{j=1}^{q'_i}\upsilon_{ji}^{(k)}-\sum_{j=1}^{m'_k}\beta_j^{(k)}$$

$$-\iota_{i^{(k)}}\sum_{j=m'_k+1}^{q'_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)}\leqslant 0$$

(G) $A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$

$+\sum_{j=1}^{m_k}\delta_j'^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}-\sum_{l=1}^{k}\lambda_j^{(i)}-\sum_{l=1}^{l}\zeta_j^{(i)}-\mu_k-\rho_k>0,$ with $k=1\cdots,r,$

$i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)}$

$B_i^{(k)}=\sum_{j=1}^{n'}\mu_j^{(k)}-\iota_i\sum_{j=n'+1}^{p'_i}\mu_{ji}^{(k)}-\iota_i\sum_{j=1}^{q'_i}\upsilon_{ji}^{(k)}+\sum_{j=1}^{n'_k}\alpha_j^{(k)}-\iota_{i^{(k)}}\sum_{j=n'_k+1}^{p'_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)}$

$+\sum_{j=1}^{m'_k}\beta_j^{(k)}-\iota_{i^{(k)}}\sum_{j=m'_k+1}^{q'_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)}-\sum_{l=1}^{k}\lambda_j'^{(i)}-\sum_{l=1}^{l}\zeta_j'^{(i)}-\mu'_k-\rho'_k>0,$ with $k=1,\cdots,s,$

$i=1,\cdots,r\ ,i^{(k)}=1,\cdots,r^{(k)}$

(H) $$\left|arg\left(z_i\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right|<\frac{1}{2}A_i^{(k)}\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,r)$$

$$\left|arg\left(z_i'\prod_{j=1}^{l}\left[1-\tau_j'(t-a)^{h_i'}\right]^{-\zeta_j'^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right|<\frac{1}{2}B_i^{(k)}\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,s)$$

(I) The multiple series occuring on the right-hand side of (3.14) is absolutely and uniformly convergent.

Proof

To prove (3.14), first, we express in serie the multivariable Aleph-function with the help of (1.6), a class of multivariable polynomials defined by Srivastava et al [4] $S_L^{h_1,\cdots,h_u}[.]$ in serie with the help of (1.32) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the Aleph-functions of r-variables and s-variables in terms of Mellin-Barnes type contour integral with the help of (1.8) and (1.20) respectively and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process . Now collect the power of $\left[1-\tau_j(t-a)^{h_i}\right]$ with $(i=1,\cdots,r;j=1,\cdots,l)$ and collect the power of (f_jt+g_j) with $j=1,\cdots,k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral in multivariable Aleph-function ,we obtain the equation (3.14).

Remarks

If a) $\rho_1=\cdots,\rho_r=\rho_1'=\cdots,\rho_s'=0$; b) $\mu_1=\cdots,\mu_r=\mu_1'=\cdots,\mu_s'=0$, we obtain the similar formulas that (3.14) with the corresponding simplifications.

4. Particular cases

a) If $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}},\iota_i,\iota_{i^{(1)}},\cdots,\iota_{i^{(s)}}\rightarrow 1$, the multivariable Aleph-functions of r and s-variables reduces to multivariable I-functions of r and s-variables defined by Sharma and al [2] respectively and we have

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{z}_1''\theta_1''(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_u''\theta_u''(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}}\end{pmatrix}$$

$$\aleph\begin{pmatrix}\mathrm{z}_1'''\theta_1'''(t-a)^{a_1'}(b-t)^{b_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_v'''\theta_v'''(t-a)^{a_v'}(b-t)^{b_v'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(v)}}\end{pmatrix}$$

$$I\left(\begin{array}{c} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$I\left(\begin{array}{c} z_1'\theta_1'(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_s'\theta_s'(t-a)^{\mu_s'}(b-t)^{\rho_s'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t$$

$$= P_1\sum_{h_1=1}^{M_1}\cdots\sum_{h_v=1}^{M_v}\sum_{k_1=0}^{\infty}\cdots\sum_{k_v=0}^{\infty}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u}z''^{R_k}B_uB_{u,v}$$

$$I_{U;U';l+k+2,l+k+1:W_1}^{0,n+n'+l+k+2:V_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_j,K_j':C_1 \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \mathrm{B}\ ;\mathrm{B}';\ \mathrm{L}_1,L_j,L_j':D_1 \end{array}\right) \qquad (4.1)$$

under the same conditions and notations that (3.14) with $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}},\iota_i,\iota_{i^{(1)}},\cdots,\iota_{i^{(s)}}\rightarrow 1$

b) If $B(L;R_1,\cdots,R_u)=\frac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(u)}}(b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{R_1\psi'_j+\cdots+R_u\psi_j^{(u)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(u)}}(d_j^{(u)})_{R_u\delta_j^{(u)}}}$ (4.2)

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]$ reduces to generalized Lauricella function defined by Srivastava et al [3]. We have

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c} z''_1\theta''_1(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z''_u\theta''_u(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}} \end{array}\right.$$

$$\left.\begin{array}{c} [(\text{-L});\text{R}_1,\cdots,R_u][(a);\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{(u)});\phi^{(u)}] \\ [(\text{c});\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)});\delta^{(u)}] \end{array}\right)$$

$$\aleph\left(\begin{array}{c} z'''_1\theta'''_1(t-a)^{a'_1}(b-t)^{b'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z'''_v\theta'''_v(t-a)^{a'_v}(b-t)^{b'_v}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(v)}} \end{array}\right)$$

$$\aleph\left(\begin{array}{c} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$\aleph\left(\begin{array}{c} z'_1\theta'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z'_s\theta'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\text{d}t$$

$$= P_1 \sum_{h_1=1}^{M_1} \cdots \sum_{h_v=1}^{M_v} \sum_{k_1=0}^{\infty} \cdots \sum_{k_v=0}^{\infty} \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} \prod_{i=1}^{v} z_i^{\prime\prime\prime \eta_{h_i,k_i}} \prod_{k=1}^{u} z^{\prime\prime R_k} B_u^{\prime} B_{u,v}$$

$$\aleph_{U;U';l+k+2,l+k+1:W_1}^{0,n+n'+l+k+2:V_1} \left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A} ; \mathrm{A} ; \mathrm{K}_1, K_2, K_j, K_j' : C_1 \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdot\cdot\cdot & \mathrm{B} ; \mathrm{B'}; \mathrm{L}_1, L_j, L_j' : D_1 \\ \cdot\cdot\cdot & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array} \right) \quad (4.3)$$

under the same notations and conditions that (3.14)

where $B_u^{\prime} = \dfrac{(-L)_{h_1R_1+\cdots+h_uR_u} B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!}$, $B[E;R_1,\ldots,R_v]$ is defined by (4.2)

Remark:

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable Aleph-functions and a class of multivariable polynomials defined by Srivastava et al [4].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of three multivariable Aleph-function and a class of multivariable polynomials defined by Srivastava et al [4] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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