On general Eulerian integral of certain products of A-functions and a multivariable Aleph-function

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ABSTRACT
The object of this paper is to establish an general Eulerian integral involving the product of two multivariable A-functions defined by Gautam et al [1], the multivariable Aleph-function and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular case concerning the multivariable H-function.

Keywords: Eulerian integral, multivariable A-function, Lauricella function of several variables, multivariable H-function, generalized hypergeometric function.

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1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable A-functions defined by Gautam et al [1], the Aleph-function of several variables and a generalized hypergeometric function with general arguments. The Aleph-function of several variables is an extension of multivariable I-function defined by Sharma et al [3].

We define : $\aleph(z_1''',\cdots,z_v''') = \aleph^{0,N:M_1,N_1,\cdots,M_v,N_v}_{P_i,Q_i,\tau_i;R:P_{i^{(1)}},Q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;P_{i^{(r)}},Q_{i^{(v)}};\tau_{i^{(v)}};R^{(v)}}\left(\begin{array}{c} z'''_1 \\ \cdot \\ \cdot \\ \cdot \\ z'''_v \end{array}\right|$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(v)})_{1,\mathfrak{n}}] \;,[\tau_i'(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(v)})_{\mathfrak{n}+1,p_i}]:$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;,[\tau_i'(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(v)})_{m+1,q_i}]:$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,n_1}],[\tau'_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ [(\mathrm{c}_j^{(v)});\gamma_j^{(v)})_{1,n_v}],[\tau'_{i^{(v)}}(c_{ji^{(v)}}^{(v)};\gamma_{ji^{(v)}}^{(v)})_{n_v+1,p_i^{(v)}}]\\ [(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,m_1}],[\tau'_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;\ [(\mathrm{d}_j^{(v)});\delta_j^{(v)})_{1,m_v}],[\tau'_{i^{(v)}}(d_{ji^{(v)}}^{(v)};\delta_{ji^{(v)}}^{(v)})_{m_v+1,q_i^{(v)}}]\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^v}\int_{L_1}\cdots\int_{L_v}\psi_1(s_1,\cdots,s_v)\prod_{k=1}^{v}\xi_k(s_k)z_k'''^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi_1(s_1,\cdots,s_v) = \frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{k=1}^{v}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i'\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\sum_{k=1}^{v}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{Q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{v}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\xi_k(s_k) = \frac{\prod_{j=1}^{M_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{N_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau'_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, P; b_j, j = 1, \cdots, Q;$

$c_j^{(k)}, j = 1, \cdots, N_k; c_{ji^{(k)}}^{(k)}, j = N_k + 1, \cdots, P_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, M_k; d_{ji^{(k)}}^{(k)}, j = M_k + 1, \cdots, Q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R\ , i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} + \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} + \tau'_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{M_k} \delta_j^{(k)}$$

$$-\tau'_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to M_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k'''| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} - \tau'_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau'_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} - \tau'_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau'_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1, \cdots, r, i = 1, \cdots, R\ , i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1''', \cdots, z_v''') = 0(|z_1'''|^{\alpha_1}, \cdots, |z_r'''|^{\alpha_r}), max(|z_1'''|, \cdots, |z_v'''|) \rightarrow 0$

$\aleph(z_1''', \cdots, z_v''') = 0(|z_1'''|^{\beta_1}, \cdots, |z_v'''|^{\beta_r}), min(|z_1'''|, \cdots, |z_v'''|) \rightarrow \infty$

where $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k$ and

$\beta_k = max[Re((c_j^{(k)} - 1) / \gamma_j^{(k)})], j = 1, \cdots, n_k$

Serie representation of Aleph-function of $u-$variables is given by

$$\aleph(z_1''',\cdots,z_v''') = \sum_{G_1,\cdots,G_v=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_v=0}^{M_v}\frac{(-)^{G_1+\cdots+G_v}}{\delta_{g_1}G_1!\cdots\delta_{g_v}G_v!}\psi_1(\eta_{G_1,g_1},\cdots,\eta_{G_v,g_v})$$

$$\times\ \xi_1(\eta_{G_1,g_1})\cdots\xi_v(\eta_{G_v,g_v})z_1^{-\eta_{G_1,g_1}}\cdots z_v^{-\eta_{G_v,g_v}} \tag{1.6}$$

Where $\psi(.,\cdots,.)$, $\theta_i(.)$,$i=1,\cdots,r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1}=\frac{d_{g_1}^{(1)}+G_1}{\delta_{g_1}^{(1)}},\cdots,\ \eta_{G_v,g_v}=\frac{d_{g_v}^{(v)}+G_v}{\delta_{g_v}^{(v)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i+p_i]\neq\delta_j^{(i)}[d_{g_i}^i+G_i]$ (1.7)
for $j\neq m_i, m_i=1,\cdots\eta_{G_i,g_i}; p_i,n_i=0,1,2,\cdots,; y_i\neq 0, i=1,\cdots,v$

The A-function is defined and represented in the following manner.

$$A(z_1',\cdots,z_s') = A_{p',q':p_1',q_1';\cdots;p_s',q_s'}^{m',n':m_1',n_1';\cdots;m_s',n_s'}\left(\begin{array}{c|c} \text{z'}_1 & (\text{a'}_j;A_j'^{(1)},\cdots,A_j'^{(s)})_{1,p'}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \text{z'}_s & (\text{b'}_j;B_j'^{(1)},\cdots,B_j'^{(s)})_{1,q'}: \end{array}\right.$$

$$\left.\begin{array}{c} (\text{c'}_j^{(1)},C_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(s)},C_j'^{(s)})_{1,p_s'} \\ \\ (\text{d'}_j^{(1)},D_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(s)},D_j'^{(s)})_{1,q_s'} \end{array}\right) \tag{1.8}$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\zeta_i(t_i)z_i'^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.9}$$

where $\phi(t_1,\cdots,t_s)$, $\zeta_i(t_i)$, $i=1,\cdots,s$ are given by :

$$\phi(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{m'}\Gamma(b_j'-\sum_{i=1}^{s}B_j'^{(i)}t_i)\prod_{j=1}^{n'}\Gamma(1-a_j'+\sum_{i=1}^{s}A_j'^{(i)}t_j)}{\prod_{j=n'+1}^{p'}\Gamma(a_j'-\sum_{i=1}^{s}A_j^{(i)}t_j)\prod_{j=m'+1}^{q'}\Gamma(1-b_j'+\sum_{i=1}^{s}B_j'^{(i)}t_j)} \tag{1.10}$$

and

$$\zeta_i(t_i)=\frac{\prod_{j=1}^{n_i'}\Gamma(1-c_j'^{(i)}+C_j'^{(i)}t_i)\prod_{j=1}^{m_i'}\Gamma(d_j'^{(i)}-D_j'^{(i)}t_i)}{\prod_{j=n_i'+1}^{p_i'}\Gamma(c_j'^{(i)}-C_j'^{(i)}t_i)\prod_{j=m_i'+1}^{q_i'}\Gamma(1-d_j'^{(i)}+D_j'^{(i)}t_i)} \tag{1.11}$$

Here $m',n',p',m_i',n_i',p_i',c_i'\in\mathbb{N}^*; i=1,\cdots,s$; $a_j',b_j',c_j'^{(i)},d_j'^{(i)},A_j'^{(i)},B_j'^{(i)},C_j'^{(i)},D_j'^{(i)}\in\mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i)z_k'|<\frac{1}{2}\eta_k\pi, \xi^*=0, \eta_i>0 \tag{1.12}$$

$$\Omega_i = \prod_{j=1}^{p'}\{A_j'^{(i)}\}^{A_j'^{(i)}}\prod_{j=1}^{q'}\{B_j'^{(i)}\}^{-B_j'^{(i)}}\prod_{j=1}^{q_i'}\{D_j'^{(i)}\}^{D_j'^{(i)}}\prod_{j=1}^{p_i'}\{C_j'^{(i)}\}^{-C_j'^{(i)}} ; i=1,\cdots,s \tag{1.13}$$

$$\xi_i^* = Im(\sum_{j=1}^{p'}A_j'^{(i)} - \sum_{j=1}^{q'}B_j'^{(i)} + \sum_{j=1}^{q_i'}D_j'^{(i)} - \sum_{j=1}^{p_i'}C_j'^{(i)}); i=1,\cdots,s \tag{1.14}$$

$$\eta_i = Re\left(\sum_{j=1}^{n'}A_j'^{(i)} - \sum_{j=n'+1}^{p'}A_j'^{(i)} + \sum_{j=1}^{m'}B_j'^{(i)} - \sum_{j=m'+1}^{q'}B_j'^{(i)} + \sum_{j=1}^{m_i'}D_j'^{(i)} - \sum_{j=m_i'+1}^{q_i'}D_j'^{(i)} + \sum_{j=1}^{n_i'}C_j'^{(i)} - \sum_{j=n_i'+1}^{p_i}C_j'^{(i)}\right)$$

$$i=1,\cdots,s \tag{1.15}$$

Consider the second multivariable A-function.

$$A(z_1'',\cdots,z_u'') = A_{p'',q'':p_1'',q_1'';\cdots;p_u'',q_u''}^{m'',n'':m_1'',n_1'';\cdots;m_u'',n_u''}\left(\begin{array}{c|c} z''_1 & (a''_j;A_j''^{(1)},\cdots,A_j''^{(u)})_{1,p''}: \\ \vdots & \\ z''_u & (b''_j;B_j''^{(1)},\cdots,B_j''^{(u)})_{1,q''}: \end{array}\right.$$

$$\left.\begin{array}{c} (c''^{(1)}_j,C_j''^{(1)})_{1,p_1''};\cdots;(c_j''^{(u)},C_j''^{(u)})_{1,p_u''} \\ (d''^{(1)}_j,D_j''^{(1)})_{1,q_1''};\cdots;(d_j''^{(u)},D_j''^{(u)})_{1,q_u''} \end{array}\right) \tag{1.16}$$

$$= \frac{1}{(2\pi\omega)^u}\int_{L_1''}\cdots\int_{L_u''}\phi'(x_1,\cdots,x_u)\prod_{i=1}^{u}\theta_i'(x_i)z_i''^{x_i}\mathrm{d}x_1\cdots\mathrm{d}x_r \tag{1.17}$$

where $\phi'(x_1,\cdots,x_u),\theta_i'(x_i), i=1,\cdots,u$ are given by :

$$\phi'(x_1,\cdots,x_u) = \frac{\prod_{j=1}^{m''}\Gamma(b_j''-\sum_{i=1}^{u}B_j''^{(i)}x_i)\prod_{j=1}^{n''}\Gamma(1-a_j''+\sum_{i=1}^{u}A_j''^{(i)}x_j)}{\prod_{j=n''+1}^{p''}\Gamma(a_j''-\sum_{i=1}^{u}A_j''^{(i)}x_j)\prod_{j=m''+1}^{q''}\Gamma\left(1-b_j''+\sum_{i=1}^{u}B_j''^{(i)}x_j\right)} \tag{1.18}$$

and

$$\theta_i'(x_i) = \frac{\prod_{j=1}^{n_i''}\Gamma(1-c_j''^{(i)}+C_j''^{(i)}x_i)\prod_{j=1}^{m_i''}\Gamma(d_j''^{(i)}-D_j''^{(i)}x_i)}{\prod_{j=n_i''+1}^{p_i''}\Gamma(c_j''^{(i)}-C_j''^{(i)}x_i)\prod_{j=m_i''+1}^{q_i''}\Gamma(1-d_j''^{(i)}+D_j''^{(i)}x_i)} \tag{1.19}$$

Here $m'',n'',p'',m_i'',n_i'',p_i'',c_i''\in\mathbb{N}^*; i=1,\cdots,u$; $a_j'',b_j'',c_j''^{(i)},d_j''^{(i)},A_j''^{(i)},B_j''^{(i)},C_j''^{(i)},D_j''^{(i)}\in\mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i')z_k''| < \frac{1}{2}\eta_k'\pi, \xi'^* = 0, \eta_i' > 0 \tag{1.20}$$

$$\Omega_i' = \prod_{j=1}^{p''}\{A_j''^{(i)}\}^{A_j''^{(i)}}\prod_{j=1}^{q''}\{B_j''^{(i)}\}^{-B_j''^{(i)}}\prod_{j=1}^{q_i''}\{D_j''^{(i)}\}^{D_j''^{(i)}}\prod_{j=1}^{p_i''}\{C_j''^{(i)}\}^{-C_j''^{(i)}}; i = 1, \cdots, u \tag{1.21}$$

$$\xi_i'^* = Im(\sum_{j=1}^{p''} A_j''^{(i)} - \sum_{j=1}^{q''} B_j''^{(i)} + \sum_{j=1}^{q_i''} D_j''^{(i)} - \sum_{j=1}^{p_i''} C_j''^{(i)}); i = 1, \cdots, u \tag{1.22}$$

$$\eta_i' = Re\left(\sum_{j=1}^{n''} A_j''^{(i)} - \sum_{j=n''+1}^{p''} A_j''^{(i)} + \sum_{j=1}^{m''} B_j''^{(i)} - \sum_{j=m''+1}^{q''} B_j''^{(i)} + \sum_{j=1}^{m_i''} D_j''^{(i)} - \sum_{j=m_i''+1}^{q_i''} D_j''^{(i)} + \sum_{j=1}^{n_i''} C_j''^{(i)} - \sum_{j=n_i''+1}^{p_i''} C_j''^{(i)}\right)$$

$$i = 1, \cdots, u \tag{1.23}$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [3 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \cdots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \cdots, r$

The Lauricella function $F_D^{(k)}$ is defined as

$$F_D^{(k)}\left[a, b_1, \cdots, b_k; c; x_1, \cdots, x_k\right] = \frac{\Gamma(c)}{\Gamma(a)\prod_{j=1}^{k}\Gamma(b_j)}\frac{1}{(2\pi\omega)^k}\int_{L_1}\cdots\int_{L_k}\frac{\Gamma\left(a+\sum_{j=1}^{k}\zeta_j\right)\Gamma(b_1+\zeta_1),\cdots,\Gamma(b_k+\zeta_k)}{\Gamma\left(c+\sum_{j=1}^{k}\zeta_j\right)}$$

$$\prod_{j=1}^{k}\Gamma(-\zeta_j)(-x_j)^{\zeta_i}\,\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_k \tag{2.2}$$

where $max\left[|arg(-x_1)|, \cdots, |arg(-x_k)|\right] < \pi, c \neq 0, -1, -2, \cdots$.

We first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha, -\sigma_1, \cdots, -\sigma_k; \alpha+\beta; -\frac{(b-a)f_1}{af_1+g_1}, \cdots, -\frac{(b-a)f_k}{af_k+g_k}\right] \tag{2.3}$$

where $a,b \in \mathbb{R}(a<b), \alpha,\beta,f_i,g_i,\sigma_i \in \mathbb{C},(i=1,\cdots,k)$; $min(Re(\alpha),Re(\beta))>0$ and

$$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava et al ([5], page 60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [4, page 454].

3. Eulerian integral

Let

$$X=m'_1,n'_1;\cdots;m'_s,n'_s;m''_1,n''_1;\cdots;m''_u,n''_u;1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{3.1}$$

$$Y=p'_1,q'_1;\cdots;p'_s,q'_s;p''_1,q''_1;\cdots;p''_u,q''_u;0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.2}$$

$$A=(a'_j;A'^{(1)}_j,\cdots,A'^{(s)}_j,0,\cdots,0,0,\cdots,0,0,\cdots,0)_{1,p'} \tag{3.3}$$

$$B=(b'_j;B'^{(1)}_j,\cdots,B'^{(s)}_j,0\cdots,0,0\cdots,0,0,\cdots,0)_{1,q'} \tag{3.4}$$

$$A'=(a''_j;0,\cdots,0,A''^{(1)}_j,\cdots,A''^{(u)}_j,0,\cdots,0,0,\cdots,0)_{1,p''} \tag{3.5}$$

$$B'=(b''_j;0,\cdots,0,B''^{(1)}_j,\cdots,B''^{(u)}_j,0,\cdots,0,0,\cdots,0)_{1,q''} \tag{3.6}$$

$$\mathrm{C}=(\mathrm{c}'^{(1)}_j,C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(s)}_j,C'^{(s)}_j)_{1,p'_s};(c''^{(1)}_j,C''^{(1)}_j)_{1,p''_1};\cdots;(c''^{(u)}_j,C''^{(u)}_j)_{1,p''_u}$$

$$(1,0);\cdots;(1,0);(1,0);\cdots;(1,0) \tag{3.7}$$

$$D=(\mathrm{d}'^{(1)}_j,D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(s)}_j,D'^{(s)}_j)_{1,q'_s};(\mathrm{d}''^{(1)}_j,D''^{(1)}_j)_{1,q''_1};\cdots;(d''^{(u)}_j,D''^{(u)}_j)_{1,q''_u};$$

$$(0,1);\cdots;(0,1);(0,1);\cdots;(0,1) \tag{3.8}$$

$$K_1=(1-\alpha-\sum_{i=1}^{v}\eta_{G_i,g_i}(\mu_i+\mu'_i);\mu_1,\cdots,\mu_s,\mu'_1,\cdots,\mu'_u,1,\cdots,1,\upsilon_1,\cdots,\upsilon_l) \tag{3.9}$$

$$K_2 = (1-\beta-\sum_{i=1}^{v}\eta_{G_i,g_i}(\rho_i+\rho_i'); \rho_1,\cdots,\rho_s,\rho_1',\cdots,\rho_u',0,\cdots,0,\tau_1,\cdots,\tau_l) \tag{3.10}$$

$$K_P = [1-A_j; 0,\cdots,0,0,\cdots,0,0,\cdots,0,1,\cdots,1]_{1,P} \tag{3.11}$$

$$K_j = [1+\sigma_j-\sum_{i=1}^{v}\eta_{G_i,g_i}(\lambda_i^{(j)}+\lambda_i'^{(j)}); \lambda_j^{(1)},\cdots,\lambda_j^{(s)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(u)},0\cdots,1,\cdots,0,\zeta_j',\cdots,\zeta_j^{(l)}]_{1,k} \tag{3.12}$$

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$$L_1 = (1-\alpha-\beta-\sum_{i=1}^{v}\eta_{G_i,g_i}(\mu_i+\mu_i'+\rho_i+\rho_i'); \mu_1+\rho_1,\cdots,\mu_s+\rho_s,\mu_1'+\rho_1',\cdots,\mu_u'+\rho_u',$$

$$1,\cdots,1,\upsilon_1+\tau_1,\cdots,\upsilon_l+\tau_l) \tag{3.13}$$

$$L_Q = [1-B_j; 0,\cdots,0,0,\cdots,0,0,\cdots,0,1\cdots,1]_{1,Q} \tag{3.14}$$

$$L_j = [1+\sigma_j-\sum_{i=1}^{v}\eta_{G_i,g_i}(\lambda_i^{(j)}+\lambda_i'^{(j)}); \lambda_j^{(1)},\cdots,\lambda_j^{(s)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(u)},0\cdots,0,\zeta_j',\cdots,\zeta_j^{(l)}]_{1,k} \tag{3.15}$$

$$A_1 = A, A'; B_1 = B, B' \tag{3.16}$$

We have the following result

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\aleph\begin{pmatrix} \text{z"'}_1(t-a)^{\mu_1+\mu_1'}(b-t)^{\rho_1+\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z"'}_v(t-a)^{\mu_v+\mu_v'}(b-t)^{\rho_v+\rho_v'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(v)}-\lambda_j'^{(v)}} \end{pmatrix}$$

$$A\begin{pmatrix} \text{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}} \end{pmatrix}$$

$$A\begin{pmatrix} \text{z'}_1(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z'}_u(t-a)^{\mu_u'}(b-t)^{\rho_u'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}} \end{pmatrix}$$

$$ {}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l} z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t = (b-a)^{\alpha+\beta-1} $$

$$ = P_1\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\sum_{h_1=1}^{M_1}\cdots\sum_{h_v=1}^{M_v}\sum_{k_1=0}^{\infty}\cdots\sum_{k_v=0}^{\infty}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v} z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u} z''^{R_k}B_uB_{u,v} $$

$$ A_{p'+p''+l+k+2,q'+q''+l+k+1;Y}^{m'+m'',n'+n''+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}_1,K_1,K_2,K_P,K_j:C \\ \cdot\;\cdot\;\cdot & \cdot \\ \cdot\;\cdot\;\cdot & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdot\;\cdot\;\cdot & \cdot \\ \cdot\;\cdot\;\cdot & \cdot \\ \frac{z_u'(b-a)^{\mu_u'+\rho_u'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(u)}}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdot\;\cdot\;\cdot & \cdot \\ \cdot\;\cdot\;\cdot & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z_1''(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & \cdot \\ \cdot\;\cdot\;\cdot & \cdot \\ \cdot\;\cdot\;\cdot & \\ \frac{z_l''(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \mathrm{B}_1,L_1,L_j,L_Q,:D \end{array}\right) \quad (3.17) $$

We obtain the A-function of $s+u+k+l$ variables.

Where $P_1=(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{h}(af_j+g_j)^{\sigma_j}\right\}$ (3.18)

$$ B_{u,v}=(b-a)^{\sum_{i=1}^{v}(\mu_i+\mu_i'+\rho_i+\rho_i')\eta_{G_i,g_i}}\left\{\prod_{j=1}^{h}(af_j+g_j)^{-\sum_{i=1}^{v}(\lambda_i+\lambda_i')\eta_{g_i,h_i}}\right\}G_v \quad (3.19) $$

where $G_v = \psi(\eta_{G_1,g_1},\cdots,\eta_{G_v,g_v}) \times \ \xi_1(\eta_{G_1,g_1})\cdots\xi_v(\eta_{G_v,g_v})$

$\psi_1, \xi_i, i=1,\cdots,v$ are defined respectively by (1.2) and (1.3)

$$B_u = \frac{(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!} \tag{3.20}$$

Provided that

(A) $m',n',p',m'_i,n'_i,p'_i,c'_i \in \mathbb{N}^*; i=1,\cdots,s$ **;** $a'_j,b'_j,c'^{(i)}_j,d'^{(i)}_j,A'^{(i)}_j,B'^{(i)}_j,C'^{(i)}_j,D'^{(i)}_j \in \mathbb{C}$

$m'',n'',p'',m''_i,n''_i,p''_i,c''_i \in \mathbb{N}^*; i=1,\cdots,u$ **;** $a''_j,b''_j,c''^{(i)}_j,d''^{(i)}_j,A''^{(i)}_j,B''^{(i)}_j,C''^{(i)}_j,D''^{(i)}_j \in \mathbb{C}$

(B) (A) $a,b \in \mathbb{R}(a<b); \mu_i,\rho_i,\mu'_j,\rho'_j\lambda^{(i)}_v;\lambda'^{(j)}_v \in \mathbb{R}^+$, $f_i,g_j,\tau_v,\sigma_j \in \mathbb{C}$ $(i=1,\cdots,r;j=1,\cdots;s;v=1,\cdots,k)$

$$(C) \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\} < 1$$

(D) $$Re\big[\alpha+\sum_{i=1}^{s}\mu_i \min_{1\leqslant j\leqslant m'_i}\frac{d'^{(i)}_j}{D'^{(i)}_j}+\sum_{i=1}^{u}\mu'_i \min_{1\leqslant j\leqslant m''_i}\frac{d''^{(i)}_j}{D''^{(i)}_j}\big]>0$$

$$Re\big[\beta+\sum_{i=1}^{s}\rho_i \min_{1\leqslant j\leqslant m'_i}\frac{d'^{(i)}_j}{D'^{(i)}_j}+\sum_{i=1}^{u}\rho'_i \min_{1\leqslant j\leqslant m''_i}\frac{d''^{(i)}_j}{D''^{(i)}_j}\big]>0$$

(E) $$\xi^*_i = Im\left(\sum_{j=1}^{p'}A'^{(i)}_j-\sum_{j=1}^{q'}B'^{(i)}_j+\sum_{j=1}^{q'_i}D'^{(i)}_j-\sum_{j=1}^{p'_i}C'^{(i)}_j\right)=0; i=1,\cdots,s$$

$$\xi'^*_i = Im\left(\sum_{j=1}^{p''}A''^{(i)}_j-\sum_{j=1}^{q''}B''^{(i)}_j+\sum_{j=1}^{q''_i}D''^{(i)}_j-\sum_{j=1}^{p''_i}C''^{(i)}_j\right)=0; i=1,\cdots,u$$

(F) $|arg(\Omega_i)z_k| < \frac{1}{2}\eta_k\pi, \xi^*=0, \eta_i>0$

$$Re\left(\sum_{j=1}^{n}A'^{(i)}_j-\sum_{j=n+1}^{p}A'^{(i)}_j+\sum_{j=1}^{m}B'^{(i)}_j-\sum_{j=m+1}^{q}B'^{(i)}_j+\sum_{j=1}^{m_i}D'^{(i)}_j-\sum_{j=m_i+1}^{q_i}D'^{(i)}_j+\sum_{j=1}^{n_i}C'^{(i)}_j-\sum_{j=n_i+1}^{p_i}C'^{(i)}_j\right)$$

$$-\mu'_i-\rho'_i-\sum_{l=1}^{k}\lambda'^{(i)}_j>0; i=1,\cdots,s$$

$$|arg(\Omega'_i)z'_k| < \frac{1}{2}\eta'_k\pi, \xi'^*=0, \eta'_i>0$$

$$Re\left(\sum_{j=1}^{n''}A''^{(i)}_j-\sum_{j=n''+1}^{p''}A''^{(i)}_j+\sum_{j=1}^{m''}B''^{(i)}_j-\sum_{j=m''+1}^{q''}B''^{(i)}_j+\sum_{j=1}^{m''_i}D''^{(i)}_j-\sum_{j=m''_i+1}^{q''_i}D''^{(i)}_j+\sum_{j=1}^{n''_i}C''^{(i)}_j-\sum_{j=n''_i+1}^{p''_i}C''^{(i)}_j\right)$$

$$-\mu_i' - \rho_i' - \sum_{l=1}^{k} \lambda_j'^{(i)} > 0 \text{ ; } i = 1, \cdots, u$$

(G) $\left| arg \left(z_i \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right| < \frac{1}{2} \eta_i \pi \quad (a \leqslant t \leqslant b; i = 1, \cdots, s)$

$$\left| arg \left(z_i' \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(i)}} \right) \right| < \frac{1}{2} \eta_i' \pi \quad (a \leqslant t \leqslant b; i = 1, \cdots, u)$$

(H) $P \leqslant Q + 1$. The equality holds, when , in addition,

either $P > Q$ and $\left| z_i'' \left(\prod_{j=1}^{k} (f_j t + g_j)^{-\zeta_j^{(i)}} \right) \right|^{\frac{1}{Q-P}} < 1 \qquad (a \leqslant t \leqslant b)$

or $P \leqslant Q$ and $\max_{1 \leqslant i \leqslant k} \left[\left| \left(z_i'' \prod_{j=1}^{k} (f_j t + g_j)^{-\zeta_j^{(i)}} \right) \right| \right] < 1 \quad (a \leqslant t \leqslant b)$

(I) The multiple series occuring on the right-hand side of (3.17) is absolutely and uniformly convergent.

Proof

First expressing the multivariable Aleph-function in serie with the help of (1.6) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the A-function of s-variables and u-variables defined by Gautam et al [1] by the Mellin-Barnes contour integral with the help of the equation (1.9) and (1.17) respectively, the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. . Now collect the power of $(f_j t + g_j)$ with $j = 1, \cdots, k$ and use the equations (2.1) and (2.2) and we obtain $k-$Mellin-Barnes contour integral. Interpreting $(r + s + k + l)-$Mellin-barnes contour integral in multivariable A-function defined by Gautam et al [1], we obtain the desired result.

4. Multivariable H-function

If $A_j'^{(i)}, B_j'^{(i)}, C_j'^{(i)}, D_j'^{(i)} \in \mathbb{R}$,$m' = 0$ and $A_j''^{(i)}, B_j''^{(i)}, C_j''^{(i)}, D_j''^{(i)} \in \mathbb{R}$ and $m'' = 0$, the multivariable A-functions reduces to multivariable H-functions defined by Srivastava et al [6]. We obtain the following formula.

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j} \aleph \begin{pmatrix} z''_1 (t-a)^{\mu_1+\mu_1'} (b-t)^{\rho_1+\rho_1'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)} - \lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z''_v (t-a)^{\mu_v+\mu_v'} (b-t)^{\rho_v+\rho_v'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(v)} - \lambda_j'^{(v)}} \end{pmatrix}$$

$$H \begin{pmatrix} z_1 (t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_s (t-a)^{\mu_s} (b-t)^{\rho_s} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(s)}} \end{pmatrix}$$

$$H\left(\begin{array}{c} z'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ z'_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l} z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=(b-a)^{\alpha+\beta-1}$$

$$=P_1\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\sum_{h_1=1}^{M_1}\cdots\sum_{h_v=1}^{M_v}\sum_{k_1=0}^{\infty}\cdots\sum_{k_v=0}^{\infty}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v} z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u} z''^{R_k}B_uB_{u,v}$$

$$H^{0,n'+n''+l+k+2;X}_{p'+p''+l+k+2,q'+q''+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}_1,K_1,K_2,K_P,K_j:C \\ \cdots & . \\ \cdots & . \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(s)}}} & . \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(u)}}} & . \\ \frac{(b-a)f_1}{af_1+g_1} & . \\ \cdots & . \\ \cdots & . \\ \frac{(b-a)f_k}{af_k+g_k} & . \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & . \\ \cdots & . \\ \cdots & \mathrm{B}_1,L_1,L_j,L_Q,:D \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \end{array}\right) \quad (4.1)$$

under the same conditions and notations that (3.17) with $A'^{(i)}_j, B'^{(i)}_j, C'^{(i)}_j, D'^{(i)}_j \in \mathbb{R}$,$m' = 0$ and $A''^{(i)}_j, B''^{(i)}_j, C''^{(i)}_j, D''^{(i)}_j \in \mathbb{R}$ and $m'' = 0$

Remark

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable A-functions defined by Gautam et al [1].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable A-functions, defined by Gautam et al [1], a expansion of mltivariable Aleph-function and a generalized hypergeometric function with general arguments. The formula established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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