

On general Eulerian integral of certain products of I-functions and a multivariable Aleph-function

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ABSTRACT

The object of this paper is to establish an general Eulerian integral involving the product of two multivariable I-functions defined by Prasad [1], the multivariable Aleph-function and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular case concerning the multivariable H-function defined by Srivastava et al [6].

Keywords: Eulerian integral, multivariable I-function, Lauricella function of several variables, multivariable H-function, generalized hypergeometric function

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1.Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable I-functions defined by Prasad [1], the Aleph-function of several variables and a generalized hypergeometric function with general arguments. The Aleph-function of several variables is an extension of multivariable I-function defined by Sharma et al [3].

$$\text{We define : } \aleph(z_1''', \dots, z_v''') = \aleph_{P_i, Q_i, \tau_i; R: P_{i(1)}, Q_{i(1)}, \tau_{i(1)}; R^{(1)}; \dots; P_{i(r)}, Q_{i(r)}, \tau_{i(r)}; R^{(v)}}^{0, N: M_1, N_1, \dots, M_v, N_v} \left(\begin{matrix} z_1''', \dots, z_v''' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z_v''' \end{matrix} \right)$$

$$[(a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(v)})_{1, n}] , [\tau'_i(a_{ji}; \alpha_{ji}^{(1)}, \dots, \alpha_{ji}^{(v)})_{n+1, p_i}] :$$

$$\dots, [\tau'_i(b_{ji}; \beta_{ji}^{(1)}, \dots, \beta_{ji}^{(v)})_{m+1, q_i}] :$$

$$\left[(c_j^{(1)}; \gamma_j^{(1)})_{1, n_1}, [\tau'_{i(1)}(c_{ji(1)}; \gamma_{ji(1)}^{(1)})_{n_1+1, p_i^{(1)}}]; \dots; [(c_j^{(v)}; \gamma_j^{(v)})_{1, n_v}, [\tau'_{i(v)}(c_{ji(v)}; \gamma_{ji(v)}^{(v)})_{n_v+1, p_i^{(v)}}]] \right]$$

$$\left[(d_j^{(1)}; \delta_j^{(1)})_{1, m_1}, [\tau'_{i(1)}(d_{ji(1)}; \delta_{ji(1)}^{(1)})_{m_1+1, q_i^{(1)}}]; \dots; [(d_j^{(v)}; \delta_j^{(v)})_{1, m_v}, [\tau'_{i(v)}(d_{ji(v)}; \delta_{ji(v)}^{(v)})_{m_v+1, q_i^{(v)}}]] \right]$$

$$= \frac{1}{(2\pi\omega)^v} \int_{L_1} \dots \int_{L_v} \psi_1(s_1, \dots, s_v) \prod_{k=1}^v \xi_k(s_k) z_k''' s_k ds_1 \dots ds_r \quad (1.1)$$

with $\omega = \sqrt{-1}$

$$\psi_1(s_1, \dots, s_v) = \frac{\prod_{j=1}^N \Gamma(1 - a_j + \sum_{k=1}^v \alpha_j^{(k)} s_k)}{\sum_{i=1}^R [\tau'_i \prod_{j=N+1}^{P_i} \Gamma(a_{ji} - \sum_{k=1}^v \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{Q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^v \beta_{ji}^{(k)} s_k)]} \quad (1.2)$$

$$\text{and } \xi_k(s_k) = \frac{\prod_{j=1}^{M_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{N_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i(k)=1}^{R^{(k)}} [\tau'_{i(k)} \prod_{j=M_k+1}^{Q_{i(k)}} \Gamma(1 - d_{ji(k)}^{(k)} + \delta_{ji(k)}^{(k)} s_k) \prod_{j=N_k+1}^{P_{i(k)}} \Gamma(c_{ji(k)}^{(k)} - \gamma_{ji(k)}^{(k)} s_k)]} \quad (1.3)$$

Suppose , as usual , that the parameters

$$a_j, j = 1, \dots, P; b_j, j = 1, \dots, Q;$$

$$c_j^{(k)}, j = 1, \dots, N_k; c_{ji^{(k)}}^{(k)}, j = N_k + 1, \dots, P_{i^{(k)}};$$

$$d_j^{(k)}, j = 1, \dots, M_k; d_{ji^{(k)}}^{(k)}, j = M_k + 1, \dots, Q_{i^{(k)}};$$

$$\text{with } k = 1 \dots, r, i = 1, \dots, R, i^{(k)} = 1, \dots, R^{(k)}$$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^N \alpha_j^{(k)} + \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} + \tau_{i^{(k)}}' \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau_{i^{(k)}}' \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leq 0 \quad (1.4)$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k -p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary , ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to M_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^r \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|arg z_k''| < \frac{1}{2} A_i^{(k)} \pi, \text{ where}$$

$$A_i^{(k)} = \sum_{j=1}^N \alpha_j^{(k)} - \tau_i' \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau_i' \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} - \tau_{i^{(k)}}' \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau_{i^{(k)}}' \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1, \dots, r, i = 1, \dots, R, i^{(k)} = 1, \dots, R^{(k)} \quad (1.5)$$

The complex numbers z_i are not zero. Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1''', \dots, z_v''') = O(|z_1'''|^{\alpha_1}, \dots, |z_r'''|^{\alpha_r}), \max(|z_1'''|, \dots, |z_v'''|) \rightarrow 0$$

$$\aleph(z_1''', \dots, z_v''') = O(|z_1'''|^{\beta_1}, \dots, |z_v'''|^{\beta_r}), \min(|z_1'''|, \dots, |z_v'''|) \rightarrow \infty$$

where $k = 1, \dots, r : \alpha_k = \min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \dots, m_k$ and

$$\beta_k = \max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \dots, n_k$$

Series representation of Aleph-function of u -variables is given by

$$\aleph(z_1''', \dots, z_v''') = \sum_{G_1, \dots, G_v=0}^{\infty} \sum_{g_1=0}^{M_1} \dots \sum_{g_v=0}^{M_v} \frac{(-)^{G_1+\dots+G_v}}{\delta_{g_1}^{G_1} \dots \delta_{g_v}^{G_v}} \psi_1(\eta_{G_1, g_1}, \dots, \eta_{G_v, g_v}) \times \xi_1(\eta_{G_1, g_1}) \dots \xi_v(\eta_{G_v, g_v}) z_1^{-\eta_{G_1, g_1}} \dots z_v^{-\eta_{G_v, g_v}} \quad (1.6)$$

Where $\psi(\cdot, \dots, \cdot), \theta_i(\cdot), i = 1, \dots, r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1, g_1} = \frac{d_{g_1}^{(1)} + G_1}{\delta_{g_1}^{(1)}}, \dots, \eta_{G_v, g_v} = \frac{d_{g_v}^{(v)} + G_v}{\delta_{g_v}^{(v)}} \quad (1.7)$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i + p_i] \neq \delta_j^{(i)}[d_{g_i}^i + G_i]$ for $j \neq m_i, m_i = 1, \dots, \eta_{G_i, g_i}; p_i, n_i = 0, 1, 2, \dots; y_i \neq 0, i = 1, \dots, v$

The multivariable I-function of r -variables is defined by Prasad [1] in term of multiple Mellin-Barnes type integral :

$$I(z_1', \dots, z_s') = I_{p_2, q_2, p_3, q_3, \dots; p_s, q_s; p^{(1)}, q^{(1)}, \dots; p^{(s)}, q^{(s)}}^{0, n_2; 0, n_3; \dots; 0, n_s; m^{(1)}, n^{(1)}, \dots; m^{(s)}, n^{(s)}} \left(\begin{matrix} z_1' \\ \vdots \\ z_s' \end{matrix} \middle| \begin{matrix} (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1, p_2}; \dots; \\ (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1, q_2}; \dots; \end{matrix} \right) \quad (1.8)$$

$$\left((a_{sj}; \alpha_{sj}^{(1)}, \dots, \alpha_{sj}^{(s)})_{1, p_s}; (a_j^{(1)}, \alpha_j^{(1)})_{1, p^{(1)}}; \dots; (a_j^{(s)}, \alpha_j^{(s)})_{1, p^{(s)}} \right) \left((b_{rj}; \beta_{sj}^{(1)}, \dots, \beta_{sj}^{(s)})_{1, q_s}; (b_j^{(1)}, \beta_j^{(1)})_{1, q^{(1)}}; \dots; (b_j^{(s)}, \beta_j^{(s)})_{1, q^{(s)}} \right)$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L_1'} \dots \int_{L_s'} \phi(t_1, \dots, t_s) \prod_{i=1}^s \zeta_i(t_i) z_i^{t_i} dt_1 \dots dt_s \quad (1.9)$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|arg z_i'| < \frac{1}{2} \Omega_i \pi, \text{ where}$$

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right) + \dots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} \right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)} \right) \quad (1.10)$$

where $i = 1, \dots, s$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the asymptotic expansion in the following convenient form :

$$I(z'_1, \dots, z'_s) = O(|z'_1|^{\alpha_1}, \dots, |z'_s|^{\alpha_r}), \max(|z'_1|, \dots, |z'_s|) \rightarrow 0$$

$$I(z'_1, \dots, z'_s) = O(|z'_1|^{\beta_1}, \dots, |z'_s|^{\beta_r}), \min(|z'_1|, \dots, |z'_s|) \rightarrow \infty$$

where $k = 1, \dots, r : \alpha'_k = \min[Re(b_j^{(k)} / \beta_j^{(k)})], j = 1, \dots, m_k$ and

$$\beta'_k = \max[Re((a_j^{(k)} - 1) / \alpha_j^{(k)})], j = 1, \dots, n_k$$

We will use these following notations in this section :

$$I(z''_1, \dots, z''_u) = I_{p'_2, q'_2; p'_3, q'_3; \dots; p'_u, q'_u; p^{(1)}, q^{(1)}; \dots; p^{(u)}, q^{(u)}}^{0, n'_2; 0, n'_3; \dots; 0, n'_u; m'^{(1)}, n'^{(1)}; \dots; m'^{(u)}, n'^{(u)}} \left(\begin{matrix} z''_1 \\ \cdot \\ \cdot \\ \cdot \\ z''_u \end{matrix} \middle| \begin{matrix} (a'_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)})_{1, p'_2}; \dots; \\ (b'_{2j}; \beta_{2j}^{(1)}, \beta_{2j}^{(2)})_{1, q'_2}; \dots; \end{matrix} \right)$$

$$\left(a'_{uj}; \alpha_{uj}^{(1)}, \dots, \alpha_{uj}^{(u)} \right)_{1, p'_u} : \left(a_j^{(1)}, \alpha_j^{(1)} \right)_{1, p^{(1)}}; \dots; \left(a_j^{(u)}, \alpha_j^{(u)} \right)_{1, p^{(u)}} \right) \quad (1.11)$$

$$\left(b'_{uj}; \beta_{uj}^{(1)}, \dots, \beta_{uj}^{(u)} \right)_{1, q'_u} : \left(b_j^{(1)}, \beta_j^{(1)} \right)_{1, q^{(1)}}; \dots; \left(b_j^{(u)}, \beta_j^{(u)} \right)_{1, q^{(u)}} \right)$$

$$= \frac{1}{(2\pi\omega)^u} \int_{L''_1} \cdots \int_{L''_u} \psi(x_1, \dots, x_u) \prod_{i=1}^u \xi_i(x_i) z_i'' x_i dx_1 \cdots dx_u \quad (1.12)$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$\text{where } |arg z_i''| < \frac{1}{2} \Omega_i'' \pi,$$

$$\begin{aligned} \Omega'_i &= \sum_{k=1}^{n'(i)} \alpha'_k{}^{(i)} - \sum_{k=n'(i)+1}^{p'(i)} \alpha'_k{}^{(i)} + \sum_{k=1}^{m'(i)} \beta'_k{}^{(i)} - \sum_{k=m'(i)+1}^{q'(i)} \beta'_k{}^{(i)} + \left(\sum_{k=1}^{n'_2} \alpha'_{2k}{}^{(i)} - \sum_{k=n_2+1}^{p'_2} \alpha'_{2k}{}^{(i)} \right) \\ &+ \cdots + \left(\sum_{k=1}^{n'_u} \alpha'_{uk}{}^{(i)} - \sum_{k=n'_u+1}^{p'_u} \alpha'_{uk}{}^{(i)} \right) - \left(\sum_{k=1}^{q'_2} \beta'_{2k}{}^{(i)} + \sum_{k=1}^{q'_3} \beta'_{3k}{}^{(i)} + \cdots + \sum_{k=1}^{q'_u} \beta'_{uk}{}^{(i)} \right) \end{aligned} \quad (1.13)$$

where $i = 1, \dots, u$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the asymptotic expansion in the following convenient form :

$$I(z''_1, \dots, z''_u) = O(|z''_1|^{\alpha'_1}, \dots, |z''_u|^{\alpha'_s}), \max(|z''_1|, \dots, |z''_u|) \rightarrow 0$$

$$I(z''_1, \dots, z''_u) = O(|z''_1|^{\beta'_1}, \dots, |z''_u|^{\beta'_s}), \min(|z''_1|, \dots, |z''_u|) \rightarrow \infty$$

where $k = 1, \dots, z : \alpha''_k = \min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \dots, m'_k$ and

$$\beta''_k = \max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \dots, n'_k$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [4, page 39 eq .30]

$$\begin{aligned} &\frac{\prod_{j=1}^P \Gamma(A_j)}{\prod_{j=1}^Q \Gamma(B_j)} {}_P F_Q [(A_P); (B_Q); -(x_1 + \cdots + x_r)] \\ &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \frac{\prod_{j=1}^P \Gamma(A_j + s_1 + \cdots + s_r)}{\prod_{j=1}^Q \Gamma(B_j + s_1 + \cdots + s_r)} \Gamma(-s_1) \cdots \Gamma(-s_r) x_1^{s_1} \cdots x_r^{s_r} ds_1 \cdots ds_r \end{aligned} \quad (2.1)$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \cdots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \dots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \dots, r$

The Lauricella function $F_D^{(k)}$ is defined as

$$\begin{aligned} F_D^{(k)}[a, b_1, \dots, b_k; c; x_1, \dots, x_k] &= \frac{\Gamma(c)}{\Gamma(a)} \frac{1}{\prod_{j=1}^k \Gamma(b_j)} \frac{1}{(2\pi\omega)^k} \int_{L_1} \cdots \int_{L_k} \frac{\Gamma(a + \sum_{j=1}^k \zeta_j) \Gamma(b_1 + \zeta_1), \dots, \Gamma(b_k + \zeta_k)}{\Gamma(c + \sum_{j=1}^k \zeta_j)} \\ &\prod_{j=1}^k \Gamma(-\zeta_j) (-x_j)^{\zeta_j} d\zeta_1 \cdots d\zeta_k \end{aligned} \quad (2.2)$$

where $\max [|arg(-x_1)|, \dots, |arg(-x_k)|] < \pi, c \neq 0, -1, -2, \dots$.

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j} dt = (b-a)^{\alpha+\beta-1} B(\alpha, \beta) \prod_{j=1}^k (af_j + g_j)^{\sigma_j} \times F_D^{(k)} \left[\alpha, -\sigma_1, \dots, -\sigma_k; \alpha + \beta; -\frac{(b-a)f_1}{af_1 + g_1}, \dots, -\frac{(b-a)f_k}{af_k + g_k} \right] \quad (2.3)$$

where $a, b \in \mathbb{R} (a < b), \alpha, \beta, f_i, g_i, \sigma_i \in \mathbb{C}, (i = 1, \dots, k); \min(Re(\alpha), Re(\beta)) > 0$ and

$$\max_{1 \leq j \leq k} \left\{ \left| \frac{(b-a)f_i}{af_i + g_i} \right| \right\} < 1$$

$F_D^{(k)}$ is a Lauricella's function of k -variables, see Srivastava et al ([5], page60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} z^r (|z| < 1) \quad (2.4)$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [4, page 454].

3. Eulerian integral

In this section , we note :

$$U = p_2, q_2; p_3, q_3; \dots; p_{s-1}, q_{s-1}; p'_2, q'_2; p'_3, q'_3; \dots; p'_{u-1}, q'_{u-1}; 0, 0; \dots; 0, 0; 0, 0; \dots; 0, 0 \quad (3.1)$$

$$V = 0, n_2; 0, n_3; \dots; 0, n_{s-1}; 0, n'_2; 0, n'_3; \dots; 0, n'_{u-1}; 0, 0; \dots; 0, 0; 0, 0; \dots; 0, 0 \quad (3.2)$$

$$X = m^{(1)}, n^{(1)}; \dots; m^{(s)}, n^{(s)}; m'^{(1)}, n'^{(1)}; \dots; m'^{(u)}, n'^{(u)}; 1, 0; \dots; 1, 0; 1, 0; \dots; 1, 0 \quad (3.3)$$

$$Y = p^{(1)}, q^{(1)}; \dots; p^{(s)}, q^{(s)}; p'^{(1)}, q'^{(1)}; \dots; p'^{(u)}, q'^{(u)}; 0, 1; \dots; 0, 1; 0, 1; \dots; 0, 1 \quad (3.4)$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}); \dots; (a_{(s-1)k}; \alpha_{(s-1)k}^{(1)}, \alpha_{(s-1)k}^{(2)}, \dots, \alpha_{(s-1)k}^{(s-1)}); (a'_{2k}; \alpha_{2k}'^{(1)}, \alpha_{2k}'^{(2)}); \dots; (a'_{(u-1)k}; \alpha_{(u-1)k}'^{(1)}, \alpha_{(u-1)k}'^{(2)}, \dots, \alpha_{(u-1)k}'^{(u-1)}) \quad (3.5)$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}); \dots; (b_{(s-1)k}; \beta_{(s-1)k}^{(1)}, \beta_{(s-1)k}^{(2)}, \dots, \beta_{(s-1)k}^{(s-1)}); (b'_{2k}; \beta_{2k}'^{(1)}, \beta_{2k}'^{(2)}); \dots; (b'_{(u-1)k}; \beta_{(u-1)k}'^{(1)}, \beta_{(u-1)k}'^{(2)}, \dots, \beta_{(u-1)k}'^{(u-1)}) \quad (3.6)$$

$$\mathfrak{A} = (a_{sk}; \alpha_{sk}^{(1)}, \alpha_{sk}^{(2)}, \dots, \alpha_{sk}^{(s)}, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0) \quad (3.7)$$

$$\mathfrak{A}' = (a'_{uk}; 0, \dots, 0, \alpha'^{(1)}_{uk}, \alpha'^{(2)}_{uk}, \dots, \alpha'^{(u)}_{uk}, 0, \dots, 0, 0, \dots, 0) \quad (3.8)$$

$$\mathfrak{B} = (b_{sk}; \beta^{(1)}_{sk}, \beta^{(2)}_{sk}, \dots, \beta^{(s)}_{sk}, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0) \quad (3.9)$$

$$\mathfrak{B}' = (b'_{uk}; 0, \dots, 0, \beta'^{(1)}_{uk}, \beta'^{(2)}_{uk}, \dots, \beta'^{(u)}_{uk}, 0, \dots, 0, 0, \dots, 0) \quad (3.10)$$

$$A' = (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \dots; (a_k^{(s)}, \alpha_k^{(s)})_{1,p^{(s)}}; (a_k'^{(1)}, \alpha_k'^{(1)})_{1,p^{(1)}}; \dots; (a_k'^{(u)}, \alpha_k'^{(u)})_{1,p^{(u)}}; \\ (1, 0); \dots; (1, 0); (1, 0); \dots; (1, 0) \quad (3.11)$$

$$B' = (b_k^{(1)}, \beta_k^{(1)})_{1,q^{(1)}}; \dots; (b_k^{(s)}, \beta_k^{(s)})_{1,q^{(s)}}; (b_k'^{(1)}, \beta_k'^{(1)})_{1,q^{(1)}}; \dots; (b_k'^{(u)}, \beta_k'^{(u)})_{1,q^{(u)}}; \\ (0, 1); \dots; (0, 1); (0, 1); \dots; (0, 1) \quad (3.12)$$

$$K_1 = (1 - \alpha - \sum_{i=1}^v \eta_{G_i, g_i}(\mu_i + \mu'_i); \mu_1, \dots, \mu_s, \mu'_1, \dots, \mu'_u, 1, \dots, 1, v_1, \dots, v_l) \quad (3.13)$$

$$K_2 = (1 - \beta - \sum_{i=1}^v \eta_{G_i, g_i}(\rho_i + \rho'_i); \rho_1, \dots, \rho_s, \rho'_1, \dots, \rho'_u, 0, \dots, 0, \tau_1, \dots, \tau_l) \quad (3.14)$$

$$K_P = [1 - A_j; 0, \dots, 0, 0, \dots, 0, 0, \dots, 0, 1, \dots, 1]_{1,P} \quad (3.15)$$

$$K_j = [1 + \sigma_j - \sum_{i=1}^v \eta_{G_i, g_i}(\lambda_i^{(j)} + \lambda_i'^{(j)}); \lambda_j^{(1)}, \dots, \lambda_j^{(s)}, \lambda_j'^{(1)} \dots, \lambda_j'^{(u)}, 0 \dots, 1, \dots, 0, \zeta'_j, \dots, \zeta_j^{(l)}]_{1,k} \quad (3.16)$$

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$$L_1 = (1 - \alpha - \beta - \sum_{i=1}^v \eta_{G_i, g_i}(\mu_i + \mu'_i + \rho_i + \rho'_i); \mu_1 + \rho_1, \dots, \mu_s + \rho_s, \mu'_1 + \rho'_1, \dots, \mu'_u + \rho'_u, \\ 1, \dots, 1, v_1 + \tau_1, \dots, v_l + \tau_l) \quad (3.17)$$

$$L_Q = [1 - B_j; 0, \dots, 0, 0, \dots, 0, 0, \dots, 0, 1 \dots, 1]_{1,Q} \quad (3.18)$$

$$L_j = [1 + \sigma_j - \sum_{i=1}^v \eta_{G_i, g_i}(\lambda_i^{(j)} + \lambda_i'^{(j)}); \lambda_j^{(1)}, \dots, \lambda_j^{(s)}, \lambda_j'^{(1)} \dots, \lambda_j'^{(u)}, 0 \dots, 0, \zeta'_j, \dots, \zeta_j^{(l)}]_{1,k} \quad (3.19)$$

$$P_1 = (b - a)^{\alpha + \beta - 1} \left\{ \prod_{j=1}^h (af_j + g_j)^{\sigma_j} \right\} \quad (3.20)$$

$$B_{u,v} = (b - a)^{\sum_{i=1}^v (\mu_i + \mu'_i + \rho_i + \rho'_i) \eta_{G_i, g_i}} \left\{ \prod_{j=1}^h (af_j + g_j)^{-\sum_{i=1}^v (\lambda_i + \lambda'_i) \eta_{g_i, h_i}} \right\}_{G_v} \quad (3.21)$$

where $G_v = \psi(\eta_{G_1, g_1}, \dots, \eta_{G_v, g_v}) \times \xi_1(\eta_{G_1, g_1}) \cdots \xi_v(\eta_{G_v, g_v})$

$\psi_1, \xi_i, i = 1, \dots, v$ are defined respectively by (1.2) and (1.3)

We have the following result

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$$

$$\Re \left(\begin{array}{c} z''_1 (t-a)^{\mu_1+\mu'_1} (b-t)^{\rho_1+\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \vdots \\ z''_v (t-a)^{\mu_v+\mu'_v} (b-t)^{\rho_v+\rho'_v} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(v)}-\lambda_j'^{(v)}} \end{array} \right)$$

$$I \left(\begin{array}{c} z_1 (t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_s (t-a)^{\mu_s} (b-t)^{\rho_s} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(s)}} \end{array} \right)$$

$$I \left(\begin{array}{c} z'_1 (t-a)^{\mu'_1} (b-t)^{\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j'^{(1)}} \\ \vdots \\ z'_u (t-a)^{\mu'_u} (b-t)^{\rho'_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j'^{(u)}} \end{array} \right)$$

$${}_P F_Q \left[(A_P); (B_Q); - \sum_{i=1}^l z''_i (t-a)^{v_i} (b-t)^{\tau_i} \prod_{j=1}^k (f_j t + g_j)^{-\zeta_j^{(i)}} \right] dt = (b-a)^{\alpha+\beta-1}$$

$$= P_1 \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \sum_{h_1=1}^{M_1} \cdots \sum_{h_v=1}^{M_v} \sum_{k_1=0}^{\infty} \cdots \sum_{k_v=0}^{\infty} \sum_{\substack{h_1 R_1 + \cdots + h_u R_u \leq L \\ R_1, \dots, R_u = 0}} \prod_{i=1}^v z_i''^{\eta_{h_i, k_i}} \prod_{k=1}^u z''^{R_k} B_u B_{u,v}$$

$$I_{U:p_s+p'_u+l+k+2,q_s+q'_u+l+k+1;Y}^{V;0,n_s+n'_u+l+k+2,X}$$

$\frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}}$ \dots $\frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(s)}}}$ $\frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}}$ \dots $\frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(u)}}}$ $\frac{(b-a)f_1}{af_1+g_1}$ \dots $\frac{(b-a)f_k}{af_k+g_k}$ $\frac{z''_1(b-a)^{\tau_1+v_1}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(1)}}}$ \dots $\frac{z''_l(b-a)^{\tau_l+v_l}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(l)}}}$

$A ; K_1, K_2, K_P, K_j, \mathfrak{A}, \mathfrak{A}'; A'$
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 $B ; L_1, L_j, L_Q, \mathfrak{B}, \mathfrak{B}'; B'$

(3.22)

We obtain the I-function of $s + u + k + l$ variables.

Provided that

(A) $a, b \in \mathbb{R}(a < b); \mu_i, \rho_i, \mu'_i, \rho'_i \lambda_v^{(i)}; \lambda_v^{(j)} \in \mathbb{R}^+, f_i, g_j, \tau_v, \sigma_j \in \mathbb{C} \ (i = 1, \dots, s; j = 1, \dots; u; v = 1, \dots, k)$

$$\textbf{(B)} \quad a'_{ij}, b'_{ik} \in \mathbb{C} \ (i = 1, \dots, s; j = 1, \dots, p'_i; k = 1, \dots, q'_i); a_j'^{(i)}, b_j'^{(k)} \in \mathbb{C}$$

$$(i = 1, \dots, s; j = 1, \dots, p'^{(i)}; k = 1, \dots, q'^{(i)})$$

$$a''_{ij}, b''_{ik}, \in \mathbb{C} \ (i = 1, \dots, u'; j = 1, \dots, p''_i; k = 1, \dots, q''_i); a_j''^{(i)}, b_j''^{(k)}, \in \mathbb{C}$$

$$(i = 1, \dots, u; j = 1, \dots, p''^{(i)}; k = 1, \dots, q''^{(i)})$$

$$\alpha_{ij}^{(k)}, \beta_{ij}'^{(k)} \in \mathbb{R}^+ \ (i = 1, \dots, s, j = 1, \dots, p_i', k = 1, \dots, s); \alpha_j'^{(i)}, \beta_j'^{(i)} \in \mathbb{R}^+ \ (i = 1, \dots, s; j = 1, \dots, p_i')$$

$$\alpha_{ii}^{\prime\prime(k)}, \beta_{ii}^{\prime\prime(k)} \in \mathbb{R}^+ \quad (i = 1, \dots, u, j = 1, \dots, p_i'', k = 1, \dots, u); \alpha_j^{\prime\prime(i)}, \beta_i^{\prime\prime(i)} \in \mathbb{R}^+ \quad (i = 1, \dots, u; j = 1, \dots, p_i'')$$

$$(C) \max_{1 \leq j \leq k} \left\{ \left| \frac{(b-a)f_i}{af_i + g_i} \right| \right\} < 1$$

$$(D) \operatorname{Re} \left[\alpha + \sum_{i=1}^s \mu_i \min_{1 \leq j \leq m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}} + \sum_{i=1}^u \mu'_i \min_{1 \leq j \leq m'^{(i)}} \frac{b_j'^{(i)}}{\beta_j'^{(i)}} \right] > 0$$

$$\operatorname{Re} \left[\beta + \sum_{i=1}^s \rho_i \min_{1 \leq j \leq m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}} + \sum_{i=1}^u \rho'_i \min_{1 \leq j \leq m'^{(i)}} \frac{b_j'^{(i)}}{\beta_j'^{(i)}} \right] > 0$$

$$(E) \Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right) + \dots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} \right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \dots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)} \right) - \mu_i - \rho_i$$

$$- \sum_{j=1}^k \lambda_j^{(i)} > 0 \quad (i = 1, \dots, s)$$

$$\Omega'_i = \sum_{k=1}^{n'^{(i)}} \alpha'_k{}^{(i)} - \sum_{k=n'^{(i)}+1}^{p'^{(i)}} \alpha'_k{}^{(i)} + \sum_{k=1}^{m'^{(i)}} \beta'_k{}^{(i)} - \sum_{k=m'^{(i)}+1}^{q'^{(i)}} \beta'_k{}^{(i)} + \left(\sum_{k=1}^{n'_2} \alpha'_{2k}{}^{(i)} - \sum_{k=n'_2+1}^{p'_2} \alpha'_{2k}{}^{(i)} \right)$$

$$+ \dots + \left(\sum_{k=1}^{n'_u} \alpha'_{uk}{}^{(i)} - \sum_{k=n'_u+1}^{p'_u} \alpha'_{uk}{}^{(i)} \right) - \left(\sum_{k=1}^{q'_2} \beta'_{2k}{}^{(i)} + \sum_{k=1}^{q'_3} \beta'_{3k}{}^{(i)} + \dots + \sum_{k=1}^{q'_u} \beta'_{uk}{}^{(i)} \right) - \mu'_i - \rho'_i$$

$$- \sum_{j=1}^k \lambda'_j{}^{(i)} > 0 \quad (i = 1, \dots, u)$$

$$(F) \left| \arg \left(z_i \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(i)}} \right) \right| < \frac{1}{2} \Omega_i \pi \quad (a \leq t \leq b; i = 1, \dots, s)$$

$$\left| \arg \left(z'_i \prod_{j=1}^k (f_j t + g_j)^{-\lambda'_j{}^{(i)}} \right) \right| < \frac{1}{2} \Omega'_i \pi \quad (a \leq t \leq b; i = 1, \dots, u)$$

$$(G) P \leq Q + 1. \text{ The equality holds, when , in addition,}$$

$$\text{either } P > Q \text{ and } \left| z_i'' \left(\prod_{j=1}^k (f_j t + g_j)^{-\zeta_j^{(i)}} \right) \right|^{\frac{1}{Q-P}} < 1 \quad (a \leq t \leq b)$$

$$\text{or } P \leq Q \text{ and } \max_{1 \leq i \leq k} \left[\left| z_i'' \prod_{j=1}^k (f_j t + g_j)^{-\zeta_j^{(i)}} \right| \right] < 1 \quad (a \leq t \leq b)$$

(H) The multiple series occurring on the right-hand side of (3.22) is absolutely and uniformly convergent.

Proof

First expressing the multivariable Aleph-function in serie with the help of (1.6) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the I-function of s-variables and u-variables defined by Prasad [1] by the Mellin-Barnes contour integral with the help of the equation (1.8) and (1.12) respectively, the generalized hypergeometric function ${}_P F_Q(\cdot)$ in Mellin-Barnes contour integral with the help of (2.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Now collect the power of $(f_j t + g_j)$ with $j = 1, \dots, k$ and use the equations (2.1) and (2.2) and we obtain k -Mellin-Barnes contour integral. Interpreting $(r + s + k + l)$ -Mellin-barnes contour integral in multivariable I-function defined by Prasad [1], we obtain the desired result.

4. Multivariable H-function

If $A = B = U = V = 0$, the multivariable I-function reduces to the multivariable H-function and we obtain

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j} \mathfrak{N} \left(\begin{matrix} z''_1 (t-a)^{\mu_1+\mu'_1} (b-t)^{\rho_1+\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ z''_v (t-a)^{\mu_v+\mu'_v} (b-t)^{\rho_v+\rho'_v} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(v)}-\lambda_j'^{(v)}} \end{matrix} \right)$$

$$H \left(\begin{matrix} z_1 (t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ z_s (t-a)^{\mu_s} (b-t)^{\rho_s} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(s)}} \end{matrix} \right)$$

$$H \left(\begin{matrix} z'_1 (t-a)^{\mu'_1} (b-t)^{\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ z'_u (t-a)^{\mu'_u} (b-t)^{\rho'_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j'^{(u)}} \end{matrix} \right)$$

$${}_P F_Q \left[(A_P); (B_Q); - \sum_{i=1}^l z_i'' (t-a)^{v_i} (b-t)^{\tau_i} \prod_{j=1}^k (f_j t + g_j)^{-\zeta_j^{(i)}} \right] dt = (b-a)^{\alpha+\beta-1}$$

$$\begin{aligned}
 &= P_1 \frac{\prod_{j=1}^Q \Gamma(B_j)}{\prod_{j=1}^P \Gamma(A_j)} \sum_{h_1=1}^{M_1} \cdots \sum_{h_v=1}^{M_v} \sum_{k_1=0}^{\infty} \cdots \sum_{k_v=0}^{\infty} \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} \prod_{i=1}^v z_i^{\eta_{h_i, k_i}} \prod_{k=1}^u z_k^{R_k} B_u B_{u,v} \\
 &H_{p_s+p'_u+l+k+2, q_s+q'_u+l+k+1; Y}^{0, n_s+n'_u+l+k+2; X} \left(\begin{array}{c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j^{(1)}}} \\ \vdots \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j^{(s)}}} \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j^{(1)'}}} \\ \vdots \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^k (af_j+g_j)^{\lambda_j^{(u)'}}} \\ \frac{(b-a)f_1}{af_1+g_1} \\ \vdots \\ \frac{(b-a)f_k}{af_k+g_k} \\ \frac{z''_1(b-a)^{\tau_1+v_1}}{\prod_{j=1}^k (af_j+g_j)^{\zeta_j^{(1)}}} \\ \vdots \\ \frac{z''_l(b-a)^{\tau_l+v_l}}{\prod_{j=1}^k (af_j+g_j)^{\zeta_j^{(l)}}} \end{array} \middle| \begin{array}{c} K_1, K_2, K_P, K_j, \mathfrak{A}, \mathfrak{A}'; A' \\ \vdots \\ L_1, L_j, L_Q, \mathfrak{B}, \mathfrak{B}'; B' \end{array} \right) \quad (4.1)
 \end{aligned}$$

under the same conditions that (3.22) with $A = B = U = V = 0$

Remark

By the following similar procedure, the results of this document can be extended to product of any finite number of multivariable I-functions defined by Prasad [1].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable I-functions, defined by Prasad [1], a expansion of mltivariable Aleph-function and a generalized hypergeometric function with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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