On general Eulerian integral of certain products of I-functions defined by Nambisan and a multivariable Aleph-function

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ABSTRACT
The object of this paper is to establish an general Eulerian integral involving the product of two multivariable I-functions defined by Nambisan et al [2], the expansion of multivariable Aleph-function and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular case concerning the multivariable H-function.

Keywords: Eulerian integral, multivariable I-function, Lauricella function of several variables, multivariable H-function, generalized hypergeometric function

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1.Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable I-functions defined by Nambisan et al [2], the Aleph-function of several variables and a generalized hypergeometric function with general arguments. The Aleph-function of several variables is an extension of multivariable I-function defined by Sharma et al [3].

We define : $\aleph(z_1''',\cdots,z_v''')=\aleph^{0,N:M_1,N_1,\cdots,M_v,N_v}_{P_i,Q_i,\tau_i;R:P_{i(1)},Q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;P_{i(r)},Q_{i(v)};\tau_{i(v)};R^{(v)}}\left(\begin{array}{c} z'''_1\\ \cdot\\ \cdot\\ \cdot\\ z'''_v\end{array}\right|$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(v)})_{1,\mathfrak{n}}]\ ,[\tau_i'(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(v)})_{\mathfrak{n}+1,p_i}]:$
$\cdots\cdots\cdots\cdots\ ,[\tau_i'(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(v)})_{m+1,q_i}]:$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,n_1}],[\tau'_{i^{(1)}}(c^{(1)}_{ji^{(1)}};\gamma^{(1)}_{ji^{(1)}})_{n_1+1,p_i^{(1)}}];\cdots;\ [(\mathrm{c}_j^{(v)});\gamma_j^{(v)})_{1,n_v}],[\tau'_{i^{(v)}}(c^{(v)}_{ji^{(v)}};\gamma^{(v)}_{ji^{(v)}})_{n_v+1,p_i^{(v)}}]\\ [(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,m_1}],[\tau'_{i^{(1)}}(d^{(1)}_{ji^{(1)}};\delta^{(1)}_{ji^{(1)}})_{m_1+1,q_i^{(1)}}];\cdots;\ [(\mathrm{d}_j^{(v)});\delta_j^{(v)})_{1,m_v}],[\tau'_{i^{(v)}}(d^{(v)}_{ji^{(v)}};\delta^{(v)}_{ji^{(v)}})_{m_v+1,q_i^{(v)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^v}\int_{L_1}\cdots\int_{L_v}\psi_1(s_1,\cdots,s_v)\prod_{k=1}^{v}\xi_k(s_k)z_k'''^{s_k}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi_1(s_1,\cdots,s_v)=\frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{k=1}^{v}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i'\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\sum_{k=1}^{v}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{Q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{v}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\xi_k(s_k)=\frac{\prod_{j=1}^{M_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{N_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau'_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-d^{(k)}_{ji^{(k)}}+\delta^{(k)}_{ji^{(k)}}s_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(c^{(k)}_{ji^{(k)}}-\gamma^{(k)}_{ji^{(k)}}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, P; b_j, j = 1, \cdots, Q;$

$c_j^{(k)}, j = 1, \cdots, N_k; c_{ji^{(k)}}^{(k)}, j = N_k + 1, \cdots, P_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, M_k; d_{ji^{(k)}}^{(k)}, j = M_k + 1, \cdots, Q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} + \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} + \tau'_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{M_k} \delta_j^{(k)}$$

$$-\tau'_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to M_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k'''| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} - \tau'_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau'_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} - \tau'_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau'_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1, \cdots, r, i = 1, \cdots, R \, , i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1''', \cdots, z_v''') = 0(|z_1'''|^{\alpha_1}, \cdots, |z_r'''|^{\alpha_r}), max(|z_1'''|, \cdots, |z_v'''|) \rightarrow 0$

$\aleph(z_1''', \cdots, z_v''') = 0(|z_1'''|^{\beta_1}, \cdots, |z_v'''|^{\beta_r}), min(|z_1'''|, \cdots, |z_v'''|) \rightarrow \infty$

where $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1) / \gamma_j^{(k)})], j = 1, \cdots, n_k$$

Serie representation of Aleph-function of $u-$variables is given by

$$\aleph(z_1''',\cdots,z_v''') = \sum_{G_1,\cdots,G_v=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_v=0}^{M_v}\frac{(-)^{G_1+\cdots+G_v}}{\delta_{g_1}G_1!\cdots\delta_{g_v}G_v!}\psi_1(\eta_{G_1,g_1},\cdots,\eta_{G_v,g_v})$$

$$\times\ \xi_1(\eta_{G_1,g_1})\cdots\xi_v(\eta_{G_v,g_v})z_1^{-\eta_{G_1,g_1}}\cdots z_v^{-\eta_{G_v,g_v}} \tag{1.6}$$

Where $\psi(.,\cdots,.), \theta_i(.), i=1,\cdots,r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1} = \frac{d_{g_1}^{(1)}+G_1}{\delta_{g_1}^{(1)}},\cdots,\ \ \eta_{G_v,g_v} = \frac{d_{g_v}^{(v)}+G_v}{\delta_{g_v}^{(v)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i+p_i] \neq \delta_j^{(i)}[d_{g_i}^i+G_i]$ (1.7)
for $j\neq m_i, m_i=1,\cdots\eta_{G_i,g_i}; p_i,n_i=0,1,2,\cdots,; y_i\neq 0, i=1,\cdots,v$

The I-function is defined and represented in the following manner.

$$I(z_1',\cdots,z_s') = I_{p',q':p_1',q_1';\cdots;p_s',q_s'}^{0,n':m_1',n_1';\cdots;m_s',n_s'}\left(\begin{array}{c|c} \text{z'}_1 & (\text{a'}_j;\alpha_j'^{(1)},\cdots,\alpha_j'^{(s)};A_j')_{1,p'}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \text{z'}_s & (\text{b'}_j;\beta_j'^{(1)},\cdots,\beta_j'^{(s)};B_j')_{1,q'}: \end{array}\right.$$

$$\left.\begin{array}{l} (\text{c'}_j^{(1)},\gamma_j'^{(1)};C_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(s)},\gamma_j'^{(s)};C_j'^{(r)})_{1,p_r'} \\ \\ (\text{d'}_j^{(1)},\delta_j'^{(1)};D_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(s)},\delta_j'^{(s)};D_j'^{(s)})_{1,q_r'} \end{array}\right) \tag{1.8}$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\zeta_i(t_i)z_i'^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.9}$$

where $\phi(t_1,\cdots,t_s), \zeta_i(t_i), i=1,\cdots,s$ are given by :

$$\phi(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{n'}\Gamma^{A_j'}\left(1-a_j'+\sum_{i=1}^{s}\alpha_j'^{(i)}t_j\right)}{\prod_{j=n'+1}^{p'}\Gamma^{A_j'}\left(a_j'-\sum_{i=1}^{s}\alpha_j'^{(i)}t_j\right)\prod_{j=m'+1}^{q'}\Gamma^{B_j'}\left(1-b_j'+\sum_{i=1}^{s}\beta_j^{(i)}t_j\right)} \tag{1.10}$$

$$\zeta_i(t_i) = \frac{\prod_{j=1}^{n_i'}\Gamma^{C_j'^{(i)}}\left(1-c_j'^{(i)}+\gamma_j'^{(i)}t_i\right)\prod_{j=1}^{m_i'}\Gamma^{D_j'^{(i)}}\left(d_j'^{(i)}-\delta_j'^{(i)}t_i\right)}{\prod_{j=n_i'+1}^{p_i'}\Gamma^{C_j'^{(i)}}\left(c_j'^{(i)}-\gamma_j'^{(i)}t_i\right)\prod_{j=m_i'+1}^{q_i'}\Gamma^{D_j'^{(i)}}\left(1-d_j'^{(i)}+\delta_j'^{(i)}t_i\right)} \tag{1.11}$$

For more details, see Nambisan et al [2].

Following the result of Braaksma [1] the I-function of r variables is analytic if

$$U_i = \sum_{j=1}^{p'} A'_j \alpha'^{(i)}_j - \sum_{j=1}^{q'} B'_j \beta'^{(i)}_j + \sum_{j=1}^{p'_i} C'^{(i)}_j \gamma'^{(i)}_j - \sum_{j=1}^{q'_i} D'^{(i)}_j \delta'^{(i)}_j \leqslant 0, i = 1, \cdots, s \tag{1.12}$$

The integral (2.1) converges absolutely if

where $|arg(z'_k)| < \frac{1}{2}\Delta_k \pi, k = 1, \cdots, s$

$$\Delta_k = -\sum_{j=n'+1}^{p'} A'_j \alpha'^{(k)}_j - \sum_{j=1}^{q'} B'_j \beta^{(k)}_j + \sum_{j=1}^{m'_k} D'^{(k)}_j \delta'^{(k)}_j - \sum_{j=m'_k+1}^{q'_k} D'^{(k)}_j \delta'^{(k)}_j + \sum_{j=1}^{n'_k} C'^{(k)}_j \gamma'^{(k)}_j - \sum_{j=n'_k+1}^{p'_k} C'^{(k)}_j \gamma'^{(k)}_j > 0 \tag{1.13}$$

Consider the second multivariable I-function.

$$I(z''_1, \cdots, z''_u) = I^{0,n'':m''_1,n''_1;\cdots;m''_u,n''_u}_{p'',q'':p''_1,q''_1;\cdots;p''_u,q''_u}\left(\begin{array}{c|c} z"_1 & (a"_j;\alpha''^{(1)}_j,\cdots,\alpha''^{(u)}_j;A''_j)_{1,p''}: \\ \cdot & \\ \cdot & \\ \cdot & \\ z"_u & (b"_j;\beta''^{(1)}_j,\cdots,\beta''^{(u)}_j;B''_j)_{1,q''}: \end{array}\right.$$

$$\left.\begin{array}{c} (c"^{(1)}_j,\gamma''^{(1)}_j;C''^{(1)}_j)_{1,p''_1};\cdots;(c''^{(u)}_j,\gamma''^{(u)}_j;C''^{(u)}_j)_{1,p''_u} \\ \\ (d"^{(1)}_j,\delta''^{(1)}_j;D''^{(1)}_j)_{1,q''_1};\cdots;(d''^{(u)}_j,\delta''^{(u)}_j;D''^{(u)}_j)_{1,q''_u} \end{array}\right) \tag{1.14}$$

$$= \frac{1}{(2\pi\omega)^u} \int_{L''_1} \cdots \int_{L''_u} \psi(x_1, \cdots, x_u) \prod_{i=1}^{u} \xi_i(x_i) z''^{x_i}_i \mathrm{d}x_1 \cdots \mathrm{d}x_u \tag{1.15}$$

where $\psi(x_1, \cdots, x_u), \xi_i(x_i), i = 1, \cdots, u$ are given by :

$$\psi(x_1, \cdots, x_u) = \frac{\prod_{j=1}^{n''} \Gamma^{A''_j}\left(1 - a''_j + \sum_{i=1}^{u} \alpha''^{(i)}_j x_j\right)}{\prod_{j=n''+1}^{p''} \Gamma^{A''_j}\left(a''_j - \sum_{i=1}^{u} \alpha''^{(i)}_j x_j\right) \prod_{j=m'+1}^{q''} \Gamma^{B''_j}\left(1 - b''_j + \sum_{i=1}^{u} \beta''^{(i)}_j x_j\right)} \tag{1.16}$$

$$\xi_i(x_i) = \frac{\prod_{j=1}^{n''_i} \Gamma^{C''^{(i)}_j}\left(1 - c''^{(i)}_j + \gamma''^{(i)}_j x_i\right) \prod_{j=1}^{m''_i} \Gamma^{D''^{(i)}_j}\left(d''^{(i)}_j - \delta''^{(i)}_j x_i\right)}{\prod_{j=n''_i+1}^{p''_i} \Gamma^{C''^{(i)}_j}\left(c''^{(i)}_j - \gamma''^{(i)}_j x_i\right) \prod_{j=m''_i+1}^{q''_i} \Gamma^{D''^{(i)}_j}\left(1 - d''^{(i)}_j + \delta''^{(i)}_j x_i\right)} \tag{1.17}$$

For more details, see Nambisan et al [2].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U'_i = \sum_{j=1}^{p''} A''_j \alpha''^{(i)}_j - \sum_{j=1}^{q''} B'_j \beta''^{(i)}_j + \sum_{j=1}^{p''_i} C''^{(i)}_j \gamma''^{(i)}_j - \sum_{j=1}^{q''_i} D''^{(i)}_j \delta''^{(i)}_j \leqslant 0, i = 1, \cdots, u \tag{1.18}$$

The integral (2.1) converges absolutely if where $|arg(z''_k)| < \frac{1}{2}\Delta''_k\pi, k = 1, \cdots, u$

$$\Delta'_k = -\sum_{j=n''+1}^{p''} A''_j\alpha''^{(k)}_j - \sum_{j=1}^{q''} B''_j\beta''^{(k)}_j + \sum_{j=1}^{m''_k} D''^{(k)}_j\delta''^{(k)}_j - \sum_{j=m''_k+1}^{q''_k} D''^{(k)}_j\delta''^{(k)}_j + \sum_{j=1}^{n''_k} C''^{(k)}_j\gamma''^{(k)}_j - \sum_{j=n''_k+1}^{p''_k} C''^{(k)}_j\gamma''^{(k)}_j > 0 \quad (1.19)$$

2. Integral representation of Lauricella function of several variables

The Lauricella function $F_D^{(k)}$ is defined as

$$F_D^{(k)}[a, b_1, \cdots, b_k; c; x_1, \cdots, x_k] = \frac{\Gamma(c)}{\Gamma(a)\prod_{j=1}^k \Gamma(b_j)} \frac{1}{(2\pi\omega)^k} \int_{L_1} \cdots \int_{L_k} \frac{\Gamma\left(a + \sum_{j=1}^k \zeta_j\right)\Gamma(b_1+\zeta_1), \cdots, \Gamma(b_k+\zeta_k)}{\Gamma\left(c + \sum_{j=1}^k \zeta_j\right)}$$

$$\prod_{j=1}^k \Gamma(-\zeta_j)(-x_j)^{\zeta_i}\,\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_k \quad (2.1)$$

where $max\left[|arg(-x_1)|, \cdots, |arg(-x_k)|\right] < \pi, c \neq 0, -1, -2, \cdots$.

In order to evaluate a number of integrals of multivariable I-functions, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^k (f_j t + g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^k (af_j+g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha, -\sigma_1, \cdots, -\sigma_k; \alpha+\beta; -\frac{(b-a)f_1}{af_1+g_1}, \cdots, -\frac{(b-a)f_k}{af_k+g_k}\right] \quad (2.2)$$

where $a, b \in \mathbb{R}(a < b), \alpha, \beta, f_i, g_i, \sigma_i \in \mathbb{C}, (i = 1, \cdots, k)$; $min(Re(\alpha), Re(\beta)) > 0$ and

$$\max_{1 \leqslant j \leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\} < 1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava et al ([6], page60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha} = \sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r (|z| < 1) \quad (2.3)$$

integrating term by term with the help of the integral given by Saigo and Saxena [3, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [6, page 60].

3. Eulerian integral

Let

$$X = m'_1, n'_1; \cdots; m'_s, n'_s; m''_1, n''_1; \cdots; m''_u, n''_u; 1, 0; \cdots; 1, 0; 1, 0; \cdots; 1, 0 \quad (3.1)$$

$$Y = p'_1, q'_1; \cdots; p'_s, q'_s; p''_1, q''_1; \cdots; p''_u, q''_u; 0, 1; \cdots; 0, 1; 0, 1; \cdots; 0, 1 \quad (3.2)$$

$$A=(a'_j;A'^{(1)}_j,\cdots,A'^{(s)}_j,0,\cdots,0,0,\cdots,0,0,\cdots,0;A'_j)_{1,p'} \tag{3.3}$$

$$B=(b'_j;B'^{(1)}_j,\cdots,B'^{(s)}_j,0\cdots,0,0\cdots,0,0,\cdots,0;B'_j)_{1,q'} \tag{3.4}$$

$$A'=(a''_j;0,\cdots,0,A''^{(1)}_j,\cdots,A''^{(u)}_j,0,\cdots,0,0,\cdots,0;A''_j)_{1,p''} \tag{3.5}$$

$$B'=(b''_j;0,\cdots,0,B''^{(1)}_j,\cdots,B''^{(u)}_j,0,\cdots,0,0,\cdots,0;B''_j)_{1,q''} \tag{3.6}$$

$$\mathrm{C}=(\mathrm{c}'^{(1)}_j,\gamma'^{(1)}_j;C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(s)}_j,\gamma'^{(s)}_j;C'^{(s)}_j)_{1,p'_s};(c''^{(1)}_j,\gamma''^{(1)}_j;C''^{(1)}_j)_{1,p''_1};\cdots;$$

$$(c''^{(u)}_j,\gamma^{(u)}_j;C''^{(u)}_j)_{1,p''_u};(1,0;1);\cdots;(1,0;1);(1,0;1);\cdots;(1,0;1) \tag{3.7}$$

$$D=(\mathrm{d}'^{(1)}_j,\delta'^{(1)}_j;D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(s)}_j,\delta'^{(s)}_j;D'^{(s)}_j)_{1,q_s};(\mathrm{d}''^{(1)}_j,\delta''^{(1)}_j;D''^{(1)}_j)_{1,q''_1};\cdots;$$

$$(\mathrm{d}''^{(u)}_j,\delta''^{(u)}_j;D''^{(u)}_j)_{1,q''_u};(0,1;1);\cdots;(0,1;1);(0,1;1);\cdots;(0,1;1) \tag{3.8}$$

$$K_1=(1-\alpha-\sum_{i=1}^{v}\eta_{G_i,g_i}(\mu_i+\mu'_i);\mu_1,\cdots,\mu_s,\mu'_1,\cdots,\mu'_u,1,\cdots,1,\upsilon_1,\cdots,\upsilon_l;1) \tag{3.9}$$

$$K_2=(1-\beta-\sum_{i=1}^{v}\eta_{G_i,g_i}(\rho_i+\rho'_i);\rho_1,\cdots,\rho_s,\rho'_1,\cdots,\rho'_u,0,\cdots,0,\tau_1,\cdots,\tau_l;1) \tag{3.10}$$

$$K_P=[1-A_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1,\cdots,1;1]_{1,P} \tag{3.11}$$

$$K_j=[1+\sigma_j-\sum_{i=1}^{v}\eta_{G_i,g_i}(\lambda^{(j)}_i+\lambda'^{(j)}_i);\lambda^{(1)}_j,\cdots,\lambda^{(s)}_j,\lambda'^{(1)}_j\cdots,\lambda'^{(u)}_j,0\cdots,\underset{\mathrm{j}}{1},\cdots,0,\zeta'_j,\cdots,\zeta^{(l)}_j;1]_{1,k} \tag{3.12}$$

$$L_1=(1-\alpha-\beta-\sum_{i=1}^{v}\eta_{G_i,g_i}(\mu_i+\mu'_i+\rho_i+\rho'_i);\mu_1+\rho_1,\cdots,\mu_s+\rho_s,\mu'_1+\rho'_1,\cdots,\mu'_u+\rho'_u,$$

$$1,\cdots,1,\upsilon_1+\tau_1,\cdots,\upsilon_l+\tau_l;1) \tag{3.13}$$

$$L_Q=[1-B_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1\cdots,1;1]_{1,Q} \tag{3.14}$$

$$L_j=[1+\sigma_j-\sum_{i=1}^{v}\eta_{G_i,g_i}(\lambda^{(j)}_i+\lambda'^{(j)}_i);\lambda^{(1)}_j,\cdots,\lambda^{(s)}_j,\lambda'^{(1)}_j\cdots,\lambda'^{(u)}_j,0\cdots,0,\zeta'_j,\cdots,\zeta^{(l)}_j;1]_{1,k} \tag{3.15}$$

$$A_1=A,A';B_1=B,B' \tag{3.16}$$

$$P_1=(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{h}(af_j+g_j)^{\sigma_j}\right\} \tag{3.17}$$

$$B_{u,v}=(b-a)^{\sum_{i=1}^{v}(\mu_i+\mu_i'+\rho_i+\rho_i')\eta_{G_i,g_i}}\left\{\prod_{j=1}^{h}(af_j+g_j)^{-\sum_{i=1}^{v}(\lambda_i+\lambda_i')\eta_{g_i,h_i}}\right\}G_v \tag{3.18}$$

where $G_v=\psi(\eta_{G_1,g_1},\cdots,\eta_{G_v,g_v})\times\ \xi_1(\eta_{G_1,g_1})\cdots\xi_v(\eta_{G_v,g_v})$

$\psi_1,\xi_i,i=1,\cdots,v$ are defined respectively by (1.2) and (1.3)

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\aleph\left(\begin{array}{c} \text{z''}_1(t-a)^{\mu_1+\mu_1'}(b-t)^{\rho_1+\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z''}_v(t-a)^{\mu_v+\mu_v'}(b-t)^{\rho_v+\rho_v'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(v)}-\lambda_j'^{(v)}} \end{array}\right)$$

$$I\left(\begin{array}{c} \text{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}} \end{array}\right)$$

$$I\left(\begin{array}{c} \text{z'}_1(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z'}_u(t-a)^{\mu_u'}(b-t)^{\rho_u'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=(b-a)^{\alpha+\beta-1}$$

$$=P_1\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\sum_{h_1=1}^{M_1}\cdots\sum_{h_v=1}^{M_v}\sum_{k_1=0}^{\infty}\cdots\sum_{k_v=0}^{\infty}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u}z''^{R_k}B_uB_{u,v}$$

$$I^{0,n'+n''+l+k+2;X}_{p'+p''+l+k+2,q'+q''+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}_1, K_1, K_2, K_P, K_j : C \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^k(af_j+g_j)^{\lambda'^{(u)}_j}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\varsigma_j^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\varsigma_j^{(l)}}} & \mathrm{B}_1, L_1, L_j, L_Q, : D \end{array}\right) \quad (3.19)$$

We obtain the I-function of $s+u+k+l$ variables.

Provided that

(A) $a, b \in \mathbb{R}(a < b); \mu_i, \rho_i, \mu'_j, \rho'_j \lambda_v^{(i)}; \lambda'^{(j)}_v \in \mathbb{R}^+$, $f_i, g_j, \tau_v, \sigma_j \in \mathbb{C}$ $(i = 1, \cdots, s; j = 1, \cdots; u; v = 1, \cdots, k)$

(B) $m_j, n_j, p_j, q_j (j = 1, \cdots, s), n', p', q' \in \mathbb{N}^*$;$\delta_j^{(i)} \in \mathbb{R}_+(j = 1, \cdots, q_i; i = 1, \cdots, s)$

$\alpha'^{(i)}_j \in \mathbb{R}_+(j = 1, \cdots, p'; i = 1, \cdots, s), \beta'^{(i)}_j \in \mathbb{R}_+(j = 1, \cdots, q'; i = 1, \cdots, s), \gamma'^{(i)}_j \in \mathbb{R}_+(j = 1, \cdots, p'_i; i = 1, \cdots, s)$

$a'_j(j = 1, \cdots, p), b'_j(j = 1, \cdots, q), c'^{(i)}_j(j = 1, \cdots, p'_i, i = 1, \cdots, s), d'^{(i)}_j(j = 1, \cdots, q'_i, i = 1, \cdots, s) \in \mathbb{C}$

The exposants

$A'_j(j = 1, \cdots, p'), B'_j(j = 1, \cdots, q'), C'^{(i)}_j(j = 1, \cdots, p'_i; i = 1, \cdots, s), D'^{(i)}_j(j = 1, \cdots, q'_i; i = 1, \cdots, s)$

of various gamma function involved in (1.10) and (1.11) may take non integer values.

$m''_j, n''_j, p''_j, q'_j(j = 1, \cdots, u), n'', p'', q'' \in \mathbb{N}^*$;$\delta''^{(i)}_j \in \mathbb{R}_+(j = 1, \cdots, q''_i; i = 1, \cdots, u)$

$\alpha_j''^{(i)} \in \mathbb{R}_+(j=1,\cdots,p'';i=1,\cdots,u), \beta_j''^{(i)} \in \mathbb{R}_+(j=1,\cdots,q'';i=1,\cdots,u), \gamma_j''^{(i)} \in \mathbb{R}_+(j=1,\cdots,p_i'';i=1,\cdots,u)$

$a_j''(j=1,\cdots,p''), b_j''(j=1,\cdots,q''), c_j''^{(i)}(j=1,\cdots,p_i'',i=1,\cdots,u), d_j''^{(i)}(j=1,\cdots,q_i'',i=1,\cdots,u) \in \mathbb{C}$

The exposants

$A_j''(j=1,\cdots,p''), B_j''(j=1,\cdots,q''), C_j''^{(i)}(j=1,\cdots,p_i'';i=1,\cdots,u), D_j''^{(i)}(j=1,\cdots,q_i'';i=1,\cdots,u)$

of various gamma function involved in (1.15) and (1.16) may take non integer values.

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\} < 1$

(D) $Re\left[\alpha+\sum_{i=1}^{s}\mu_i\min_{1\leqslant j\leqslant m_i'}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}+\sum_{i=1}^{u}\mu_i'\min_{1\leqslant j\leqslant m_i''}\frac{d_j''^{(i)}}{\delta_j''^{(i)}}\right] > 0$

$$Re\left[\beta+\sum_{i=1}^{s}\rho_i\min_{1\leqslant j\leqslant m_i'}\frac{d_j'^{(i)}}{\delta_j'^{(i)}}+\sum_{i=1}^{u}\rho_i'\min_{1\leqslant j\leqslant m_i''}\frac{d_j''^{(i)}}{\delta_j''^{(i)}}\right] > 0$$

(E) $U_i=\sum_{j=1}^{p'}A_j'\alpha_j'^{(i)}-\sum_{j=1}^{q'}B_j'\beta_j'^{(i)}+\sum_{j=1}^{p_i'}C_j'^{(i)}\gamma_j'^{(i)}-\sum_{j=1}^{q_i'}D_j'^{(i)}\delta_j'^{(i)}\leqslant 0, i=1,\cdots,s$

$$U_i'=\sum_{j=1}^{p''}A_j''\alpha_j''^{(i)}-\sum_{j=1}^{q''}B_j'\beta_j''^{(i)}+\sum_{j=1}^{p_i''}C_j''^{(i)}\gamma_j''^{(i)}-\sum_{j=1}^{q_i''}D_j''^{(i)}\delta_j''^{(i)}\leqslant 0, i=1,\cdots,u$$

(F) $\Delta_k=-\sum_{j=n'+1}^{p'}A_j'\alpha_j'^{(k)}-\sum_{j=1}^{q'}B_j'\beta_j'^{(k)}+\sum_{j=1}^{m_k'}D_j'^{(k)}\delta_j'^{(k)}-\sum_{j=m_k'+1}^{q_k'}D_j'^{(k)}\delta_j'^{(k)}+\sum_{j=1}^{n_k'}C_j'^{(k)}\gamma_j'^{(k)}-\sum_{j=n_k'+1}^{p_k'}C_j'^{(k)}\gamma_j'^{(k)}$

$$-\mu_i-\rho_i-\sum_{l=1}^{k}\lambda_j^{(i)} > 0\,;i=1,\cdots,s$$

$$\Delta_k'=-\sum_{j=n''+1}^{p''}A_j''\alpha_j''^{(k)}-\sum_{j=1}^{q''}B_j''\beta_j''^{(k)}+\sum_{j=1}^{m_k''}D_j''^{(k)}\delta_j''^{(k)}-\sum_{j=m_k''+1}^{q_k''}D_j''^{(k)}\delta_j''^{(k)}+\sum_{j=1}^{n_k''}C_j''^{(k)}\gamma_j''^{(k)}-\sum_{j=n_k''+1}^{p_k''}C_j''^{(k)}\gamma_j''^{(k)}$$

$$-\mu_i'-\rho_i'-\sum_{l=1}^{k}\lambda_j'^{(i)} > 0\,;i=1,\cdots,u$$

(G) $\left|arg\left(z_i\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right| < \frac{1}{2}\Delta_i\pi \quad (a\leqslant t\leqslant b; i=1,\cdots,s)$

$$\left|arg\left(z_i'\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right| < \frac{1}{2}\Delta_i'\pi \quad (a\leqslant t\leqslant b; i=1,\cdots,u)$$

(H) $P\leqslant Q+1$. The equality holds, when , in addition, either $P > Q$ and $\left| z_i'' \left(\prod_{j=1}^{k} (f_j t + g_j)^{-\zeta_j^{(i)}} \right) \right|^{\frac{1}{Q-P}} < 1 \qquad (a \leqslant t \leqslant b)$

or $P \leqslant Q$ and $\max_{1 \leqslant i \leqslant k} \left[\left| \left(z_i'' \prod_{j=1}^{k} (f_j t + g_j)^{-\zeta_j^{(i)}} \right) \right| \right] < 1 \qquad (a \leqslant t \leqslant b)$

(I) The multiple series occuring on the right-hand side of (3.19) is absolutely and uniformly convergent.

Proof

First expressing the the multivariable Aleph-function in serie with the help of (1.6) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the I-function of s-variables and u-variables defined by Nambisan et al [2] by the Mellin-Barnes contour integral with the help of the equation (1.9) and (1.15) respectively, the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process.. Now collect the power of $(f_j t + g_j)$ with $j = 1, \cdots, k$ and use the equations (2.1) and (2.2) and we obtain $k-$Mellin-Barnes contour integral. Interpreting $(r+s+k+l)-$Mellin-barnes contour integral in multivariable I-function defined by Nambisan et al [2], we obtain the desired result.

4.Particular case

$A_j' = B_j' = C_j'^{(i)} = D_j'^{(i)} = A_j'' = B_j'' = C_j''^{(i)} = D_j''^{(i)} = 1$, The multivariable I-functions defined by Nambisan reduces to multivariable H-function defined by Srivastava et al [7]. We have.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j} \aleph \left(\begin{array}{c} \text{z''}_1 (t-a)^{\mu_1+\mu_1'}(b-t)^{\rho_1+\rho_1'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z''}_v (t-a)^{\mu_v+\mu_v'}(b-t)^{\rho_v+\rho_v'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(v)}-\lambda_j'^{(v)}} \end{array} \right)$$

$$H \left(\begin{array}{c} \text{z}_1 (t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z}_s (t-a)^{\mu_s}(b-t)^{\rho_s} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(s)}} \end{array} \right) H \left(\begin{array}{c} \text{z'}_1 (t-a)^{\mu_1'}(b-t)^{\rho_1'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z'}_u (t-a)^{\mu_u'}(b-t)^{\rho_u'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(u)}} \end{array} \right)$$

$${}_PF_Q \left[(A_P); (B_Q); -\sum_{i=1}^{l} z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i} \prod_{j=1}^{k} (f_j t + g_j)^{-\zeta_j^{(i)}} \right] \mathrm{d}t = (b-a)^{\alpha+\beta-1}$$

$$= P_1 \frac{\prod_{j=1}^{Q} \Gamma(B_j)}{\prod_{j=1}^{P} \Gamma(A_j)} \sum_{h_1=1}^{M_1} \cdots \sum_{h_v=1}^{M_v} \sum_{k_1=0}^{\infty} \cdots \sum_{k_v=0}^{\infty} \sum_{R_1,\cdots,R_u=0}^{h_1 R_1+\cdots h_u R_u \leqslant L} \prod_{i=1}^{v} z_i'''^{\eta_{h_i,k_i}} \prod_{k=1}^{u} z''^{R_k} B_u B_{u,v}$$

$$H^{0,n'+n''+l+k+2;X}_{p'+p''+l+k+2,q'+q''+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}_1, K_1, K_2, K_P, K_j : C \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^k(af_j+g_j)^{\lambda'^{(u)}_j}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^k(af_j+g_j)^{\zeta_j^{(l)}}} & \mathrm{B}_1, L_1, L_j, L_Q, : D \end{array}\right) \quad (4.1)$$

under the same conditions and notations that (3.19) with $A'_j = B'_j = C'^{(i)}_j = D'^{(i)}_j = A''_j = B''_j = C''^{(i)}_j = D''^{(i)}_j = 1$

Remark

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions defined by Nambisan et al et al [2].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable I-functions, defined by Nambisan et al [2], a expansion of mltivariable Aleph-function and a generalized hypergeometric function with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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