On general Eulerian integral of certain product of multivariable Aleph-function, the classes of polynomials and generalized hypergeometric function

F.Y. AYANT[1]

1 Teacher in High School , France

ABSTRACT
The object of this paper is to establish an general Eulerian integral involving the product of the multivariable Aleph-function, the general classes of multivariable polynomials and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular case concerning the multivariable I-function defined by Sharma et al [3] and the Srivastava-Daoust polynomial [4].

Keywords: Eulerian integral, multivariable I-function, Lauricella function of several variables, multivariable Aleph-function, generalized hypergeometric function

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1.Introduction

In this paper, we evaluate a general Eulerian integral concerning the product of the multivariable Aleph-function, a generalized hypergeometric function and the classes of multivariable polynomials.

We define : $$\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}':m'_1,n'_1,\cdots,m'_r,n'_r}_{p'_i,q'_i,\tau'_i;R':p'_{i^{(1)}},q'_{i^{(1)}},\tau'_{i^{(1)}};R'^{(1)};\cdots;p'_{i^{(r)}},q'_{i^{(r)}};\tau'_{i^{(r)}};R'^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & \end{array}\right.$$

$$\begin{array}{l}[(\mathrm{a'}_j;\alpha'^{(1)}_j,\cdots,\alpha'^{(r)}_j)_{1,\mathfrak{n}'}] \;,[\tau'_i(a_{ji};\alpha'^{(1)}_{ji},\cdots,\alpha'^{(r)}_{ji})_{\mathfrak{n}'+1,p'_i}]:\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;,[\tau'_i(b_{ji};\beta'^{(1)}_{ji},\cdots,\beta'^{(r)}_{ji})_{m'+1,q'_i}]:\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{c'}^{(1)}_j);\gamma'^{(1)}_j)_{1,n'_1}],[\tau'_{i^{(1)}}(c'^{(1)}_{ji^{(1)}};\gamma'^{(1)}_{ji^{(1)}})_{n'_1+1,p'^{(1)}_i}];\cdots;[(\mathrm{c'}^{(r)}_j);\gamma'^{(r)}_j)_{1,n'_r}],[\tau'_{i^{(r)}}(c'^{(r)}_{ji^{(r)}};\gamma'^{(r)}_{ji^{(r)}})_{n'_r+1,p'^{(r)}_i}]\\ [(\mathrm{d'}^{(1)}_j);\delta'^{(1)}_j)_{1,m'_1}],[\tau'_{i^{(1)}}(d'^{(1)}_{ji^{(1)}};\delta'^{(1)}_{ji^{(1)}})_{m'_1+1,q'^{(1)}_i}];\cdots;[(\mathrm{d'}^{(r)}_j);\delta'^{(r)}_j)_{1,m'_r}],[\tau'_{i^{(r)}}(d'^{(r)}_{ji^{(r)}};\delta'^{(r)}_{ji^{(r)}})_{m'_r+1,q'^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}'}\Gamma(1-a'_j+\sum_{k=1}^{r}\alpha'^{(k)}_j s_k)}{\sum_{i=1}^{R}{}'[\tau'_i\prod_{j=\mathfrak{n}'+1}^{p'_i}\Gamma(a'_{ji}-\sum_{k=1}^{r}\alpha'^{(k)}_{ji}s_k)\prod_{j=1}^{q'_i}\Gamma(1-b'_{ji}+\sum_{k=1}^{r}\beta'^{(k)}_{ji}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m'_k}\Gamma(d'^{(k)}_j-\delta'^{(k)}_j s_k)\prod_{j=1}^{n'_k}\Gamma(1-c'^{(k)}_j+\gamma'^{(k)}_j s_k)}{\sum_{i^{(k)}=1}^{R'^{(k)}}[\tau'_{i^{(k)}}\prod_{j=m'_k+1}^{q'^{(k)}_i}\Gamma(1-d'^{(k)}_{ji^{(k)}}+\delta'^{(k)}_{ji^{(k)}}s_k)\prod_{j=n'_k+1}^{p'^{(k)}_i}\Gamma(c'^{(k)}_{ji^{(k)}}-\gamma'^{(k)}_{ji^{(k)}}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$a'_j, j = 1, \cdots, p'; b_j, j = 1, \cdots, q';$

$c'^{(k)}_j, j = 1, \cdots, n'_k; c'^{(k)}_{ji^{(k)}}, j = n'_k + 1, \cdots, p'_{i^{(k)}};$

$d^{(k)}_{ji^{(k)}}, j = m_k + 1, \cdots, q_{i^{(k)}}; d'^{(k)}_j, j = 1, \cdots, m'_k;$

with $k = 1 \cdots, r, i = 1, \cdots, R', i^{(k)} = 1, \cdots, R'^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U'^{(k)}_i = \sum_{j=1}^{\mathfrak{n}'} \alpha'^{(k)}_j + \tau'_i \sum_{j=\mathfrak{n}'+1}^{p'_i} \alpha'^{(k)}_{ji} + \sum_{j=1}^{n'_k} \gamma'^{(k)}_j + \tau'_{i^{(k)}} \sum_{j=n'_k+1}^{p'_{i^{(k)}}} \gamma'^{(k)}_{ji^{(k)}} - \tau'_i \sum_{j=1}^{q'_i} \beta'^{(k)}_{ji} - \sum_{j=1}^{m'_k} \delta'^{(k)}_j$$

$$-\tau'_{i^{(k)}} \sum_{j=m'_k+1}^{q'_{i^{(k)}}} \delta'^{(k)}_{ji^{(k)}} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R' , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R'^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d'^{(k)}_j - \delta'^{(k)}_j s_k)$ with $j = 1$ to m_k are separated from those $\Gamma(1 - a'_j + \sum_{i=1}^{r} \alpha'^{(k)}_j s_k)$of with $j = 1$ to n and $\Gamma(1 - c'^{(k)}_j + \gamma^{(k)}_j s'_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A'^{(k)}_i \pi$, where

$$A'^{(k)}_i = \sum_{j=1}^{\mathfrak{n}'} \alpha'^{(k)}_j - \tau'_i \sum_{j=\mathfrak{n}'+1}^{p'_i} \alpha^{(k)}_{ji} - \tau'_i \sum_{j=1}^{q'_i} \beta'^{(k)}_{ji} + \sum_{j=1}^{n'_k} \gamma'^{(k)}_j - \tau'_{i^{(k)}} \sum_{j=n'_k+1}^{p'_{i^{(k)}}} \gamma'^{(k)}_{ji^{(k)}}$$

$$+\sum_{j=1}^{m'_k} \delta'^{(k)}_j - \tau'_{i^{(k)}} \sum_{j=m'_k+1}^{q'_{i^{(k)}}} \delta'^{(k)}_{ji^{(k)}} > 0, \quad \text{with } k = 1, \cdots, r, i = 1, \cdots, R', i^{(k)} = 1, \cdots, R'^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$

where $k = 1, \cdots, r : \alpha'_k = min[Re(d'^{(k)}_j / \delta'^{(k)}_j)], j = 1, \cdots, m'_k$ and

$\beta'_k = max[Re((c'^{(k)}_j - 1)/\gamma'^{(k)}_j)], j = 1, \cdots, n'_k$

We will use these following notations in this paper

$$U = p'_i, q'_i, \tau'_i; R' \; ; \; V = m'_1, n'_1; \cdots ; m'_r, n'_r \tag{1.6}$$

$$\mathrm{W} = p'_{i^{(1)}}, q'_{i^{(1)}}, \tau'_{i^{(1)}}; R'^{(1)}, \cdots, p'_{i^{(r)}}, q'_{i^{(r)}}, \tau'_{i^{(r)}}; R'^{(r)} \tag{1.7}$$

$$A = \{(a'_j; \alpha'^{(1)}_j, \cdots, \alpha'^{(r)}_j)_{1,n'}\}, \{\tau'_i(a'_{ji}; \alpha'^{(1)}_{ji}, \cdots, \alpha'^{(r)}_{ji})_{n'+1,p'_i}\} \tag{1.8}$$

$$B = \{\tau'_i(b'_{ji}; \beta'^{(1)}_{ji}, \cdots, \beta'^{(r)}_{ji})_{m'+1,q'_i}\} \tag{1.9}$$

$$C = \{(c'^{(1)}_j; \gamma'^{(1)}_j)_{1,n'_1}\}, \tau'_{i^{(1)}}(c'^{(1)}_{ji^{(1)}}; \gamma'^{(1)}_{ji^{(1)}})_{n'_1+1,p'_{i^{(1)}}}\}$$

$$\{(c'^{(r)}_j; \gamma'^{(r)}_j)_{1,n'_r}\}, \tau'_{i^{(r)}}(c'^{(r)}_{ji^{(r)}}; \gamma'^{(r)}_{ji^{(r)}})_{n'_r+1,p'_{i^{(r)}}}\} \tag{1.10}$$

$$D = \{(d'^{(1)}_j; \delta'^{(1)}_j)_{1,m_1}\}, \tau'_{i^{(1)}}(d'^{(1)}_{ji^{(1)}}; \delta'^{(1)}_{ji^{(1)}})_{m'_1+1,q'_{i^{(1)}}}\}, \cdots$$

$$, \{(d'^{(r)}_j; \delta'^{(r)}_j)_{1,m'_r}\}, \tau'_{i^{(r)}}(d'^{(r)}_{ji^{(r)}}; \delta'^{(r)}_{ji^{(r)}})_{m'_r+1,q'_{i^{(r)}}}\} \tag{1.11}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_r) = \aleph^{0,\mathfrak{n}':V}_{U:W}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A : C} \\ \cdot & \\ \cdot & \cdots \\ \cdot & \\ \mathrm{z}_r & \mathrm{B : D} \end{array}\right) \tag{1.12}$$

The generalized polynomials of multivariables defined by Srivastava [6], is given in the following manner :

$$S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_\mathfrak{v}}_{N_1,\cdots,N_v}[y_1, \cdots, y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_\mathfrak{v}]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_\mathfrak{v} K_v}}{K_v!}$$

$$A[N_1, K_1; \cdots ; N_v, K_v] y_1^{K_1} \cdots y_v^{K_v} \tag{1.13}$$

where $\mathfrak{M}_1, \cdots, \mathfrak{M}_\mathfrak{v}$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots ; N_v, K_v]$ are arbitrary constants, real or complex.

Srivastava and Garg [6] introduced and defined a general class of multivariable polynomials as follows

$$S^{h_1,\cdots,h_u}_L[z_1, \cdots, z_u] = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} (-L)_{h_1R_1+\cdots+h_uR_u} B(E; R_1, \cdots, R_u) \frac{z_1^{R_1} \cdots z_u^{R_u}}{R_1! \cdots R_u!} \tag{1.14}$$

The coefficients are $B[E; R_1, \ldots, R_v]$ arbitrary constants, real or complex.

We will note $a_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_\mathfrak{v} K_v}}{K_v!} A[N_1, K_1; \cdots ; N_v, K_v]$ and

$$b_u = \frac{(-E)_{F_1L_1+\cdots+F_uL_u} B(E; L_1, \cdots, L_u)}{L_1! \cdots L_u!} \tag{1.15}$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [7 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t\ =(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right) \tag{2.2}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i,\tau_j,h_j\in\mathbb{C},\lambda_j\in\mathbb{R}^+(i=1,\cdots,k;j=1,\cdots,l)$

$$min(Re(\alpha),Re(\beta))>0,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1,\ \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,$$

and $F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[5,page 454] given by :

$$F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\prod_{j=1}^{l}\Gamma(\lambda_j)\prod_{j=1}^{k}\Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}}\int_{L_1}\cdots\int_{L_{l+k}}\frac{\Gamma\left(\alpha+\sum_{j=1}^{l}h_jw_j+\sum_{j=1}^{k}w_{l+j}\right)}{\Gamma\left(\alpha+\beta+\sum_{j=1}^{l}h_jw_j+\sum_{j=1}^{k}w_{l+j}\right)}\prod_{j=1}^{l}\Gamma(\lambda_j+w_j)\prod_{j=1}^{k}\Gamma(-\sigma_j+w_{l+j})$$

$$\prod_{j=1}^{l+k}\Gamma(-w_j)z_1^{w_1}\cdots z_l^{w_l}z_{l+1}^{w_{l+1}}\cdots,w_{l+k}^{w_{l+k}}\,\mathrm{d}w_1\cdots\mathrm{d}w_{l+k} \tag{2.3}$$

Here the contour $L_j's$ are defined by $L_j=L_{w\zeta_j\infty}(Re(\zeta_j)=v_j'')$ starting at the point $v_j''-\omega\infty$ and terminating at the point $v_j''+\omega\infty$ with $v_j''\in\mathbb{R}(j=1,\cdots,l)$ and each of the remaining contour $L_{l+1},\cdots,L_{l+k}$ run from $-\omega\infty$ to $\omega\infty$

(2.2) can be easily established by expanding $\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [1, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [5, page 454].

3. Eulerian integral

In this section , we evaluate a general Eulerian integral with the product of two multivariable Aleph-functions, class of multivariable polynomials and generalized hypergeometric function. We note

$$\theta_i=\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}},\zeta_j^{(i)}>0(i=1,\cdots,r)\,;\,\theta'_i=\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{\prime(i)}},\zeta_j^{\prime(i)}>0(i=1,\cdots,s)$$

$$\theta''_i=\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{\prime\prime(i)}},\zeta_j^{\prime\prime(i)}>0(i=1,\cdots,u)$$

$$\theta'''_i=\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{\prime\prime\prime(i)}},\zeta_j^{\prime\prime\prime(i)}>0(i=1,\cdots,v) \tag{3.1}$$

$$K_1=(1-\alpha-\sum_{i=1}^{u}R_ia_i-\sum_{i=1}^{v}K_ia'_i;\mu_1,\cdots,\mu_r,\mu'_1,\cdots,\mu'_s,h_1,\cdots,h_l,1,\cdots,1) \tag{3.2}$$

$$K_2=(1-\beta-\sum_{i=1}^{u}R_ib_i-\sum_{i=1}^{v}K_ib'_i;\rho_1,\cdots,\rho_r,\rho'_1,\cdots,\rho'_s,0,\cdots,0,0\cdots,0) \tag{3.3}$$

$$K_P=[1-A_j;0,\cdots,0,1,\cdots,1,0,\cdots,0,0,\cdots,0]_{1,P} \tag{3.4}$$

$$K_j=[1-\lambda_j-\sum_{i=1}^{u}R_i\zeta_j^{\prime\prime(i)}-\sum_{i=1}^{v}K_i\zeta_j^{\prime\prime\prime(i)};\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j^{\prime(1)}\cdots,\zeta_j^{\prime(s)},$$

$$0,\cdots,\underset{\text{j}}{1},\cdots,0,0\cdots,0]_{1,l} \tag{3.5}$$

$$K'_j=[1+\sigma_j-\sum_{i=1}^{u}R_i\lambda_j^{\prime\prime(i)}-\sum_{i=1}^{v}K_i\lambda_j^{\prime\prime\prime(i)};\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j^{\prime(1)}\cdots,\lambda_j^{\prime(s)},$$

$$0,\cdots,0,0\cdots,\underset{\text{j}}{1},\cdots,0]_{1,k} \tag{3.6}$$

$$L_1=(1-\alpha-\beta-\sum_{i=1}^{u}R_i(a_i+b_i)-\sum_{i=1}^{v}K_i(a'_i+b'_i);\mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu'_1+\rho'_1,\cdots,\mu'_r+\rho'_r,$$

$$h_1,\cdots,h_l,1,\cdots,1) \tag{3.7}$$

$$L_Q=[1-B_j;0,\cdots,0,1,\cdots,1,0,\cdots,0,0\cdots,0]_{1,Q} \tag{3.8}$$

$$L_j=[1-\lambda_j-\sum_{i=1}^{u}R_i\zeta_j^{\prime\prime(i)}-\sum_{i=1}^{v}K_i\zeta_j^{\prime\prime\prime(i)};\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j^{\prime(1)}\cdots,\zeta_j^{\prime(s)},0,\cdots,0,0\cdots,0]_{1,l} \tag{3.9}$$

$$L'_j = [1+\sigma_j - \sum_{i=1}^{u} R_i \lambda''^{(i)}_j - \sum_{i=1}^{v} K_i \lambda'''^{(i)}_j; \lambda^{(1)}_j, \cdots, \lambda^{(r)}_j, \lambda'^{(1)}_j \cdots, \lambda'^{(s)}_j, 0, \cdots, 0, 0, \cdots, 0]_{1,k} \quad (3.10)$$

$$V_1 = V; 1,0; \cdots; 1,0; 1,0; \cdots; 1,0; 1,0; \cdots; 1,0 \ ; W_1 = W; 0,1; \cdots; 0,1; 0,1; \cdots; 0,1; 0,1; \cdots; 0,1 \quad (3.11)$$

$$C_1 = C; (1,0), \cdots, (1,0); (1,0), \cdots, (1,0); (1,0); \cdots; (1,0);$$

$$D_1 = D; (0,1), \cdots, (0,1); (0,1), \cdots, (0,1); (0,1); \cdots; (0,1) \quad (3.12)$$

V, W, C and D are defined by (1.6), (1.7), (1.10) and (1.11) respectively

$$B_{u,v} = (b-a)^{\sum_{i=1}^{v} K_i(a'_i+b'_i)+\sum_{i=1}^{u}(a_i+b_i)R_i} \left\{ \prod_{j=1}^{h} (af_j+g_j)^{-\sum_{i=1}^{v} K_i \lambda'''_i - \sum_{i=1}^{u} \lambda''_i R_i} \right\} \quad (3.13)$$

We have the general Eulerian integral

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{l} [1-\tau_j(t-a)^{h_i}]^{-\lambda_j} \prod_{j=1}^{k} (f_j t+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u} \begin{pmatrix} z''_1 \theta''_1 (t-a)^{a_1}(b-t)^{b_1} \prod_{j=1}^{k}(f_j t+g_j)^{-\lambda''^{(1)}_j} \\ \cdot \\ \cdot \\ \cdot \\ z''_u \theta''_u (t-a)^{a_u}(b-t)^{b_u} \prod_{j=1}^{k}(f_j t+g_j)^{-\lambda''^{(u)}_j} \end{pmatrix}$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v} \begin{pmatrix} z'''_1 \theta'''_1 (t-a)^{a'_1}(b-t)^{b'_1} \prod_{j=1}^{k}(f_j t+g_j)^{-\lambda'''^{(1)}_j} \\ \cdot \\ \cdot \\ \cdot \\ z'''_v \theta'''_v (t-a)^{a'_v}(b-t)^{b'_v} \prod_{j=1}^{k}(f_j t+g_j)^{-\lambda'''^{(v)}_j} \end{pmatrix}$$

$$\aleph \begin{pmatrix} z_1 \theta_1 (t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^{k}(f_j t+g_j)^{-\lambda^{(1)}_j} \\ \cdot \\ \cdot \\ \cdot \\ z_r \theta_r (t-a)^{\mu_r}(b-t)^{\rho_r} \prod_{j=1}^{k}(f_j t+g_j)^{-\lambda^{(r)}_j} \end{pmatrix}$$

$${}_PF_Q \left[(A_P); (B_Q); -\sum_{i=1}^{s} z'_i \theta'_i (t-a)^{\mu'_i}(b-t)^{\rho'_i} \prod_{j=1}^{k} (f_j t+g_j)^{-\lambda'^{(i)}_j} \right] \mathrm{d}t =$$

$$(b-a)^{\alpha+\beta-1} \frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)} \prod_{j=1}^{h} (af_j+g_j)^{\sigma_j} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} \prod_{i=1}^{v} z'''^{K_i}_i \prod_{k=1}^{u} z''^{R_k} a_v b_u B_{u,v}$$

$$\aleph^{0,n+P+k+k+2:V_1}_{U_{P+l+k+2,Q+k+l+1}:W_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\mathrm{K}_1,K_2,K_P,K_j,K'_j,C_1 \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^k(af_j+g_j)^{\lambda'^{(s)}_j}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \mathrm{B}\ ,\mathrm{L}_1,L_Q,L_j,L'_j,D_1 \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (3.14)$$

where $U_{P+k+l+2,P+k+l+1} = U + p'_i + P + k + l + 2, q'_i + Q + k + l + 1, \tau'_i; R'$

This result is an extansion the formula given by Saxena et al [2]. Provided that

(A) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\} < 1, \ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\} < 1,$

(B) $Re[\alpha + \sum_{i=1}^r \mu_i \min_{1\leqslant j\leqslant m'_i} \frac{d'^{(i)}_j}{\delta'^{(i)}_j}] > 0$ and $Re[\beta + \sum_{i=1}^r \rho_i \min_{1\leqslant j\leqslant m'_i} \frac{d'^{(i)}_j}{\delta'^{(i)}_j}] > 0$

(C) $Re\left(\alpha + \sum_{i=1}^v K_i a'_i + \sum_{i=1}^u R_i a_i + \sum_{i=1}^r \mu_i s_i + \sum_{i=1}^l h_i w_i\right) > 0\,; Re\left(\beta + \sum_{i=1}^v K_i b'_i + \sum_{i=1}^u R_i b_i + \sum_{i=1}^r \rho_i s_i\right) > 0$

$Re\left(\lambda_j + \sum_{i=1}^v K_i \lambda'''^{(i)}_j + \sum_{i=1}^u R_i \lambda''^{(i)}_j + \sum_{i=1}^r s_i \zeta^{(i)}_j\right) > 0 (j=1,\cdots,l)$

$Re\left(-\sigma_j + \sum_{i=1}^v K_i \lambda'''^{(i)} + \sum_{i=1}^u R_i \lambda''^{(i)}_j + \sum_{i=1}^r s_i \lambda^{(i)}_j\right) > 0 (j=1,\cdots,k)$

(D) $A'^{(k)}_i = \sum_{j=1}^{\mathfrak{n}'} \alpha'^{(k)}_j - \tau'_i \sum_{j=\mathfrak{n}'+1}^{p'_i} \alpha^{(k)}_{ji} - \tau'_i \sum_{j=1}^{q'_i} \beta'^{(k)}_{ji} + \sum_{j=1}^{n'_k} \gamma'^{(k)}_j - \tau'_{i(k)} \sum_{j=n'_k+1}^{p'_{i(k)}} \gamma'^{(k)}_{ji(k)}$

$$+\sum_{j=1}^{m'_k}\delta'^{(k)}_j - \tau'_{i(k)}\sum_{j=m'_k+1}^{q'_{i(k)}}\delta'^{(k)}_{ji^{(k)}} - \mu_i - \rho_i - \sum_{l=1}^{k}\lambda_l^{(i)} > 0 \quad (i=1,\cdots,r)$$

(E) $$\left|arg\left(z_i\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right| < \frac{1}{2}A_i'^{(k)}\pi \quad (a\leqslant t\leqslant b; i=1,\cdots,r)$$

(F) $P\leqslant Q+1$. The equality holds, when , in addition,

either $P>Q$ and $\left|\left(z'_i\sum_{i=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right|^{\frac{1}{Q-P}} <1 \ (a\leqslant t\leqslant b)$

or $P\leqslant Q$ and $\max_{1\leqslant i\leqslant k}\left[\left|\left(z'_i\sum_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right|\right] < 1 \ (a\leqslant t\leqslant b)$

Proof

To prove (3.14), first expressing a class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ defined by Srivastava [4] in serie with the help of (1.13), a class of multivariable polynomials $S_L^{h_1,\cdots,h_u}[.]$ defined by Srivastava et al [6] in serie with the help of (1.14) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the Aleph-functions of r-variables in terms of Mellin-Barnes type contour integral with the help of (1.1)and the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Now collect the power of $\left[1-\tau_j(t-a)^{h_i}\right]$ with $(i=1,\cdots,r;j=1,\cdots,l)$ and collect the power of (f_jt+g_j) with $j=1,\cdots,k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral to multivariable Aleph-function, we obtain the equation (3.14).

4. Particular cases

a) If $\tau'_i,\tau'_{i(1)},\cdots,\tau'_{i(r)} \rightarrow 1$, the multivariable Aleph-function of s-variables reduces to multivariable I-function of s-variables defined by Sharma and al [3] and we have

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\begin{pmatrix} z''_1\theta''_1(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z''_u\theta''_u(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}}\end{pmatrix}$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} z'''_1\theta'''_1(t-a)^{a'_1}(b-t)^{b'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z'''_v\theta'''_v(t-a)^{a'_v}(b-t)^{b'_v}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(v)}}\end{pmatrix}$$

$$I\begin{pmatrix} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{pmatrix}$$

$$ {}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{s} z_i'\theta_i'(t-a)^{\mu_i'}(b-t)^{\rho_i'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right]\mathrm{d}t=$$

$$(b-a)^{\alpha+\beta-1}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}(af_j+g_j)^{\sigma_j}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}z_i'''^{K_i}\prod_{k=1}^{u}z''^{R_k}a_vb_uB_{u,v}$$

$$I_{U_{P+l+k+2,Q+k+l+1}:W_1}^{0,n+P+k+k+2:V_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_P,K_j,K_j',C_1 \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \mathrm{B}\ ,\ \mathrm{L}_1,L_Q,L_j,L_j',D_1 \end{array}\right) \quad (4.1)$$

under the same conditions and notations that (3.14) with $\tau_i',\tau_{i^{(1)}}',\cdots,\tau_{i^{(r)}}' \to 1$

b) If $B(L;R_1,\cdots,R_u) = \dfrac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(u)}}(b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{R_1\psi'_j+\cdots+R_u\psi_j^{(u)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(u)}}(d_j^{(u)})_{R_u\delta_j^{(u)}}}$ (4.2)

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]$ reduces to generalized Lauricella function defined by Srivastava et al [5]. We have

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c} z''_1\theta''_1(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z''_u\theta''_u(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}} \end{array}\right|$$

$$\left.\begin{array}{c} [(\text{-L});\text{R}_1,\cdots,R_u][(a);\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{(u)});\phi^{(u)}] \\ [(\text{c});\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)});\delta^{(u)}] \end{array}\right)$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c} z'''_1\theta'''_1(t-a)^{a'_1}(b-t)^{b'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z'''_v\theta'''_v(t-a)^{a'_v}(b-t)^{b'_v}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(v)}} \end{array}\right)$$

$$\aleph\left(\begin{array}{c} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{s}z'_i\theta'_i(t-a)^{\mu'_i}(b-t)^{\rho'_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right]\text{dt} =$$

$$(b-a)^{\alpha+\beta-1}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}(af_j+g_j)^{\sigma_j}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_\mathfrak{v}]}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}z_i'''^{K_i}\prod_{k=1}^{u}z''^{R_k}a_v b'_u B_{u,v}$$

$$\aleph_{U_{P+l+k+2,Q+k+l+1}:W_1}^{0,n+P+k+k+2:V_1}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_P,K_j,K'_j,C_1 \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(s)}_j}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \\ \cdots & \mathrm{B}\ ,\ \mathrm{L}_1,L_Q,L_j,L'_j,D_1 \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \tag{4.3}$$

under the same conditions and notations that (3.14)

where $b'_u=\frac{(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!}$, $B[E;R_1,\ldots,R_v]$ is defined by (4.2)

Remark:

By the following similar procedure, the results of this document can be extented a class of multivariable polynomials defined by Srivastava et al [6] and Srivastava [4]. The formula (3.14) is an extension of result concerning the multivariable H-function defined by Srivastava et al [8]. For more details, see Saigo et al [2].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of the multivariable Aleph-function, the classes of multivariable polynomials and generalized hypergeometric function with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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Personal adress : 411 Avenue Joseph Raynaud
Le parc Fleuri , Bat B
83140 , Six-Fours les plages
Tel : 06-83-12-49-68
Department : VAR
Country : FRANCE