On $W_4 - \varphi$ −Recurrent Trans-Sasakian Manifolds

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Abstract: *The aim of the present paper is to study on a type of $W_4 - \varphi -$recurrent trans-Sasakian manifolds.*

Keywords: *Trans-Sasakian manifold, W_4 curvature tensor, Locally $\varphi -$ symmetric trans-Sasakian manifold, Characteristic vector field.*

I. Introduction

T. Takahashi [12] introduced the notion of locally $\varphi -$symmetric Sasakian manifold in 1977. U.C. De et. al.[14] studied the $\varphi -$recurrent Sasakian manifold. Also a new class of almost contact metric structures which was a generalization of Sasakian [2], $\alpha -$ Sasakian [6], Kenmotsu [6], $\beta -$ Kenmotsu [6] and cosympletic [6] manifolds, which was called trans-Sasakian manifold [8] was introduced by J. A. Oubinain in 1985. Later on many authors ([3], [4], [5], [6], [9], [10], [11], [12]) have studied various type of properties in trans-Sasakian manifold.

In the present paper Section−2 is concerned with preliminaries. Section−3 is devoted to the study of $W_4 - \varphi -$recurrent trans-Sasakian manifold which satisfies the condition $\varphi \text{grad}(\alpha) = (2n-1)\text{grad}\beta$, and proved that such a manifold is an Einstein manifold.

It is shown that in a conformal $W_4 - \varphi -$recurrent trans-Sasakian manifold (M^{2n+1}, g), $n \geq 1$, the characteristic vector field ξ and the vector field ρ associated to the $1-$form A are co-directional.

II. Preliminaries

A $(2n+1)$ dimensional, $(n \geq 1)$ almost contact metric manifold M with almost contact metric structure (φ, ξ, η, g), where φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \varphi(\xi) = 0,\ \eta o \varphi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in \chi(M)$, is called trans-Sasakian manifold [1] if and only if

$$(\nabla_X \varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi(X), \tag{2.4}$$

for some smooth functions α and β on M. From (2.4) it follows that

$$\nabla_X \xi = -\alpha \varphi X + \beta(X - \eta(X)\xi), \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y). \tag{2.6}$$

In [12], the authors obtained some results which shall be useful for next section. They are

$$R(X,Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) + (Y\alpha)\varphi X$$

$$-(X\alpha)\varphi Y+(Y\beta)\varphi^2X-(X\beta)\varphi^2Y, \tag{2.7}$$
$$R(\xi,X)\xi=(\alpha^2-\beta^2-\xi\beta)(\eta(X)\xi-X), \tag{2.8}$$
$$2\alpha\beta+\xi\alpha=0, \tag{2.9}$$

$$S(X,\xi)=(2n(\alpha^2-\beta^2)-\xi\beta)\eta(X)-(2n-1)X\beta-(\varphi X)\alpha, \tag{2.10}$$

$$Q\xi=(2n(\alpha^2-\beta^2)-\xi\beta)\xi-(2n-1)\text{grad}\beta+\varphi(\text{grad}\alpha). \tag{2.11}$$

When $\varphi\text{grad}(\alpha)=(2n-1)\text{grad}\beta$, then (2.10) and (2.11) reduces to
$$S(X,\xi)=2n(\alpha^2-\beta^2)\eta(X), \tag{2.12}$$
$$Q\xi=2n(\alpha^2-\beta^2)\xi. \tag{2.13}$$

A trans-Sasakian manifold is said to be locally φ −symmetric [12] if
$$\varphi^2((\nabla_W R)(X,Y)Z)=0, \tag{2.14}$$

for all vector fields X,Y,Z,W orthogonal to ξ.
A trans-Sasakian manifold is said to be $W_4-\varphi$-recurrent manifold if there exists a non-zero 1-form A such that
$$\varphi^2((\nabla_W W_4)(X,Y)Z)=A(W)W_4(X,Y)Z, \tag{2.15}$$

for $X,Y,Z,W\in\chi(M)$, where the 1-form A is defined as
$$g(X,\rho)=A(X),\forall X\in\chi(M), \tag{2.16}$$

ρ being the vector field associated to the 1-form A and W_4 is a W_4 curvature tensor given by [7]
$$W_4(X,Y)Z=R(X,Y)Z-\frac{1}{n-1}[g(Y,Z)QX-g(X,Z)QY] \tag{2.17}$$

where R is the curvature tensor, S is the Ricci-tensor and $\mathbf{r}$ is the scalar curvature. Also,
$$g(QX,Y)=S(X,Y), \tag{2.18}$$
Q being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S.
The above results will be useful in the next section.

III. $W_4-\varphi$-recurrent Trans-Sasakian manifold

In this section let us consider a trans-Sasakian manifold which is $W_4-\varphi$-recurrent. Then by equation (2.1) and (2.15) we have
$$-(\nabla_W W_4)(X,Y)Z+\eta((\nabla_W W_4)(X,Y)Z)\xi=A(W)W_4)(X,Y)Z. \tag{3.1}$$

From (3.1) it follows that

$$-g((\nabla_W W_4)(X,Y)Z,U)+\eta((\nabla_W W_4)(X,Y)Z)\eta(U)=A(W)g(W_4)(X,Y)Z,U). \tag{3.2}$$

Let $\{e_i\}$, $i=1,2,\dots,2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $X=U=\{e_i\}$, in (3.2) and taking summation over i, $1\le i\le 2n+1$, we have
$$\nabla_W S(Y,Z)=-\frac{1}{n(n-1)}[(r-2n(\alpha^2-\beta^2-\xi\beta))\nabla_W g(Y,Z)+2n(\alpha^2-\beta^2-\xi\beta)\nabla_W\eta(Y)\eta(Z)]$$
$$-A(W)\frac{1}{n(n-1)}\left[nS(Y,Z)-\left(r-2n(\alpha^2-\beta^2-\xi\beta)\right)g(Y,Z)+2n(\alpha^2-\beta^2-\xi\beta)\eta(Y)\eta(Z)\right] \tag{3.3}$$

Replacing Z by ξ and using (2.1), (2.3) and (2.12) we get
$$\nabla_W S(Y,\xi)=-\frac{1}{n(n-1)}(r\nabla_W\eta(Y)\eta(Z))-A(W)\frac{1}{n(n-1)}[2n(\alpha^2-\beta^2)-r]\nabla_W\eta(Y) \tag{3.4}$$

We know that

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{3.5}$$

Using (2.5) and (2.12) in the above quation (3.5) we get,

$$(\nabla_W S)(Y,\xi) = 2n(\alpha^2-\beta^2)[-\alpha g(\varphi W,Y) + \beta g(W,Y)] + \alpha S(Y,\varphi W) - \beta S(Y,W) \tag{3.6}$$

Using (3.6) in (3.4), we obtain

$$2n(\alpha^2-\beta^2)[-\alpha g(\varphi W,Y) + \beta g(W,Y)] + \alpha S(Y,\varphi W) - \beta S(Y,W)$$
$$= -\frac{1}{n(n-1)}(r\nabla_W \eta(Y)\eta(Z)) - A(W)\frac{1}{n(n-1)}[2n(\alpha^2-\beta^2) - r]\nabla_W \eta(Y). \tag{3.7}$$

Replacing Y and W by φY and φW respectively, we get

$$S(Y,W) = 2n(\alpha^2-\beta^2)g(Y,W) \tag{3.8}$$

and

$$S(\varphi Y,W) = 2n(\alpha^2-\beta^2)g(\varphi Y,W). \tag{3.9}$$

Hence we can state the following theorem:

Theorem 3.1. A $W_4-\varphi$-recurrent trans-Sasakian manifold (M^{2n+1},g) satisfying $\varphi\,\mathrm{grad}\,(\alpha) = (2n-1)\mathrm{grad}\beta$, is an Einstein manifold.

Now from (3.1) and (2.16) we have

$$(\nabla_W R)(X,Y)Z = +\eta((\nabla_W R)(X,Y)Z)\xi - \frac{1}{(n-1)}[(\nabla_W g)(Y,Z)\eta(QX)$$
$$-(\nabla_W g)(X,Z)\eta(QZ)\xi - A(W)[R(X,Y)Z$$
$$-\frac{1}{(n-1)}[g(Y,Z)QX - g(X,Z)QZ] + \frac{1}{(n-1)}[(\nabla_W g)(Y,Z)QX$$
$$-(\nabla_W g)(X,Y)QZ]. \tag{3.10}$$

Using Bianchi's identity in (3.10) and putting $Y = Z = \{e_i\}$, where e_i be an orthonormal basis of the tangent space at any point of the manifold, and taking summation over i, $1 \le i \le 2n+1$, we obtain

$$A(W)\eta(X)) = A(X)\eta(W) \tag{3.11}$$

Putting again $X = \xi$ and using (2.1) and (2.3) we obtain

$$A(W) = \eta(W)\eta(\rho), \tag{3.12}$$

for any vector field W and ρ being the vector field associated to the 1-form A, defined as (2.16). Thus we can state the following theorem:

Theorem 3.2. In a $W_4-\varphi$-recurrent trans-Sasakian manifold (M^{2n+1},g), $n \ge 1$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are opposite directional and the 1-form A is given by

$$A(W) = \eta(W)\eta(\rho).$$

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