Evolution of geometric properties on solution curve of KdV equation

Ling XU[#1]
[#]*Collage of Sciences of Jiangsu University*
China ,Jiangsu ,Zhenjiang

Abstract---*This paper discusses some properties of solution curve of the Cauchy problem on the KdV equation by Lagrange coordinate, obtained the evolution consequence of monotonicity, concavity and the isolated extreme points. Namely under the enough smooth condition, the isolated extreme points , monotonicity and convexity of initial solution can be inherited to the solution curve at any* $t > 0$.

Keywords: *KdV equation ; solution curve ; monotonicity ; convexity ; isolated extreme point.*

Introduction

No matter in mathematical theories or application field ,the KdV equation is a very important equation. This equations are used to describe the shallow water wave phenomenon with small amplitude in physical systems.The background of the equation goes back to the concept of solitary wave is put forward .Solitary wave was first found by Russell in 1834 . Until 1895,Korteweg and de Vries researched the Shallow water wave and established a one-dimensional mathematical model under the assumption of small amplitude and the long wave approximation.The equation they established is $\eta_t + 6\eta\eta_x + \eta_{xxx} = 0$,which is called the KdV equation.

Many important results on KdV equation had be obtained. In 1983,Kato proved that the $\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, x,t \in R \\ u(x,0) = \varphi \end{cases}$ have only one solution $u \in C(IT; H^s) \cap C^1(IT; H^{s-3})$ which was only decided by $|\varphi|_1$ （$|\ |_s$ means $H^s - norm$） with $IT \equiv [0,T]$, $T > 0$ in[1] .

Suppose $A(r)$ is a Fréchet space if for some $r_0 > 0$, $\varphi \in A(r_0)$, there is a $r_1 > 0$ and the solution of problem $u \in C(IT; A(r_1))$ in [2].

In 1970 ,Anders founded the KdV equation $u_t = uu_x + \delta u_{xxx}$ has only one solution if $f(x)$ is a period function with the period 1 and $f_{xxx}(x) \in L_2$,where $\delta \neq 0$, $u(x,t) = u(x+1,t)$ and $u(x,0) = f(x)$ in [3].

In 1991,Kenig,Ponce and Vega proved that the $\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, x,t \in R \\ u(x,0) = u_0(x) \end{cases}$ is local well-posed in $C([-T,T]: H^s(R))$ when $s > -\frac{3}{4}$ in[4].

For the high order KdV equation and the discrete KdV equation ,researchers also get many results . Readers refer to see[5],[7] and [8]and those references within.

In 2011,Hannah，Himonas and Petronilho studied the regularity of period gKdV equation during the Gevrey space and the corresponding results are obtained, the norm of $G^{\delta,s}$ is defined by $\|u_0\|^2_{G^{\sigma,s}} = \int (1+|\xi|)^{2s} e^{2\sigma(1+|\xi|)} |\hat{u}_0(\xi)|^2 d\xi$ in [9].

On the other hand, by studying the relationship between the point on the curve and the time variable, we can obtain the evolution process of the solution curve or the solution surface with time. It is meaningful to understand the equations and its related problems.

Gage proved that each level curve of evolution equation like $X_t(t,\varphi) = k(t,\varphi)N(t,\varphi)$ converges to a circle before disappearing in[10].

Marcos and Ralph studied the curvature equation,which is given by

$$\frac{\partial u}{\partial t} = \frac{\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial y}\right)^2 - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial x}\right)^2}{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}.$$

The initial curve u(x,y,0) has a local isolated extremum (x_m ,y_m) and assume that the level curves near the extremum are closed and convex , then the solution u(x,y,t) admits a local extremum ($x_m(t)$,$y_m(t)$) close to(x_m ,y_m)for any small t>0 in [11].

In this paper ,we will study the following problem :

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

where (x,y) are the space variables and t the time variable. We will investigate the process of evolution of monotonicity, isolated extreme point and concavity of the solution curve. This discussion will be helpful for solving practical problem

In order to the accuracy and convenience, we first review the following concepts .

Definition 1.1.*The x_0 is called maximum point(minimum point) of f if $\forall x \in U^{\circ}(x_0,h) \subset I$, $f(x) < f(x_0)$ (or $f(x) > f(x_0)$) ,where I is the domain interval of function f .*

Definition 1.2*.The U is called the monotone increasing interval of $f(x)$ if $\forall x_1, x_2 \in U$, $x_1 < x_2$,then $f(x_1) < f(x_2)$.*

Definition1.3*.The function f is called convex function if $\forall x_1, x_2 \in I$, $\lambda \in (0,1)$*

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2) \tag{1}$$

Conversely the function f *is called concave function if* $\forall x_1, x_2 \in I$, $\lambda \in (0,1)$

$$f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda) f(x_2) \qquad (2)$$

where I *is the interval of function* f .

In the next ,we will prove that under the assumption of sufficient smooth, the isolated extreme point,local monotonicity and local concavity of initial solution curve could be inherited at any t>0.

Theorem and proving

First , introduce following lemmas:

Lemma 2.1[12] $\begin{cases} u_t + u^k u_x + u_{xxx} = 0, t \in R, x \in R, k \in N \\ u(0) = \varphi \end{cases}$, $u \in H^s(R)$ *is global well-posed when* $s \geq 1$, $k = 1,2,3$.

Lemma 2.2. *Assume* $u_0 \in H^4(R)$ *,if* $u(x,t)$ *is the solution of* (1.1) *, then for any* $t > 0$ *,* $\|u(x,t)\|_{H^4}$ *is bounded .*

Proof. Do product on the left of the first equation of (1.1) by $u, u_{xx}, u_{x^4}, u_{x^6}$ and u_{x^8} respectively, then integrate them, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^4}^2 = \int_R \left(uu_x u_{xx} - uu_x u_{x^4} + 3u_x u_{xx} u_{x^4} + uu_{xxx} u_{x^4} - 10 u_{xx} u_{xxx} u_{x^4} - \frac{7}{2} u_x u_{x^4}^2 \right) dx \qquad (3)$$

$$\frac{d}{dt}\|u\|_{H^4}^2 \leq C \int_R |uu_x u_{xx}| + |uu_x u_{x^4}| + |u_x u_{xx} u_{x^4}| + |uu_{xxx} u_{x^4}| + |u_{xx} u_{xxx} u_{x^4}| + |u_x u_{x^4}^2| dx \qquad (4)$$

$$\|u\|_{H^4}^2 \geq \max_{0 \leq i \leq 4, 0 \leq j \leq 4, i, j \in z} \left\{ \|u_{x^i}\| \|u_{x^j}\| \right\}$$

$$\frac{d}{dt}\|u\|_{H^4}^2 \leq C \|u\|_{H^4}^4$$

$$\|u(x,t)\|_{H^4}^2 \leq \frac{\|u(x,0)\|_{H^4}^2}{1 - C\|u(x,0)\|_{H^4}^2 t}, \quad C \geq 20 \qquad (5)$$

Let $t_1 = \frac{1}{2C\|u(x,0)\|_{H^4}^2}$, then

$$\|u(x,t)\|_{H^4}^2 \leq 2\|u(x,0)\|_{H^4}^2$$

$$\exists t_1, 0 < t \leq t_1, \|u(x,t)\|_{H^4} < +\infty$$

When $t > t_1$, use t_1 replace $t = 0$,so we have

$$\|u(x,t)\|_{H^4}^2 \le \frac{\|u(x,t_1)\|_{H^4}^2}{1 - C\|u(x,t_1)\|_{H^4}^2 (t - t_1)} \quad (6)$$

.

Let $t - t_1 = \dfrac{1}{2C\|u(x,t_1)\|_{H^4}^2}$, so $\|u(x,t)\|_{H^4}^2 \le 2\|u(x,t_1)\|_{H^4}^2$.

Let $t_2 = \dfrac{1}{2C\|u(x,t_1)\|_{H^4}^2} + t_1$, then $\|u(x,t)\|_{H^4} < +\infty$ for all $t_1 < t < t_2$.

Repeat the steps above, we find there always have

$$\|u(x,t)\|_{H^4}^2 \le \frac{\|u(x,t_n)\|_{H^4}^2}{1 - C\|u(x,t_n)\|_{H^4}^2 (t - t_n)},$$

$$t - t_n = \frac{1}{2C\|u(x,t_n)\|_{H^4}^2}, t_{n+1} = \frac{1}{2C\|u(x,t_n)\|_{H^4}^2} + t_n.$$

For all $t_n < t < t_{n+1}$, $\|u(x,t)\|_{H^4}^2 \le 2\|u(x,t_n)\|_{H^4}^2$,that means $\|u(x,t)\|_{H^4} < +\infty$.

So for any t_n, $\exists \Delta t_n$, when $t \in U(t_n, \Delta t_n)$, $\|u(x,t)\|_{H^4} < +\infty$ if $u_0(x) \in H^4(R)$.

We denote

$$w(u,y,x,z,t) = u_{yyy}(\lambda(y(x,t) + y(z,t)),t) - \lambda(u(y(x,t),t) - u(y(z,t),t)) \quad (7)$$

If $u_0 \in H^4(R)$,then $w(u,y,x,z,t)$ is bounded.

In order to explain and understand the process of evolution of curve of KdV equation ,we will use Lagrange coordinates

$$\begin{cases} y_t(x,t) = u[t, y(x,t)] \\ y(0,x) = x \end{cases} \quad (8)$$

.

Lemma 2.3 *$y(x_1,t) > y(x_2,t)$ for any $t > 0$ if $x_1 > x_2 \in R$.*

Proof. Because $y_t(x,t) = u(t, y(x,t))$

$$y_{tx}(x,t) = u_x(y(x,t),t)\frac{dy}{dx}, \text{ so } \left(\frac{dy}{dx}\right)_t = u_x(t, y(x,t))\frac{dy}{dx}$$

$y_x = e^{\int_0^t u_x(\tau, y(x,\tau))d\tau}$, $y_x > 0$ .Therefore , $y(x_1,t) > y(x_2,t)$ when $x_1 > x_2$.

Theorem 2.1 *The monotonicity of $u(y(x,t),t)$ still hold on $(y(x_1,t), y(x_2,t))$ if $u_0(x)$ is monotone on*

(x_1, x_2) *for any* $t > 0$*,where* $u(y(x,t),t)$ *is the solution of* (1.1) *and* $u_0 \in H^4(R)$.

Proof. Assume $u_0(x)$ is monotonic increase on (x_1, x_2), and s_1 is the first time the monotonicity of $u(y(x,t),t)$ has changed on (x_1, x_2).So $\exists m < n \in (x_1, x_2)$ $u(y(m,s_1),s_1) \le u(y(n,s_1),s_1)$. But when $t < s_1$, $u(y(m,s_1),s_1) > u(y(n,s_1),s_1)$.

From *lemma 2.2* we know $\exists \Delta s_1$, when $t \in U(s_1, \Delta s_1)$, $\|u(y(x,t),t)\|_{H^4} < +\infty$ .Then linearize $u(y(m,s_1),s_1) - u(y(n,s_1),s_1)$ at time $s_1 - \Delta s \in U(s_1, \Delta s_1)$

$$\begin{aligned} & u(y(m,s_1),s_1) - u(y(m,s_1),s_1) \\ & = u(y(m,s_1 - \Delta s),s_1 - \Delta s_1) - u(y(m,s_1 - \Delta s),s_1 - \Delta s_1) \\ & - [u_{yyy}(y(m,\xi_1),\xi_1) - u_{yyy}(y(n,\xi_1),\xi_1)]\Delta s + o(\Delta s) \end{aligned} \quad (9)$$

Where $\xi_1 \in (s_1 - \Delta s, s_1) \subset U(s_1, \Delta s_1)$.

Because $u(y(m,s_1 - \Delta s),s_1 - \Delta s) - u(y(n,s_1 - \Delta s),s_1 - \Delta s) > 0$,so there exist appropriate small s_1^* (such as let $s_1^* = \frac{1}{2}\frac{u(y(m,s_1 - s_1^*),s_1 - s_1^*) - u((n,s_1 - s_1^*),s_1 - s_1^*)}{|u_{yyy}(y(m,\xi_1),\xi_1^*) - u_{yyy}(y(n,\xi_1),\xi_1^*)|}$) let

$$\begin{aligned} & u(y(m,s_1 - s_1^*),s_1 - s_1^*) - u((n,s_1 - s_1^*),s_1 - s_1^*) \\ & - [u(y(m,\xi_1^*),\xi_1^*) - u((n,\xi_1^*),\xi_1^*)]s_1^* + o(\Delta s) > 0 \end{aligned} \quad (10)$$

Where $\xi_1^* \in (s_1 - s_1^*, s_1) \subset U(s_1, \Delta s_1)$.

This is contradictory to the hypothesis, so we know the monotonicity of $u(y(x,t),t)$ not change on (x_1, x_2) at time s_1 . That means the monotonicity of $u(y(x,t),t)$ would never change on $(y(x_1,t), y(x_2,t))$ for any $t > 0$ if $u_0(x)$ is monotone on (x_1, x_2).

Theorem 2.2. $u(y(x_0,t),t) > u(y(x,t),t)$ (*or* $u(y(x_0,t),t) < u(y(x,t),t)$) *if any* $x \in U^\circ(x_0,h), u_0(x_0) > u_0(x)$ (*or* $u_0(x_0) < u_0(x)$)*for any* $t > 0$*,where* $u(y(x,t),t)$ *is the solution of* (1.1) *and* $u_0 \in H^4(R)$.

Proof. Assume x_0 is an isolated maximum point of $u_0(x)$.By the smoothness and continuity of $u_0(x)$ we know $\exists h_1$, $x \in U^\circ(x_0,h)$, $u_0(x) < u_0(x_0)$, $u_0(x)$ is increasing on $(x_0 - h, x_0)$,and decreasing on

(x_0, x_0+h).Then from *lemma 2.3* and *theorem 2.1* we know the monotonicity of $u(y(x,t),t)$ on $(y(x_0-h,t), y(x_0,t))$ and $(y(x_0,t), y(x_0+h,t))$ will not change. So for any $t>0$, $x \in U^{\circ}(x_0,h)$, $y(x_0,t)$ still be an isolated maximum point of $u(y(x,t),t)$ on $(y(x_0-h,t), y(x_0+h,t))$.

From *theorem 2.1* and *theorem 2.2* we know the isolated extreme point of initial curve will not change at any $t>0$.

Theorem 2.3 $u(y(x,t),t)$ *still be convex (or be concave) on* $(y(x_1,t), y(x_2,t))$ *if* $u_0(x)$ *is convex (or is concave) for any* $t>0$ *on* (x_1,x_2)*,where* $u(y(x,t),t)$ *is the solution of* (1.1) *and* $u_0 \in H^4(R)$.

Proof. Assume $u_0(x)$ is convex on (x_1,x_2) and T_1 is the first time the concavity of $u(y(x,t),t)$ change on (x_1,x_2).So

$u(\lambda y(x,T_1)+(1-\lambda)y(z,T_1),T_1) \geq \lambda u(y(x,T_1),T_1)+(1-\lambda)u(y(z,T_1),T_1)$ with $x,z \in (x_1,x_2)$ .From *lemma 2.2* we know $\exists \Delta T_1$, then $\|u(y(x,t),t)\|_{H^4} < +\infty$ when $t \in U(T_1,\Delta T_1)$ .For convenience of calculation, let $\lambda = \frac{1}{2}$,we get

$$u\left(\frac{1}{2}(y(x,T_1)+y(z,T_1)),T_1\right)-\frac{1}{2}(u(y(x,T_1),T_1)+u(y(z,T_1),T_1)) \geq 0. \qquad (11)$$

The concavity of $u(y(x,t),t)$ would not change when $t<T_1$, then linearize

$$u\left(\frac{1}{2}(y(x,T_1)+y(z,T_1)),T_1\right)-\frac{1}{2}(u(y(x,T_1),T_1)+u(y(z,T_1),T_1))$$

at $T_1-\Delta T \in (T_1-\Delta T_1, T_1)$,

we get

$$u\left(\frac{1}{2}(y(x,T_1)+y(z,T_1)),T_1\right)-\frac{1}{2}(u(y(x,T_1),T_1)+u(y(z,T_1),T_1))$$

$$=u\left(\frac{1}{2}(y(x,T_1-\Delta T)+y(z,T_1-\Delta T)),T_1-\Delta T\right)$$

$$-\frac{1}{2}(u(y(x,T_1-\Delta T),T_1-\Delta T)+u(y(z,T_1-\Delta T),T_1-\Delta T))$$

$$-w(u,y,x,z,\omega_1)\Delta T+o(\Delta t), \qquad (12)$$

where $\omega_1\in(T_1-\Delta T,T_1)\subset U(T_1,\Delta T_1)$.

But from the hypothesis we know

$$u\left(\frac{1}{2}(y(x,T_1-\Delta T)+y(z,T_1-\Delta T)),T_1-\Delta T\right)$$
$$-\frac{1}{2}(u(y(x,T_1-\Delta T),T_1-\Delta T)+u(y(z,T_1-\Delta T),T_1-\Delta T))<0$$

so there exist appropriate small time T_1^* and

$$u\left(\frac{1}{2}(y(x,T_1-\Delta T)+y(z,T_1-T_1^*)),T_1-T_1^*\right)$$
$$-\frac{1}{2}(u(y(x,T_1-T_1^*),T_1-T_1^*)+u(y(z,T_1-T_1^*),T_1-T_1^*))$$
$$-w(u,y,x,z,\omega_1^*)\Delta T+o(\Delta t)<0 \qquad (13)$$

Where $\omega_1^*\in(T_1-T_1^*,T_1)\subset U(T_1,\Delta T_1)$.

This is contradictory to the hypothesis, so we know the concavity of $u(y(x,t),t)$ not change on (x_1,x_2) at time T_1. That means the concavity of $u(y(x,t),t)$ would never change on $(y(x_1,t),y(x_2,t))$ for any $t>0$ if $u_0(x)$ is convex on (x_1,x_2),.

This paper discuss the evolution of geometric properties of KdV equation by Lagrange coordinates , explains how would the solution curve change .

Acknowledgement

This work is supported by The Natural Science Foundation of China（No:11371175）.

References

[1] T Kato,On the Cauchy problem for the (generalized) Korteweg-de Vries equation,Advances in Mathematics supplementary studies,studies in Applied Math.8, pp.93-128,1989.

[2] T Kato,K.Masuda,Nonlinear Evolution Equations and Analyticity.I,Analyse nonlinéaire,pp.455-467,1986.

[3] ANDERS SJÖBERG，On the Korteweg-de Vries Equation:Existence and Uniqueness,J.Math.Anal.Appl.29, pp.569-579 ,1970.

[4] KENIG C E, PONCE G , VEGA L,Oscillatory integrals and regularity of dispersive ,equations[J].Indiana Univ Math,40(1),pp.33-69,1991.

[5] Jennifer Gorsky，A.Alexandrou Himonas，Well-posedness of KdV with higher ,dispersion.Mathematics and Computers in Simulation 80,pp.173–183,2009.

[6] Junfeng Li,Shaoguang Shi,Local well-posedness for the dispersion generalized periodic KdV equation,J.Math.Anal.Appl.379 ,pp.706-718,2011.

[7] Chulkwang Kwak，Local well-posedness for the fifth-order KdV equations on T,Journal of Differential Equations 260,pp.7683-7737,2016.

[8] Sunghyun Hong, Chulkwang Kwak，Global well-posedness and nonsqueezing propertyfor the higher-order KdV-type flow，J.Math.Anal.Appl.441

pp.140-166,2016.

[9] Heather Hannah, A Alexandrou Himonas , Gerson Petronilho，Gevrey regularity of the periodic gKdV equation，Journal of Differential Equations 250,pp. 2581-2600,2011.

[10] M.E.Gage,Curve shortening makes convex curves circular,Invent. Math. 76, pp.357-364,1984.

[11] Marcos Craizer , Ralph Teixeira,Evolution of an extremum by curvature motion,J.Math.Anal.Appl.293,pp.721-737,2004.

[12] Luc Molinet,F rancis Ribaud,On the Cauchy problem for the generalized Korteweg-de Vries equation,Communications in Partial Differential Equations.28(11-12),pp.2065-2091,2003.