New Oscillation Conditions for Second Order Non-Linear Neutral Difference Equations with Damping Term

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Abstract. This paper deals with the asymptotic and oscillatory properties of solutions of a class of second order non-linear damped neutral difference equations of the form

$$\Delta[r(n)(\Delta z(n)^{\alpha}] + a(n+1)(\Delta z(n+1))^{\alpha} + q(n)f\big(x(n+\sigma)\big) = 0; \qquad n \geq n_0, \qquad (*)$$

where $z(n) = x(n) - p(n)x(n-\tau), \alpha \geq 1$ is a ratio of positive odd integers, $\{p(n)\}$, $\{q(n)\}$, $\{r(n)\}$ and $\{a(n)\}$ are sequences of real numbers, τ and σ are integers, and $f: R \to R$ is a real valued continuous function. We established some new sufficient conditions under which every solution of $(*)$ is either oscillatory or tends to zero as $n \to \infty$. The results are illustrated with examples.

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1. Introduction

In this paper, we deals with the asymptotic and oscillatory properties of a class of second order nonlinear damped neutral difference equations of the form

$$\Delta[r(n)(\Delta z(n)^{\alpha}] + a(n+1)(\Delta z(n+1))^{\alpha} + q(n)f\big(x(n+\sigma)\big) = 0; \qquad n \geq n_0, \qquad (1.1)$$

where

$$z(n) = x(n) - p(n)x(n-\tau), \qquad (1.2)$$

Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n), \alpha \geq 1$ is a ratio of positive odd integers, $\{p(n)\}$, $\{q(n)\}$, $\{r(n)\}$ and $\{a(n)\}$ are sequences of real numbers, and $f: R \to R$ is a real valued continuous function with $uf(u) > 0$ for $u \neq 0$.

Throughout this paper, the following conditions are assumed to be hold:

(H_1) $\{p(n)\}$ is a sequence of nonnegative real number and there exist a constant p_0 such that $0 \leq p(n) \leq p_0 < 1$;

(H_2) $\{r(n)\}$ is a sequence of positive real numbers;

(H_3) $\{a(n)\}$ is a sequence of real numbers such that $a(n) + r(n) > 0$,

(H_4) $\{q(n)\}$ is a sequence of nonnegative real numbers and $\{q(n)\}$ is not identically zero for all sufficiently large n;

(H_5) $R(n, n_0) = \sum_{s=n\_0}^{n-1} \frac{1}{(b(s)r(s))^{\frac{1}{\alpha}}} \to \infty \ as\ n \to \infty$, where

$$b(n) = \prod_{s=n_0}^{n} \left(1 + \frac{a(s)}{r(s)}\right)$$

(H_6) There exist a constant $k > 0$ such that $\frac{f(u)}{u^\beta} \geq k$ for all $u \neq 0$, where β is a ratio of positive odd integers with $\alpha \geq \beta$;

(H_7) τ is a nonnegative integer and σ is an integers.

By a solution of (1.1), we mean a real sequence $\{x(n)\}$ which is defined for $n^* \geq min\{n_0, n_0 - \tau, n_0 + \sigma\}$ and satisfies (1.1) for $n \geq n^*$. We consider only such solution which are nontrivial for all large n. A solution $\{x(n)\}$ of (1.1) is said to be nonoscillatory if the terms $x(n)$ of the sequence are eventually positive eventually negative. Otherwise it is called oscillatory.

Recently, neutral delay difference equations, that is, difference equations in which the highest order difference of the unknown sequence appears both with and without delays, have considerable attention in the study of qualitative properties of these equations. The problem of asymptotic and oscillatory properties of solutions of neutral difference equations is of both theoretical and practical interest. One reason for this is that they arise, for example, in applications to electric networks containing lossless transmission lines such networks appear in highspeed computers where lossless transmission lines are used to inter connect switching circuits. They also occur in problems dealing with vibrating masses attached to an elastic bar and in the solution of variational problems with time delays.

On reviewing the literature, it becomes apparent that most results concerning the oscillation of all solutions of (1.1) are for the special case when $a(n) = 0$. Regarding the oscillation of undamped neutral difference equations, that is, special case of (1.1) with $a(n) = 0$, many papers have been published for different cases of $p(n)$ such as $-1 \leq p(n) \leq 0$, $-\infty < p_0 \leq p(n) \leq 0$ and $0 \leq p(n) \leq p_0 < 1$. We refer the reader to [3, 4, 7, 11, 18] and the references cited therein as examples of recent results on this topic.

In [9], we established sufficient conditions under which every solution of (1.1) is either oscillatory or tends to zero and derived sufficient conditions for oscillation of all bounded solutions of (1.1) for the case $\alpha = 1, a(n) \equiv 0$ and $f(u) = u$.

Saker et al. [20] established sufficient conditions which ensures oscillation of all solutions of the equation (1.1) for the case $p(n) \equiv 0, a(n) \geq 0$ and $\sigma = 1$.

Tunc et al. [19] consider the following second order nonlinear damped neutral differential equation of the form

$$(r(t)(z'(r))^\alpha)' + h(t)\left(z'(t)\right)^\alpha + q(t)f\left(x\left(\delta(t)\right)\right) = 0, \qquad t \geq t_0 > 0, \tag{1.3}$$

and they established sufficient conditions under which every solution of (1.3) is either oscillatory or tends to zero as $t \to \infty$. Motivated by the above observations, we establish sufficient conditions under which every solution of (1.1) is either oscillatory or tends to zero as $n \to \infty$.. Our obtained results are discrete analogues of the some well-known results due to [19]. For the general background of difference equations, one can refer to the papers [8, 10, 12-15, 17, 21], monographs [1, 2, 5] and references cited therein.

2. Main Results

To simplify the formulation of our results, we will use the following notations:

$$(i) \quad R(n,n_1) = \sum_{s=n_1}^{n-1} \frac{1}{(b(s)r(s))^{\frac{1}{\alpha}}} \text{ for } n_0 \le n_1 < n < \infty \tag{2.1}$$

$$(ii) \quad \theta(n,n_*) = \frac{R(n+\sigma,n_*)}{R(n+1,\ n_*)}, \text{for sufficiently large } n; \tag{2.2}$$

$$\phi(n,n_*) = \begin{cases} 1; & \sigma \ge 1 \\ \theta^\beta(n,n_*); & \sigma \le 0 \end{cases} \tag{2.3}$$

Also for any sequence $\{u(n)\}$, we get

$$\big(u(n)\big)_+ = \max\{0,\mathrm{u(n)}\}.$$

The following lemmas are very useful to prove our main results.

Lemma 2.1. [6]. *If A and B are nonnegative and $\lambda > 1$; then*

$$\lambda AB^{\lambda-1} - A^\lambda \le (\lambda-1)B^\lambda; \tag{2.4}$$

where the equality holds if and only if $A = B$.

Lemma 2.2. [6] *If A and B are positive real numbers and $\lambda > 0$, then*

$$A^\lambda - B^\lambda \ge \lambda B^{\lambda-1}(A-B) \text{ if } \lambda \ge 1$$

or

$$A^\lambda - B^\lambda \ge \lambda A^{\lambda-1}(A-B) \text{ if } \ 0 < \lambda \le 1.$$

There is obviously equality when $\lambda = 1$ or $A = B$.

Theorem 2.3. *Assume that there exist a positive sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that for all sufficiently large N_* and for $N > N_*$,*

$$\limsup_{n\to\infty} \sum_{s=N}^{n} \left[kb(s)\eta(s)q(s)\phi(s,N_*) - \frac{C^*\big(\Delta\eta(s)\big)_+}{R^\beta(s+1,N_*)}\right] = \infty, \tag{2.5}$$

with $C^* > 0$, then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of (1.1). Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1). Then there exist an integer $n_1 \ge n_0$ such that

$$x(n) > 0, \ \ x(n-\tau) > 0 \ \text{ and } \ x(n+\sigma) > 0 \ \text{ for } \ n \ge n_1. \tag{2.6}$$

Multiplying (1.1) by $b(n)$ we have

$$\Delta\big[b(n)r(n)\big(\Delta z(n)\big)^\alpha\big] + q(n)b(n)f\big(x(n+\sigma)\big) = 0,$$

which implies that

$$\Delta\big(b(n)r(n)\big(\Delta z(n)\big)^\alpha\big) = -q(n)b(n)f\big(x(n+\sigma)\big) \le 0; \ \text{ for } n \ge n_1, \tag{2.7}$$

so $\{b(n)r(n)\big(\Delta z(n)\big)^\alpha\}$ is eventually decreasing sequence, say for $n \ge n_2 \ge n_1$. We claim that

$$\Delta z(n) > 0 \ \text{ for } n \ge n_2. \tag{2.8}$$

If this is not so, then there exists $n_3 \ge n_2$ such that $\Delta z(n_3) \le 0$. In view of (2.7), there is $n_4 \ge n_3$ such that

$$b(n)r(n)\big(\Delta z(n)\big)^\alpha \le b(n_4)r(n_4)\big(\Delta z(n_4)\big)^2$$

$$= C < 0 \text{ for } n \geq n_4.$$

Hence

$$\Delta z(n) \leq C^{\frac{1}{\alpha}} \frac{1}{\left(b(n)r(n)\right)^{\frac{1}{\alpha}}}, \tag{2.9}$$

from which it follows that

$$z(n) \leq z(n_4) + C^{\frac{1}{\alpha}} \sum_{s=n_4}^{n-1} \frac{1}{\left(b(s)r(s)\right)^{\frac{1}{\alpha}}} \text{ for } n \geq n_4. \tag{2.10}$$

In view of (H_5) and (2.10), we follows that

$$\lim_{n\to\infty} z(n) = -\infty. \tag{2.11}$$

Thus, there are two cases to consider.

Case 1: If $\{x(n)\}$ is unbounded, then there exists a sequence $\{n_k\}$ such that $\lim_{n\to\infty} n_k = \infty$ and , $\lim_{n\to\infty} x(n_k) = \infty$, where

$$x(n_k) = \max\{x(s): n_0 \leq s \leq n_k\}, \tag{2.12}$$

Then from (1.2), we have

$$\begin{aligned} z(n_k) &= x(n_k) - p(n_k)x(n_k - \tau) \\ &\geq x(n_k)\left(1 - p(n_k)\right) \\ &\geq x(n_k)(1 - p_0) > 0, \end{aligned}$$

which contradicts (2.11).

Case 2: If $\{x(n)\}$ is bounded, then, in view of the definition of $z(n)$ and the fact that $0 \leq p(n) \leq p_0 < 1$, it follows that $\{z(n)\}$ is bounded, which again contradicts (2.11). Thus, in view of Cases 1 and 2, we conclude that (2.8) holds.

Hence from (2.7) and (2.8) and the definition of $z(n)$, we conclude that there exists $n_2 \geq n_1$ such that, for $n \geq n_2$, either

$$\begin{aligned} &z(n) > 0, \\ &\Delta z(n) > 0, \\ &\Delta\left(b(n)r(n)\left(\Delta z(n)\right)^{\alpha}\right) \leq 0 \end{aligned} \tag{2.13}$$

or

$$\begin{aligned} &z(n) < 0, \\ &\Delta z(n) > 0, \\ &\Delta\left(b(n)r(n)\left(\Delta z(n)\right)^{\alpha}\right) \leq 0. \end{aligned} \tag{2.14}$$

Assume that (2.13) holds. We note that $x(n) \geq z(n)$ and set

$$w(n) = \eta(n) \frac{b(n)r(n)\left(\Delta z(n)\right)^{\alpha}}{z^{\beta}(n)} \text{ for } n \geq n_2. \tag{2.15}$$

Then $w(n) > 0$ for $n \geq n_2$ and, form (1.1) and (2.15) we obtain

$$\Delta w(n) = \eta(n) \frac{\Delta\left(b(n)r(n)\left(\Delta z(n)\right)^{\alpha}\right)}{z^{\beta}(n)} - \eta(n) \frac{b(n+1)r(n+1)\left(\Delta z(n+1)\right)^{\alpha}}{z^{\beta}(n)z^{\beta}(n+1)} \Delta z^{\beta}(n)$$

$$+\frac{b(n+1)r(n+1)\big(\Delta z(n+1)\big)^{\alpha}}{z^{\beta}(n+1)}\Delta\eta(n). \tag{2.16}$$

By using Lemma 2.2, we have

$$\Delta z^{\beta}(n)=z^{\beta}(n+1)-z^{\beta}(n)\geq \beta z^{\beta-1}(n)\Delta z(n). \tag{2.17}$$

Using (2.17) and (2.16), we get

$$\Delta w(n)\leq \frac{\eta(n)\Delta\Big(b(n)r(n)\big(\Delta z(n)\big)^{\alpha}\Big)}{z^{\beta(n)}}-\beta\eta(n)\frac{b(n+1)r(n+1)\big(\Delta z(n+1)\big)^{\alpha}}{z(n)z^{\beta}(n+1)}\Delta z(n)$$

$$+\frac{b(n+1)r(n+1)\big(\Delta z(n+1)\big)^{\alpha}}{z^{\beta}(n+1)}\Delta\eta(n)$$

$$\leq -\eta(n)\frac{q(n)b(n)f\big(x(n+\sigma)\big)}{z^{\beta}(n)}-\frac{\beta\eta(n)b(n+1)r(n+1)\big(\Delta z(n+1)\big)^{\alpha}}{z(n)z^{\beta}(n+1)}\Delta z(n)$$

$$+\frac{b(n+1)r(n+1)\big(\Delta z(n+1)\big)^{\alpha}}{z^{\beta}(n+1)}\big(\Delta\eta(n)\big)_{+}$$

$$\leq k\frac{-\eta(n)q(n)b(n)x^{\beta}(n+\sigma)}{z^{\beta}(n)}-\frac{\beta\eta(n)b(n+1)r(n+1)\big(\Delta z(n+1)\big)^{\alpha}}{z(n)z^{\beta}(n+1)}\Delta z(n)$$

$$+\frac{b(n+1)r(n+1)\big(\Delta z(n+1)\big)^{\alpha}}{z^{\beta}(n+1)}\big(\Delta\eta(n)\big)_{+} \tag{2.18}$$

$$\leq \frac{-k\eta(n)q(n)b(n)z^{\beta}(n+\sigma)}{z^{\beta}(n+1)}-\frac{\beta\eta(n)b(n+1)r(n+1)\big(\Delta z(n+1)\big)^{\alpha}}{z(n)z^{\beta}(n+1)}\Delta z(n)$$

$$+\frac{b(n+1)r(n+1)\big(\Delta z(n+1)\big)^{\alpha}}{z^{\beta}(n+1)}\big(\Delta\eta(n)\big)_{+}. \tag{2.19}$$

In view of the fact that $\Delta z(n)>0$ for $n\geq n_2$ it follows from (2.19) that

$$\Delta w(n)\leq b(n+1)r(n+1)\left[\frac{\Delta z(n+1)}{z(n+1)}\right]^{\alpha}z^{\alpha-\beta}(n+1)\big(\Delta\eta(n)\big)_{+}$$

$$-kb(n)\eta(n)q(n)\left[\frac{z(n+\sigma)}{z(n)}\right]^{\beta}. \tag{2.20}$$

Since $\{b(n)r(n)\big(\Delta z(n)\big)^{\alpha}\}$ is nonincreasing, we see that

$$z(n+1)\geq z(n+1)-z(n_2)$$

$$=\sum_{s=n_2}^{n}\frac{\Big(b(s)r(s)\big(\Delta z(s)\big)^{\alpha}\Big)^{\frac{1}{\alpha}}}{\big(b(s)r(s)\big)^{\frac{1}{\alpha}}}$$

$$\geq \Big(b(n)r(n)\big(\Delta z(n)\big)^{\alpha}\Big)^{\frac{1}{\alpha}}\sum_{s=n_2}^{n}\frac{1}{\big(b(s)r(s)\big)^{\frac{1}{\alpha}}},$$

$$\geq \big(b(n+1)r(n+1)\big)^{\frac{1}{\alpha}}\Delta z(n+1)\sum_{s=n_2}^{n}\frac{1}{\big(b(s)r(s)\big)^{\frac{1}{\alpha}}},$$

from which it follows that

$$\frac{\Delta z(n+1)}{z(n+1)} \leq \frac{1}{\left(b(n+1)r(n+1)\right)^{\frac{1}{\alpha}} R(n+1,n_2)}. \tag{2.21}$$

Using the fact that $\left\{b(n)r(n)\left(\Delta z(n)\right)^{\alpha}\right\}$ is eventually decreasing for $n \geq n_2$, we have

$$0 < b(n)r(n)\left(\Delta z(n)\right)^{\alpha} \leq b(n_2)r(n_2)\left(\Delta z(n_2)\right)^{\alpha} = C_1, \quad n \geq n_2 \tag{2.22}$$

and thus we have, for all $n \geq n_3 = n_2 + 1$, that

$$\begin{aligned} z(n+1) &\leq z(n_2) + C_1^{\frac{1}{\alpha}} \sum_{s=n_2}^{n} \frac{1}{\left(b(s)r(s)\right)^{\frac{1}{\alpha}}} \\ &\leq z(n_2) + C_1^{\frac{1}{\alpha}} R(n+1,n_2) \\ &= \left[\frac{z(n_2)}{R(n+1,n_2)} + C_1^{\frac{1}{\alpha}}\right] R(n+1,n_2) \\ &= \left[\frac{z(n_2)}{R(n_3+n_2)} + C_1^{\frac{1}{\alpha}}\right] R(n+1,n_2) \\ &= C_2 R(n+1,n_2) \end{aligned} \tag{2.23}$$

holds, where $C_2 = \frac{z(n_2)}{R(n_3+n_2)} + C_1^{\frac{1}{\alpha}}$. Using (2.21) and (2.23) in (2.20) we obtain

$$\Delta w(n) \leq \frac{C_3\left(\Delta \eta(n)\right)_+}{R^{\beta}(n+1,n_2)} - kb(n)\eta(n)q(n)\left(\frac{z(n+\sigma)}{z(n+1)}\right)^{\beta} \quad \text{for } n \geq n_2, \tag{2.24}$$

where $C_3 = C_2^{\alpha-\beta}$.

Now if $\sigma \geq 1$, in view of the fact that $\{z(n)\}$ is increasing, we have

$$\frac{z(n+\sigma)}{z(n+1)} \geq 1. \tag{2.25}$$

Using (2.25) in (2.24), we get

$$\Delta w(n) \leq \frac{C_3\left(\Delta \eta(n)\right)_+}{R^{\beta}(n+1,n_2)} - kb(n)\eta(n)q(n). \tag{2.26}$$

Now, if $\sigma \leq 0$, we can choose $n_3 > n_2$ such that $n + \sigma \geq n_2$, for all $n \geq n_3$. Thus from the fact that $\left\{b(n)r(n)\left(\Delta z(n)\right)^{\alpha}\right\}$ is eventually decreasing, we have

$$\begin{aligned} z(n+1) - z(n+\sigma) &= \sum_{s=n+\sigma}^{n} \frac{\left(b(s)r(s)\left(\Delta z(s)\right)^{\alpha}\right)^{\frac{1}{\alpha}}}{\left(b(s)r(s)\right)^{\frac{1}{\alpha}}} \\ &\leq \left(b(n+\sigma)r(n+\sigma)\left(\Delta z(n+\sigma)\right)^{\alpha}\right)^{\frac{1}{\alpha}} \sum_{s=n+\sigma}^{n} \frac{1}{\left(b(s)r(s)\right)^{\frac{1}{\alpha}}} \\ &= \left(b(n+\sigma)r(n+\sigma)\left(\Delta z(n+\sigma)\right)^{\alpha}\right)^{\frac{1}{\alpha}} R(n+1,n+\sigma); \end{aligned} \tag{2.27}$$

that is

$$\frac{z(n+1)}{z(n+\sigma)} \leq 1 + \frac{\left(b(n+\sigma)r(n+\sigma)\left(\Delta z(n+\sigma)\right)^{\alpha}\right)^{\frac{1}{\alpha}}}{z(n+\sigma)} \left[R(n+1,n_2) - R(n+\sigma,n_2)\right]. \tag{2.28}$$

Again

$$z(n+\sigma) \geq z(n+\sigma) - z(n_2)$$

$$= \sum_{s=n_2}^{n+\sigma-1} \frac{\left(b(s)r(s)\left(\Delta z(s)\right)^{\alpha}\right)^{\frac{1}{\alpha}}}{\left(b(s)r(s)\right)^{\frac{1}{\alpha}}} \tag{2.29}$$

$$\geq \left(b(n+\sigma)r(n+\sigma)\left(\Delta z(n+\sigma)\right)^{\alpha}\right)^{\frac{1}{\alpha}} R(n+\sigma, n_2)$$

that is,

$$\frac{\left(b(n+\sigma)r(n+\sigma)\left(\Delta z(n+\sigma)\right)^{\alpha}\right)^{\frac{1}{\alpha}}}{z(n+\sigma)} \leq \frac{1}{R(n+\sigma, n_2)}; \text{ for } n \geq n_3. \tag{2.30}$$

From (2.28) and (2.30), we have

$$\frac{z(n+1)}{z(n+\sigma)} \leq \frac{R(n+1, n_2)}{R(n+\sigma, n_2)}$$

or

$$\frac{z(n+\sigma)}{z(n+1)} \leq \frac{R(n+\sigma, n_2)}{R(n+1, n_2)} = \theta(n, n_2). \tag{2.31}$$

Using (2.31) in (2.24), we see that

$$\Delta w(n) \leq \frac{C_3\left(\Delta\eta(n)\right)_+}{R^{\beta}(n+1, n_2)} - kb(n)\eta(n)q(n)\theta^{\beta}(n, n_2). \tag{2.32}$$

Combining (2.26) and (2.32), we see that

$$\Delta w(n) \leq -kb(n)\eta(n)q(n)\phi(n, n_2) + \frac{C_3\left(\Delta\eta(n)\right)_+}{R^{\beta}(n+1, n_2)} \text{ for } n \geq n_3. \tag{2.33}$$

Summing the inequality (2.33) from n_3 to $n-1$, yields

$$\sum_{s=n_3}^{n-1} \left[kb(s)\eta(s)q(s)\phi(s, n_2) - \frac{C_3\left(\Delta\eta(s)\right)_+}{R^{\beta}(s+1, n_2)}\right] \leq w(n_3) - w(n) < w(n_3), \tag{2.34}$$

which contradicts condition (2.5).

Now, let (2.14) hold. In view of $z(n) < 0$ and $\Delta z(n) > 0$, we have

$$\lim_{n\to\infty} z(n) = l \leq 0, \tag{2.35}$$

where l is a constant, and so $\{z(n)\}$ is bounded for sufficiently large n. We assert that $\{x(n)\}$ is also bounded. Otherwise if $\{x(n)\}$ is unbounded, then there exists a sequence $\{n_k\}$ such that $\lim_{k\to\infty} n_k = \infty$ and $\lim_{k\to\infty} x(n_k) = \infty$, where $x(n_k)$ is defined by (2.12) and so, from the definition of $z(n)$ and $\tau \leq 0$, we see that

$$z(n_k) = x(n_k) - p(n_k)x(n_k - \tau)$$

$$\geq \left(1 - p(n_k)\right)x(n_k - \tau) > 0, \tag{2.36}$$

which contradicts the fact that $z(n) < 0$ for $n \geq n_2$, and so $\{x(n)\}$ is bounded. Therefore, we have

$$\limsup_{n\to\infty} x(n) = \mu, \qquad 0 \leq \mu < \infty. \tag{2.37}$$

If $\mu > 0$, then there exists a sequence $\{n_m\}$ such that $\lim_{m\to\infty} n_m = \infty$ and $\lim_{m\to\infty} x(n_m) = \mu$. Let $\epsilon = \mu(1 - p_0)/2p_0$; then for all large n, we have $x(n_m - \tau) < \mu + \epsilon$. From this and the definition of $z(n)$, we obtain

$$0 \geq \lim_{n\to\infty} z(n_m) \geq \lim_{m\to\infty} x(n_m) - p_0(\mu+\epsilon) = \frac{\mu(1-p_0)}{2} > 0, \tag{2.38}$$

which contradicts the fact that $z(n) < 0$, and hence $\limsup_{n\to\infty} x(n) = 0$. Now, in view of the fact that $x(n) > 0$, we conclude that $\lim_{n\to\infty} x(n) = 0$, which completes the proof of Theorem 2.3.

Theorem 2.4. *If there exists a positive sequence* $\{\eta(n)\}_{n=n_0}^{\infty}$ *such that for all sufficiently large* N_* *and for* $N > N_*$.

$$\limsup_{n\to\infty} \sum_{s=N}^{n} \left[k\eta(s)b(s)q(s)\phi(s,N_*) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{C^* b(s)r(s)\left(\left(\Delta\eta(s)\right)_+\right)^{\alpha+1}}{\left(\beta\eta(s)\right)^\alpha R^{\beta-\alpha}(s+1,N_*)} \right] = \infty, \tag{2.39}$$

where $C^* > 0$, *then every solution of (1.1) is either oscillatory or tends to zero.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of (1.1). Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive. Then there exist an integer $n_1 \geq n_0$ such that

$$x(n) > 0, \;\; x(n-\tau) > 0, \qquad \text{and } \; x(n+\sigma) > 0 \;\text{ for }\; n \geq n_1. \tag{2.40}$$

Proceeding as in the proof of Theorem 2.3, we see that (2.13) or (2.14) holds. If (2.13) holds, as in the proof of Theorem 2.3, we obtain (2.17), (2.20), (2.21), (2.23), (2.25) and (2.31). Using (2.15) and (2.31) in (2.17), we have,

$$\Delta w(n) \leq -kb(n)\eta(n)q(n)\phi(n,n_2) - \frac{\beta\eta(n)b(n+1)r(n+1)\left(\Delta z(n+1)\right)^\alpha \Delta z(n)}{z(n)z^\beta(n+1)} + \frac{w(n+1)}{\eta(n+1)}\left(\Delta\eta(n)\right)_+. \tag{2.41}$$

Using (2.15), (2.23), $\Delta z(n) > 0$ and the fact that $\frac{\beta}{\alpha} - 1 \leq 0$ in the middle term of the right hand side of (2.41), we have

$$\Delta w(n) \leq -kb(n)\eta(n)q(n)\phi(n,n_2) - \frac{\beta\eta(n)C_2^{\frac{\beta}{\alpha}-1} R^{\frac{\beta}{\alpha}-1}(n+1,n_2)}{\eta^{\frac{\alpha+1}{\alpha}}(n+1)\left(b(n)r(n)\right)^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(n+1) + \frac{w(n+1)}{\eta(n+1)}\left(\Delta\eta(n)\right)_+,$$

or

$$\Delta w(n) \leq -kb(n)\eta(n)q(n)\phi(n,n_2) + \frac{\left(\Delta\eta(n)\right)_+}{\eta(n+1)} w(n+1) - \frac{C_4\beta\eta(n)R^{\frac{\beta}{\alpha}-1}(n+1,n_2)}{\eta^{\frac{\alpha+1}{\alpha}}(n+1)\left(b(n)r(n)\right)^{\frac{1}{\alpha}}} w^{\frac{\alpha+1}{\alpha}}(n+1), \tag{2.42}$$

for $n \geq n_3$ where $C_4 = C_2^{\frac{\beta}{\alpha}-1}$. Letting

$$A = \left(\frac{C_4\beta\eta(n)R^{\frac{\beta}{\alpha}-1}(n+1,n_2)}{\eta^{\frac{\alpha+1}{\alpha}}(n+1)\left(b(n)r(n)\right)^{\frac{1}{\alpha}}}\right)^{\frac{1}{\lambda}} w(n+1),$$

$$B=\left\{\frac{1}{\lambda}\left(\frac{\eta^{\frac{\alpha+1}{\alpha}}(n+1)\big(b(n)r(n)\big)^{\frac{1}{\alpha}}}{C_4\beta\eta(n)R^{\frac{\beta}{\alpha}-1}(n+1,n_2)}\right)^{\frac{1}{\lambda}}\frac{\big(\Delta\eta(n)\big)_+}{\eta(n+1)}\right\}^{\alpha}$$

and $\lambda=\frac{\alpha+1}{\alpha}$ in Lemma 2.1, (2.42) implies

$$\Delta w(n)\le -k\eta(n)b(n)q(n)\phi(n,n_2)$$
$$+\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{b(n)r(n)\left(\big(\Delta\eta(n)\big)_+\right)^{\alpha+1}}{\big(C_4\beta\eta(n)\big)^{\alpha}R^{\beta-\alpha}(n+1,n_2)},\quad \text{for } n\ge n_3. \qquad (2.43)$$

Summing the above inequality from n_3 to $n-1$, we have

$$\sum_{s=n_3}^{n}\left[k\eta(s)b(s)q(s)\phi(s,n_2)-\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{C_3b(s)r(s)\left(\big(\Delta\eta(s)\big)_+\right)^{\alpha+1}}{\big(\beta\eta(s)\big)^{\alpha}R^{\beta-\alpha}(s+1,n_2)}\right]\le w(n_3)<\infty, \quad (2.44)$$

which contradicts condition (2.39).

Finally, if (2.14) holds, proceeding as in the proof Theorem 2.3, we see that $\lim_{n\to\infty}x(n)=0$, which completes the proof of Theorem 2.4.

Theorem 2.5. *Assume that $\alpha\ge\beta\ge1$. Suppose that there exists a positive sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that for all sufficiently large N_* and for $N>N_*$.*

$$\limsup_{n\to\infty}\sum_{s=N}^{n}\left[k\eta(s)b(s)q(s)\phi(s,N_*)-\frac{C^*b(s)r(s)^{\frac{1}{\alpha}}\left(\big(\Delta\eta(s)\big)_+\right)^{2}}{4\beta\eta(s)R^{\beta-1}(s+1,N_*)}\right]=\infty, \quad (2.45)$$

where $C^*>0$, then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of (1.1). Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive. Then there exist an integer $n_1\ge n_0$ such that

$$x(n)>0,\ \ x(n-\tau)>0,\qquad \text{and } x(n+\sigma)>0 \ \text{ for } n\ge n_1. \qquad (2.46)$$

Proceeding as in the proof of Theorem 2.3 and 2.4, we see that (2.13) or (2.14) holds. If (2.13) holds, as in the proof of Theorem 2.4, we obtain (2.42) which can be rewritten as

$$\Delta w(n)\le -k\eta(n)b(n)q(n)\phi(n,n_2)+\frac{\big(\Delta\eta(n)\big)_+}{\eta(n+1)}w(n+1)$$
$$-\frac{C_4\beta\eta(n)R^{\frac{\beta}{\alpha}-1}(n+1,n_2)}{\eta^{\frac{\alpha+1}{\alpha}}(n+1)\big(b(n)r(n)\big)^{\frac{1}{\alpha}}}w^{\frac{1}{\alpha}-1}(n+1)w^{2}(n+1), \qquad (2.47)$$

for $n\ge n_3$. From (2.15), we have

$$w^{\frac{1}{\alpha}-1}(n+1)=\big(\eta(n+1)b(n+1)r(n+1)\big)^{\frac{1}{\alpha}-1}\left(\frac{\big(\Delta z(n+1)\big)^{\alpha}}{z^{\beta}(n+1)}\right)^{\frac{1}{\alpha}-1}$$
$$=\big(\eta(n+1)b(n+1)r(n+1)\big)^{\frac{1}{\alpha}-1}\frac{\big(\Delta z(n+1)\big)^{1-\alpha}}{z^{\beta\left(\frac{1}{\alpha}-1\right)}(n+1)}$$

$$= \left(\eta(n+1)b(n+1)r(n+1)\right)^{\frac{1}{\alpha}-1} \frac{z^{\beta-\frac{\beta}{\alpha}}(n+1)}{\left(\Delta z(n+1)\right)^{\alpha-1}}$$

$$= \left(\eta(n+1)b(n+1)r(n+1)\right)^{\frac{1}{\alpha}-1} \left(\frac{z(n+1)}{\Delta z(n+1)}\right)^{\alpha-1} z^{(\alpha-1)\left(\frac{\beta}{\alpha}-1\right)}(n+1). \quad (2.48)$$

From (2.21), we have

$$\frac{z(n+1)}{\Delta z(n+1)} \geq \left(b(n+1)r(n+1)\right)^{\frac{\alpha-1}{\alpha}} R^{\alpha-1}(n+1,n_2). \quad (2.49)$$

From (2.23) and the fact that $(\alpha-1)\left(\frac{\beta}{\alpha}-1\right) \leq 0$, we obtain,

$$z^{(\alpha-1)\left(\frac{\beta}{\alpha}-1\right)}(n+1) \leq {C_2}^{(\alpha-1)\left(\frac{\beta}{\alpha}-1\right)} R^{(\alpha-1)\left(\frac{\beta}{\alpha}-1\right)}(n+1,n_2). \quad (2.50)$$

Substituting (2.49) and (2.50) into (2.48), we have

$$w^{\frac{1}{\alpha}-1}(n+1) \geq \eta^{\frac{1}{\alpha}-1}(n+1){C_2}^{(\alpha-1)\left(\frac{\beta}{\alpha}-1\right)} R^{\beta-\frac{\beta}{\alpha}}(n+1,n_2). \quad (2.51)$$

Using (2.51) in (2.47), we obtain

$$\Delta w(n) \leq -k\eta(n)b(n)q(n)\phi(n,n_2) + \frac{\left(\Delta\eta(n)\right)_+}{\eta(n+1)} w(n+1)$$
$$- \frac{\beta\eta(n)R^{\beta-1}(n+1,n_2)}{C_3\eta^2(n+1)\left(b(n)r(n)\right)^{\frac{1}{\alpha}}} w^2(n+1). \quad (2.52)$$

Completing sequence with respect to $w(n+1)$, it follows from (2.52), that

$$\Delta w(n) \leq -k\eta(n)b(n)q(n)\phi(n,n_2) + \frac{C_3\left(b(n)r(n)\right)^{\frac{1}{\alpha}}\left(\left(\Delta\eta(n)\right)_+\right)^2}{4\beta\eta(n)R^{\beta-1}(n+1,n_2)}. \quad (2.53)$$

Summing the last inequality from n_3 to $n-1$, we obtain

$$\sum_{s=n_3}^{n}\left[k\eta(s)b(s)q(s)\phi(n,n_2) - \frac{C_3 b(s)r(s)^{\frac{1}{\alpha}}\left(\left(\Delta\eta(s)\right)_+\right)^2}{4\beta\eta(s)R^{\beta-1}(s+1,n_2)}\right] \leq w(n_3) < \infty, \quad (2.54)$$

which contradicts condition (2.45).

Finally, if (2.14) holds, proceeding as in the proof of Theorem 2.3, we see that $\lim_{n\to\infty} x(n) = 0$, which completes the proof of Theorem 2.4.

Remark 2.6. *If $\alpha = \beta$, then we have $C^* = 1$ in Theorem 2.3-2.5.*

3. Examples

Examples 3.1. *Consider the following second order neutral advanced difference equation of the form*

$$\Delta^2\left[x(n) - \frac{1}{2}x(n-1)\right] - \frac{1}{2}\Delta\left[x(n+1) - \frac{1}{2}x(n)\right] + \frac{2^n}{2^n-1}x(n+1) = 0; n = 1,2,\ldots. \quad (3.1)$$

Clearly, we have $r(n) = 1$, $a(n) = -\frac{1}{2}$, $q(n) = \frac{2^n}{2^n-1}$, $p(n) = \frac{1}{2}$, $\tau = 1$, $\sigma = 1$, $f(u) = u$ and $\alpha = \beta = 1$, $C^* = 1$.

Also we can see that

$$b(n) = \frac{1}{2^n},$$

$$R(n+1,N_*) = R(n+1,1) = 2^{n+1} - 2;$$

and

$$\phi(n,N_*) = \phi(n,1) = 1.$$

Choose $\eta(n) = 2^n$, *then* $\Delta\eta(n) = 2^n$. *Now,*

$$\limsup_{n\to\infty} \sum_{s=n}^{n} \left[k\eta(s)b(s)q(s)\phi(s,N_*) - \frac{C^*\big(\Delta\eta(s)\big)_+}{R^\beta(s+1,N_*)} \right]$$

$$= \limsup_{n\to\infty} \sum_{s=N}^{n} \left(\frac{2^s}{2^{s+1}-2} \right)$$

$$= \limsup_{n\to\infty} \sum_{s=N}^{n} \frac{1}{2} = \infty,$$

which implies that by Theorem 2.3, every solution of (3.1) is either oscillatory to tends to zero.

Example 3.2. *Consider the following second order neutral delay difference equation*

$$\Delta\left[n^3 \left(\Delta \left(x(n) - \frac{1}{3}x(n-1) \right) \right)^3 \right] + (n+1)^2 \left(\Delta \left(x(n+1) - \frac{1}{3}x(n) \right) \right)^3$$

$$+ \frac{1}{128n} x^3(n-1) = 0; \quad n = 1,2,\ldots. \tag{3.2}$$

Here , we have $r(n) = n^3$, $a(n) = n^2$, $p(n) = \frac{1}{3}$, $q(n) = \frac{1}{28n}$, $\tau = 1$, $\sigma = 1$, $f(u) = u^3$, $k = 1$ *and* $\alpha = \beta = 3$. *Let us choose* $\eta(n) = n$. *Then* $\Delta\eta(n) = 1$. *Now*

$$b(n) = \prod_{s=1}^{n} \left(1 + \frac{a(s)}{b(s)} \right) = n+1.$$

Also,

$$R(n,1) = \sum_{s=1}^{n-1} \frac{1}{\big(b(s)r(s)\big)^{\frac{1}{\alpha}}}$$

$$= \sum_{s=1}^{n-1} \frac{1}{\big((s+1)s^3\big)^{\frac{1}{3}}} \to \infty \text{ as } n \to \infty.$$

Now

$$\phi(n,N_*) = \phi(n,1) = \theta^\beta(n,n_*) = 1.$$

Now we can show that

$$\limsup_{n\to\infty} \sum_{s=N}^{n} \left[k\eta(s)b(s)q(s)\phi(s,N_*) - \frac{\alpha^\alpha C^* b(s)r(s)\left(\big(\Delta\eta(s)\big)_+\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\beta\eta(s))^\alpha R^{\beta-\alpha}(s+1,n_*)} \right]$$

$$\limsup_{n\to\infty} \sum_{s=N}^{n} \left[\frac{s+1}{128} - \frac{s+1}{256} \right] = \infty.$$

Then by Theorem 2.4 every solution of (3.2) is either oscillatory or teds to zero.

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