NUMERICAL SOLUTION OF FRACTIONAL ORDER DELAY DIFFERENTIAL EQUATION USING SHIFTED CHEBYSHEV POLYNOMIALS OF SECOND KIND

AJMAL ALI

ABSTRACT. The main aim of this article is to present an efficient numerical method to solve the Delay Differential Equation of fractional order.We use the fractional derivative in Caputo's sense.The properties of Chebyshev polynomials of second kind are utilized to reduce Delay Fractional Differential Equation (DFDE) to a linear or non-linear easily solvable system of algebraic equations.Numerically illustrative solved examples are present.The results shows that proposed method is very effective and simple.Thats reveals the validity and applicability of method.

1. INTRODUCTION

In recent years,fractional calculus has become more important than the ordinary calculus.The ordinary calculus have achieved the discovery at its peak point.That's why,the mathematician and researchers feel the need of fractional calculus.Fractional differential Equations (FDEs) is generalization of Ordinary Differential Equations (ODEs) because Fractional Differential Equations describe values on each point continually and distinguished the gaps between the two integers.This is the reason that after the discovery of fractional calculus,it is observed that Fractional Differential Equation has more real in natural phenomena than to the Ordinary Differential Equations[6].

The study of Fractional Differential Equations in fractional calculus is one of the most popular subject in many mathematical scientific areas including the image processing,earthquake and biomedical engineering ,viscoelasticity [3],finance [24],hydrology [4] and control system [1].
To obtain the exact analytic solutions of Fractional Differential Equation,it is very difficult and some time impossible to deal with the complexities computations in these equations.So it is better,to look for some useful approximations and numerically techniques such as variation iteration method [8],homotopy perturbation method[26],collection method [16-20],Galerkin method,Laplace transform and Fourier transform methods and other methods[12-13].

The organization of this paper is as follows:Insection 2,we introduce some definitions regarding to fractional derivatives and fractional order delay differential equations.In section 3,we introduce chebyshev polynomials of second kind and find the approximate formula for fractional derivative.In section 3,we give the procedure for solving delay fractional order differential equation.In section 4,we present two numerical examples to show the validity of the method..Finally in last section, we give some remarks about calculations and graphs in our paper.

2. PRELIMINARIES

Definition 2.1. *The Caputo's fractional derivative of order α is defined as*

$$D^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}$$

Key words and phrases. Chebyshev polynomial of second kind,Caputo's fractional derivative,Fractional Delay Differential Equation.

Where $\alpha > 0$, $m-1 < \alpha < m, m \in N$, $x > 0$
Caputo's fractional derivative operator D^{α} *is linear operator.i.e*

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x)$$

Where λ *and* μ *are constants.*
Moreover if C *is any constant then* $D^{\alpha} = 0$

$$D^{\alpha}x^{n} = \begin{cases} 0, \; for \; n \in W \;\; n < \lceil \alpha \rceil \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, \; for \; n \in W \;\; n \geq \lceil \alpha \rceil \end{cases}$$

We use the ceiling function $\lceil \alpha \rceil$ *to denote the smallest integer greater that or equal to* α *and* $W = \{0, 1, 2, 3 \cdots\}$. *Recall that for* $\alpha \in W$ *the Caputo's fractional differential operator coincides with the usual differential operator of integer order[17-20].*

The main goal in this article is concerned with the application of chebyshev polynomial of second kind to obtain the numerical solution of fractional order Delay Differential Equation of the form;

$$D^{\alpha}u(x) = f(x, u(x), u(g(x))), \; a \leq x \leq b, \; 1 < \alpha \leq 2 \cdots (1)$$

with the following boundary conditions;

$$u(a) = \gamma_0, \; u(b) = \gamma_1, \; u(x) = \psi(x), \; x \in [a^*, a] \cdots (2)$$

Where α refers to fractional order of spatial derivatives.The function g is called the delay function and it is assumed to be continues on the interval $[a, b]$ and satisfies the inequality $a^* \leq g(x) \leq x, x \in [a, b], \psi \in C[a^*, a]$ [16]. Note that for $\alpha = 2$ the equation(1) is the classical second-order delay differential equation;

$$u^{''}(x) = f(x, u(x), u(g(x))), \; a \leq x \leq b$$

3. Properties of Chebyshev Polynomials of second kind

3.1. Chebyshev Polynomials of second kind. The Chebyshev polynomial of second kind $U_n(x)$ are the orthogonal polynomials of degree n in x defined on $[1, -1]$. It is the special case of Jacobi's polynomials $P_n^{\alpha,\beta}$ in which $\alpha = \beta = 1/2$ i.e

$$U_n(x) = (n+1)\frac{P_n^{1/2,1/2}(x)}{P_n^{1/2,1/2}(x)}$$

The Chebyshev polynomial of second kind $U_n(x)$ has trigonometric definition involving half angle i.e $\frac{\theta}{2}$. Let $x = cos\theta$ and $\theta \in [0, \pi]$ the above equation will take the following form

$$U_n(x) = \frac{sin\theta(n+1)}{sin\theta}$$

The Chebyshev polynomial of second kind $U_n(x)$ are the orthogonal with respect to the weights $\sqrt{1-x^2}$ on the interval $[-1, 1]$.The condition of orthogonality is given by the relation,

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx = \begin{cases} 0 : n \neq m \\ \frac{\Pi}{2} : n = m \end{cases}$$

The Chebyshev polynomial of second kind $U_n(x)$ can be generated by the following recurrence relations

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), n = 2, 3...$$

with $U_0(x) = 1, U_1 = 2x$.The analytic form of The Chebyshev polynomial of second kind $U_n(x)$ of degree n is given by

$$U_n(x) = \sum_{k=0}^{\lceil \frac{n}{2} \rceil}(-1)^k 2^{n-2k}\frac{\Gamma(n-k+1)x^{n-2k}}{\Gamma(k+1)\Gamma(n-2k+1)}, n > 0$$

Where $\lceil \frac{n}{2} \rceil$ denotes the integral part of $\frac{n}{2}$. The function $u(x)$ which belong to the space of integrable functions in $[-1,1]$ may be expressed as in term of Chebyshev polynomials of second kind $U_n(x)$ as:

$$u(x) = \sum_{i=0}^{\infty} c_i U_i(x)$$

Where the coefficients $c_i, i = 0, 1, 2..$ are given by;

$$c_i = \frac{2}{\Pi} \int_{-1}^{1} u(\frac{1+x}{2}) \sqrt{1-x^2} U_i(x) dx$$

3.2. **shifted Chebyshev Polynomials of second kind.** The calculations in the interval $[-1,1]$ become more complicated than the calculations made in the interval $[0,1]$.In order to use the polynomials defined in section 3.1 on the interval $[0,1]$ we define the so called the shifted chebyshev polynomials of second kind denoted by $U_n^*(x)$ for all $x \in [0,1]$ by change of variable $s = 2x - 1$ or $x = 1/2(s+1)$,thus we can write $U_n^*(x) = U_n(2x-1)$.

The Chebyshev polynomial of second kind $U_n^*(x)$ are the orthogonal with respect to the weights $\sqrt{1-x^2}$ on the interval $[0,1]$.The condition of orthogonality is given by the relation,

$$\int_{-1}^{1} U_n^*(x) U_m^*(x) \sqrt{x-x^2} dx = \begin{cases} 0 : n \neq m \\ \frac{\Pi}{8} : n = m \end{cases}$$

The Chebyshev polynomial of second kind $U_n(x)$ can be generated by the following recurrence relations

$$U_n^*(x) = 2(2x-1)U_{n-1}^*(x) - U_{n-2}^*(x), n = 2, 3...$$

with $U_0^*(x) = 1, U_1^*(x) = 4x - 2$. The analytic form of The Chebyshev polynomial of second kind $U_n^*(x)$ of degree n is given by

$$U_n^*(x) = \sum_{k=0}^{n} (-1)^k 2^{2n-2k} \frac{\Gamma(2n-k+2) x^{n-k}}{\Gamma(k+1)\Gamma(2n-2k+2)}, n > 0$$

The function $u(x)$ which belong to the space of integrable functions in $[0,1]$ may be expressed as in term of Chebyshev polynomials of second kind $U_n^*(x)$ as:

$$u(x) = \sum_{i=0}^{\infty} c_i U_i^*(x) \cdots (3)$$

Where the coefficients $c_i, i = 0, 1, 2..$ are given by;

$$c_i = \frac{8}{\Pi} \int_{0}^{1} u(x) \sqrt{x-x^2} U_i^*(x) dx$$

In practical, we consider the first $m+1$ terms of the shifted Chebyshev polynomials of second kind So Eq.(3) will take the form

$$u_m(x) = \sum_{i=0}^{m} c_i U_i^*(x) \cdots (4)$$

4. Derivation of approximate formula for Fractional Derivative Using Chebyshev Polynomials of second kind

In this section,we derive an approximate formula for fractional derivative of $u(x)$ using the shifted chebyshev polynomials of second kind.

Theorem 4.1. *Let $u(x)$ be approximated by Chebyshev polynomials of second kind as in Eq(4) and also suppose that $\alpha > 0$,then*

$$D^{\alpha}(u_m(x)) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=0}^{i-\lceil\alpha\rceil} c_i \Omega_{i,k}^{(\alpha)} x^{i-k-\alpha}$$

where $\Omega_{i,k}^{(\alpha)}$ is given by

$$\Omega_{i,k}^{(\alpha)} = (-1)^k 2^{2i-2k} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\alpha)}$$

Proof. Since the Caputo's fractional differentiation is linear,so apply the fractional derivative on both sides of Eq(4), we have

$$D^{\alpha}(u_m(x)) = \sum_{i=0}^{m} c_i D^{\alpha}(U_i^*(x)) \cdots (5)$$

From definition 2.1, for $i = 0, 1, ...\lceil\alpha\rceil - 1, \alpha > 0$, we have , $D^{\alpha}(U_i^*(x)) = 0$

Also for $i = \lceil\alpha\rceil + \lceil\alpha\rceil + 1....m$ by property of linearity,we have

$$D^{\alpha}(U_i^*(x)) = \sum_{k=0}^{i} c_i (-1)^k 2^{2i-2k} \frac{\Gamma(2i-k+2) D^{\alpha} x^{i-k}}{\Gamma(k+1)\Gamma(2i-2k+2)}$$

by definition 2.1 we have

$$D^{\alpha}(U_i^*(x)) = \sum_{k=0}^{i-\lceil\alpha\rceil} c_i (-1)^k 2^{2i-2k} \frac{\Gamma(2i-k+2)\Gamma(i-k+1) x^{i-k-\alpha}}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k+1-\alpha)} \cdots (6)$$

Using values from Eq (6)into Eq (5),we have the desire result.

□

5. Procedure solution of the Fractional Delay Differential Equation

Consider the fractional delay differential equation of the type given in section 2 Eq (1).In order to use the Chebyshev polynomials of second kind,we first approximate $u(x)$ as;

$$u_m(x) = \sum_{i=0}^{m} c_i U_i^*(x) \cdots (7)$$

From Eq(1) and Eq(7) and Theorem (4.1) we have

$$\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=0}^{i-\lceil\alpha\lceil} c_i \Omega_{i,k}^{(\alpha)} x^{i-k-\alpha} = f(x, \sum_{i=0}^{m} c_i U_i^*(x), \sum_{i=0}^{m} c_i U_i^*(g(x)) \cdots (8)$$

We collect the equation Eq(8) at $(m+1-\lceil\alpha\rceil)$ points $x_p, p = 0, 1, ...m - \lceil\alpha\rceil$ as;

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=0}^{i-\lceil\alpha\lceil} c_i\Omega_{i,k}^{(\alpha)}x_p^{i-k-\alpha} = f(x_p,\sum_{i=0}^{m} c_iU_i^*(x_p),\sum_{i=0}^{m} c_iU_i^*(g(x_p))\cdots(9)$$

For suitable collection points we use the roots of the shifted Chebyshev polynomial of second kind $U^*_{m+1-\lceil\alpha\rceil}(x)$. Also by substituting the Eq.(7) in the boundary conditions (2),we can obtain $\lceil\alpha\rceil$ equations as follows:

$$\sum_{i=0}^{m}(-1)^{i+2}(i+1)c_i = \gamma_0,\sum_{i=0}^{m}(i+1)c_i = \gamma_1\cdots(10)$$

From Eq(10) together with $\lceil\alpha\rceil$ equations of boundary conditions gives $(m+1)$ linear or non-linear algebraic equations which can b solved easily for unknown $c_n, n = 0,1,2...m$. Thus $u(x)$ can be calculated.

6. Numerical Implementation

In this section,we solve numerically the fractional order delay differential equations using properties of Chebyshev polynomial of second kind.

Example 6.1. *Consider the following linear fractional delay differential equation;*

$$D^\alpha u(x) = u(x-\frac{1}{2}) + u^3(x) + \frac{2}{\Gamma(1.5)}x^{0.5} - (x-\frac{1}{2})^2 - x^6,\ 1<\alpha\leq 2$$

with the boundary conditions;

$$u(0) = 0, u(1) = 1\cdots(11)$$

The exact solution of this problem is $u(x) = x^2$

We implement the suggested method for $\alpha = 1.5$ *with* $m = 3$ *we approximate the solution as,*

$$u_3(x) = \sum_{i=0}^{3} c_iU_i^*(x)$$

Using Eq(9) we have

$$\sum_{i=2}^{3}\sum_{k=0}^{i-2} c_i\Omega_{i,k}^{(1.5)}x_p^{i-k-1.5} = \sum_{i=0}^{3} c_iU_i^*(x_p-0.5) + (\sum_{i=0}^{3} U_i^*(x_p))^3 + g(x_p)\cdots(12)$$

Where

$$g(x_p) = \frac{2}{\Gamma(1.5)}x_p^{0.5} - (x_p-\frac{1}{2})^2 - x_p^6$$

With $p = 0,1$ *where* x_p *are the roots of shifted Chebyshev polynomial of second kind* $U_2^*(x)$ *and their values are:*

$$x_0 = 0.7500, x_1 = 0.2500$$

By using Eq(10) and (11) we have

$$c_0 - 2c_1 + 3c_2 - 4c_3 = 0\cdots(13)$$

$$c_0 + 2c_1 + 3c_2 + 4c_3 = 1\cdots(14)$$

After solving Eq(12),(13) and (14) together, we find the table (1) of approximate values of $u(x)$ for $m = 3$.

The table (1) shows the comparison between the estimated values and exact values at each indicated point for $m = 3$. The exact values of $u(x)$ is denoted by u_{ext} and approximate values of $u(x)$ at $\alpha = 1.25$ is denoted by $u_{apx}^{\alpha=1.25}$. Similarly approximate values of $u(x)$ at $\alpha = 1.5$, $\alpha = 1.75$ and $\alpha = 2$ are denoted by $u_{apx}^{\alpha=1.5}$, $u_{apx}^{\alpha=1.75}$ and $u_{apx}^{\alpha=2}$ respectively. The error $|Error|_1$ denotes the absolute value of the error between exact value u_{ext} and approximate value $u_{apx}^{\alpha=1.25}$. Similarly $|Error|_2$, $|Error|_3$ and $|Error|_4$ denotes the absolute value of the error between exact value u_{ext} and approximate values $u_{apx}^{\alpha=1.5}$, $u_{apx}^{\alpha=1.75}$ and $u_{apx}^{\alpha=2}$ respectively. More convenient, we can write

$$|Error|_1 = |u_{ext} - u_{apx}^{\alpha=1.25}|, |Error|_2 = |u_{ext} - u_{apx}^{\alpha=1.5}|$$
$$|Error|_3 = |u_{ext} - u_{apx}^{\alpha=1.75}|, |Error|_4 = |u_{ext} - u_{apx}^{\alpha=2}|$$

.

x	u_{ext}	$u_{apx}^{\alpha=1.25}$	$u_{apx}^{\alpha=1.5}$	$u_{apx}^{\alpha=1.75}$	$u_{apx}^{\alpha=2}$	$\|Error\|_1$	$\|Error\|_2$	$\|Error\|_3$	$\|Error\|_4$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1000	0.010	0.110	0.0520	0.0230	0.0110	0.1000	0.0420	0.0130	$1.000\ e^{-3}$
0.2000	0.040	0.230	0.1400	0.0730	0.0420	0.1900	0.1000	0.0330	$2.000\ e^{-3}$
0.3000	0.090	0.370	0.2580	0.1550	0.0930	0.2800	0.1680	0.0650	$3.000\ e^{-3}$
0.4000	0.160	0.510	0.4020	0.2600	0.1640	0.3500	0.2420	0.1000	$4.000\ e^{-3}$
0.5000	0.250	0.680	0.5700	0.3790	0.2550	0.4300	0.3200	0.6290	$5.000\ e^{-3}$
0.6000	0.360	0.870	0.7580	0.5300	0.3640	0.5100	0.3980	0.1700	$4.000\ e^{-3}$
0.7000	0.490	1.110	0.9700	0.6900	0.4940	0.6200	0.4800	0.2000	$4.000\ e^{-3}$
0.8000	0.640	1.360	1.2010	0.8800	0.6440	0.7200	0.5610	0.2400	$4.000\ e^{-3}$
0.9000	0.810	1.600	1.4600	1.0900	0.8110	0.7900	0.6500	0.2800	$1.000\ e^{-3}$
1.0000	1	1.900	1.7450	1.3150	0.9990	0.9000	0.7450	0.315	$1.000\ e^{-3}$

TABLE 1. Comparison b/w exact and approximate values at $\alpha = 1.25, 1.5, 1.75, 2$ and $m = 3$

Example 6.2. *Consider the following linear fractional delay differential equation;*

$$D^{\alpha}u(x) = \frac{1}{2}e^{\frac{x}{2}}u(\frac{x}{2}) + \frac{1}{2}u(x),\ 0 < \alpha \leq 1$$

With the boundary conditions:

$$u(0) = 0, u(1) = 1 \cdots (15)$$

The exact solution of this problem is $u(x) = e^{x}$
We implement the suggested method with $m = 3$ and $\alpha = 0.25, 0.5, 0.75, 1$ as follows,

$$u_3(x) = \sum_{i=0}^{3} c_i U_i^*(x)$$

Using Eq(9) we have

$$\sum_{i=1}^{3}\sum_{k=0}^{i-1} c_i \Omega_{i,k}^{(0.25)} x_p^{i-k-0.25} = \frac{1}{2} e^{\frac{x_p}{2}} \sum_{i=0}^{3} c_i U_i^*(\frac{x_p}{2}) + \frac{1}{2}\sum_{i=0}^{3} c_i U_i^*(x_p) \cdots (16)$$

With $p = 0, 1$ *where* x_p *are the roots of shifted Chebyshev polynomial of second kind* $U_2^*(x)$ *and their values are:*

$$x_0 = 0.7500, x_1 = 0.2500$$

By using Eq(10) and (15)

$$c_0 - 2c_1 + 3c_2 - 4c_3 = 0 \cdots (17)$$

$$c_0 + 2c_1 + 3c_2 + 4c_3 = 1 \cdots (18)$$

After solving Eq(16),(17) and (18) together we find the table (2) of approximate values for $m = 3$.

The table (2) represent different values of the solution of the problem in Example (2) with different values of α.

The table (2) shows the comparison between the estimated values and exact values at each indicated point for $m = 3$.*The exact values of* $u(x)$ *is denoted by* u_{ext} *and approximate values of* $u(x)$ *at* $\alpha = 0.25$ *is denoted by* $u_{apx}^{\alpha=0.5}$. *Similarly approximate values of* $u(x)$ *at* $\alpha = 0.5$,$\alpha = 0.75$ *and* $\alpha = 1$ *are denoted by* $u_{apx}^{\alpha=0.5}$, $u_{apx}^{\alpha=0.75}$ *and* $u_{apx}^{\alpha=1}$ *respectively. The error* $|Error|_1$ *denotes the absolute value of the error between exact value* u_{ext} *and approximate value* $u_{apx}^{\alpha=0.25}$.*Similarly* $|Error|_2$,$|Error|_3$ *and* $|Error|_4$ *denotes the absolute value of the error between exact value* u_{ext} *and approximate values* $u_{apx}^{\alpha=0.5}$,$u_{apx}^{\alpha=0.75}$ *and* $u_{apx}^{\alpha=1}$ *respectively.More convenient,we can write*

$$|Error|_1 = |u_{ext} - u_{apx}^{\alpha=0.25}|, |Error|_2 = |u_{ext} - u_{apx}^{\alpha=0.5}|$$
$$|Error|_3 = |u_{ext} - u_{apx}^{\alpha=0.75}|, |Error|_4 = |u_{ext} - u_{apx}^{\alpha=1}|$$

.

x	u_{ext}	$u_{apx}^{\alpha=1}$	$u_{apx}^{\alpha=0.75}$	$u_{apx}^{\alpha=0.50}$	$u_{apx}^{\alpha=0.25}$	$\|Error\|_1$	$\|Error\|_2$	$\|Error\|_3$	$\|Error\|_4$
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	$0.0000\ e^{-0}$
0.1000	1.1052	1.1150	1.2160	1.4500	1.6020	0.4968	0.3448	0.1108	$9.8000\ e^{-03}$
0.2000	1.2214	1.2300	1.3910	1.7010	1.9990	0.7776	0.4796	0.1696	$8.6000\ e^{-03}$
0.3000	1.3499	1.3500	1.5650	1.9260	2.3460	0.9961	0.5761	0.2151	$1.0000\ e^{-04}$
0.4000	1.4918	1.4900	1.7450	2.1420	2.6780	1.1862	0.6502	0.2532	$1.8000\ e^{-03}$
0.5000	1.6487	1.6400	1.9320	2.3560	2.9650	1.3163	0.7073	0.2833	$8.7000\ e^{-03}$
0.6000	1.8221	1.8010	2.1280	2.5730	3.2410	1.4189	0.7509	0.3059	$2.1100\ e^{-02}$
0.7000	2.0138	2.0110	2.3340	2.7940	3.51240	1.4986	0.7802	0.3202	$2.8000\ e^{-03}$
0.8000	2.2255	2.2400	2.5530	3.0220	3.7800	1.5545	0.7965	0.3275	$1.4500\ e^{-02}$
0.9000	2.4596	2.4500	2.7840	3.2600	3.9654	1.5058	0.8004	0.3244	$9.6000\ e^{-03}$
1.0000	2.7183	2.7350	3.0290	3.5080	4.1020	1.3837	0.7897	0.3107	$1.6700\ e^{-02}$

TABLE 2. Comparison b/w exact and approximate values at $\alpha = 0.25, 0.50, 0.75, 1$ and $m = 3$

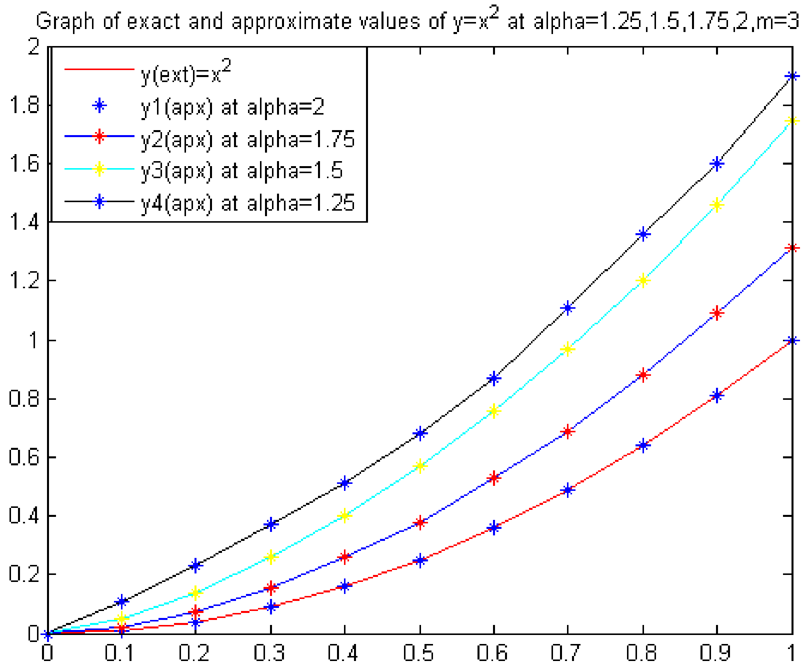

FIGURE 1. graph of example 1 by present method.

7. CONCLUSION AND REMARKS

In this paper,we are implemented Chebyshev polynomials of second kind to solve Delay Differential Equations of fractional order.The fractional derivative is applied in Caputo's sense.The properties of Chebyshev polynomials are used to reduce Delay Fractional Differential Equations into easily solvable linear or nonlinear algebraic equations.Two numerically solved examples shows the present method is well organized applied and calculated approximate values are in excellent agreement with the exact solutions and hence this approach can be solve the problem efficiently.In our suggested method,the graphs of approximate values at indicated values of α converge to the graph of the exact solution.This shows that our suggested method is more effective,valid and applicable.All numerical results are obtained using MATLAB Programming.

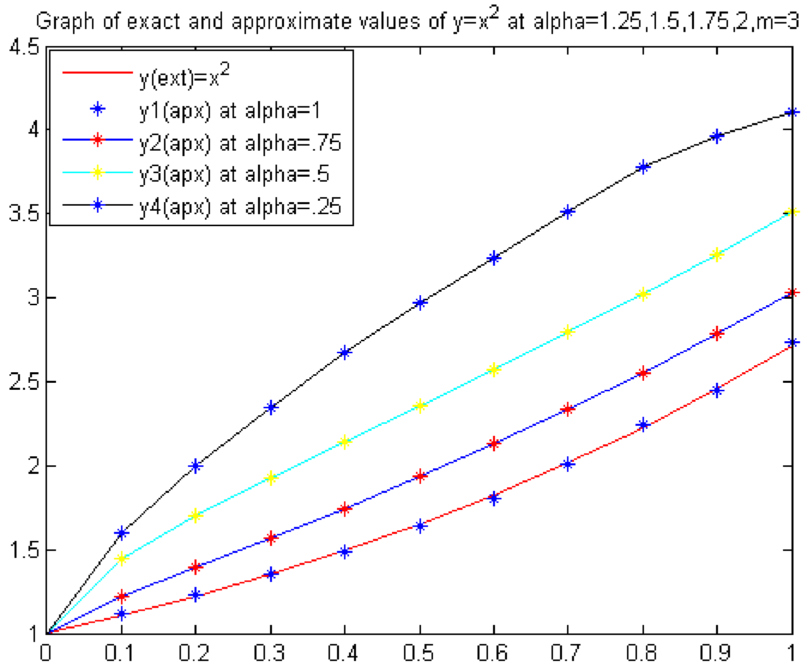

FIGURE 2. graph of example 2 by present method.

REFERENCES

[1] O. P. Agrawal, O. Defterli, and D. Baleanu,Fractional optimal control problems with several state and control variables,Journal of Vibration and Control, vol. 16, no. 13, pp. 19671976, 2010

[2] Ali Mehmet Akinlar,Aydin Secer and Mustafa Bayram,Numerical Solution of fractional Benney equation,Appl.Math.Inf.Sci.8,No.4,2014 pp 1633-1637

[3] R.L.Bagley,P.J. Torvik,Theoretical basis for the application of fractional calculus to viscoelasticity,Journal of Rheology,Vol.27,No.3, pp201-210,2013

[4] D. A. Benson, S. W. Wheatcraft, and M. M. Meerschaert,Application of a fractional advection-dispersion equation, Water Resources Research, vol. 36, no. 6, pp. 14031412, 2000

[5] Chang-Ming Chen,F. Liu,V.Anh and I.Turner, Numerical methods for solving a two dimensional variable-order anomalous subdiffusion equations, Mathematics of Computation Vol 81,No.277,Jan 2012, pp 345-366

[6] Concepcion A.Monje, Yangquan Chen,M.Vinangre,Dingyu Xue,Vicente Feliu,Fractional order systems and controls:fundamentals and applications Adva. Indus.Con.Springer-Verlag.London,2010

[7] Eid H Doha, Ali H Bhrawy, Dumitru Baleanu and Samer S Ezz-Eldien,The operational matrix formulation of the Jacobi tau approximation for space fractional diffusion equation,Springer open journal,Doha et al.Advances in difference equations,2014, 2014:231

[8] He, J.H.Approximate analytical solution for fractional derivatives in porous media. Computer Methods in Applied Mechanics and Engineering, 167 (1-2),1998: pp 57-68.

[9] I.I. Gorial,Numerical Solution for fractional Partial Differential euations of two sided,Journal of Al-Nahrain University Vol.12 (2), June,2009,pp 128-131

[10] A. A. Hemeda,Homotopy Perturbation Method for Solving Partial Differential Equations of Fractional Order,Int. Journal of Math. Analysis, Vol. 6, 2012, no. 49, 2431 - 2448

[11] Ibrahim Karatay,Nurdane Kale,Serife R.Bayramoglu,A New Difference Scheme for time fractional Heat equations based on the crank-nicholson method, An Int J for Theory and Applications, Vol 16,No.4 (2013)

[12] Ibrahim Karatay and Nurdane Kale,Finite difference method of fractional parabolic partial differential equations with variable coefficients, Int J. of Contemporary Mathematics Sciences,Vol. 9,2014.No.16,pp 767-776

[13] Ibrahim Karatay,Nurdane Kale,Serife R.Bayramoglu,A characheristics difference scheme for time fractional heat equations based on the crank-nicholson difference scheme,Hindawi publishing corporation Abstract and Applied Analysis Vol. 2012,Article ID 548292,11pages doi:10.1155/2012/548292

[14] M.A Iqbal, Ayyaz Ali and S.T. Mohyud-Din,Chebyshev Wavelets Method for Fractional Delay Differential Equations,International Journal of Modern Applied Physics, 2013, 4(1): 49-61

[15] H.Jafari, M. Soleymanivaraki, M. A.Firoozjaee,Legendre Wavelets for Solving Fractional Differential Equations,Journal of Applied Mathematics, Vol.7, No.4(27), Winter 2011, pp 65-70

[16] M.M Khader,A.S Hendy,The approximate and exact solutions of the fractional-order delay differential equations using legendre seudospectral method, Int J. Pure and Applied Mathematics,Vol 74 No.3 2012,pp 287-297

[17] M.M Khader,Numerical solution of Nonlinear Multi-order fractional differential equations by implementation of the operational derivative, Studies in Nonlinear Sciences 2(1):05-12,2011

[18] M.M Khadar,N.H Sweilam,T.A Assiri, On numerical solution for the fractional wave equation using legendre psedospectral method, Int J. Pure and Applied Mathematics,Vol 84 No.4 2013,pp 307-319

[19] M.M Khadar,N.H Sweilam,A.M.S. Mahdy,An Efficient method for solving the fractional diffusion equation, Int J. Pure and Applied Mathematics,Vol 1, No.2 2011,pp 1-12

[20] M.M. Khader,On the numerical solutions for the fractional diffusion equation,Communications in Nonlinear Science and Numerical Simulation,doi: 10.1016/j.cnsns.2010.09.007

[21] Mahmoud N. Sherif,Ibrahim Abouelfarag,T.S. Amer,Numerical solution of fractional Delay dufferential euations using spline functions, International Journal of Pure and Applied Mathematics,Volume 90 No. 1 2014, 73-83

[22] M. M. Meerschaert, H.PScheffler,C.Tadjeran,Finite difference methods for two-dimensional fractional dispersion equation,Journal of Computational Physics 211 (2006) 249261

[23] Osama H. Mohammed and Abbas I. Khlaif,Adomian Decomposition Method for Solving Delay Differential Equations of Fractional Order,IOSR Journal of Mathematics,Volume 10, Issue 6 Ver. I (Nov - Dec. 2014), PP 01-05

[24] E. Scalas, R. Gorenflo, and F. Mainardi,Fractional calculus and continuous-time finance, Physica A,vol. 284, no. 14, pp. 376384, 2000

[25] M.Seifollahi,A.S Shamloo,Numerical solution of non linear muti-order fractional differential equations by operational matrix of chebyshev polynomials,World Applied Programming Vol(3),March 2013,85-92

[26] Sweilam, N.H., M.M. Khader and R.F. Al-Bar,Homotopy perturbation method for multidimensional nonlinear coupled system of parabolic and hyperbolic equations,Topological Methods in Nonlinear Analysis, 31: 295-304.

[27] Wanhai Geng,Yiming Chen,Wavelet Method for Nonlinear Partial Differential Equations of Fractional Order,Computer and Information Science, Vol. 4, No. 5; September 2011,doi:10.5539/cis.v4n5p28

Department of Mathematics,Nizwa College of Technology, Sultanate of Oman