A Note on Invariant Submanifolds of *(LCS)ₙ*-Manifold

D. Nirmala and C.S. Bagewadi
Department of Mathematics, Kuvempu University,
Shankaraghatta - 577 451, Shimoga, Karnataka, INDIA.

Abstract**—** *The object of the present paper is to obtain a necessary condition for an invariant submanifold of* $(LCS)_n$ *-manifold satisfying the conditions* $Q(\sigma, R) = 0, Q(S, \sigma) = 0$ *and* $Q(\sigma, C) = 0$ *to be totally geodesic, where* $S, R, C,$ *are the Ricci tensor, curvature tensor and concircular curvature tensor respectively and* σ *is the second fundamental form.*

Keywords — *invariant submanifold,* $(LCS)_n$ *-manifold, totally geodesic.*

I. INTRODUCTION

This The study of LP-Sasakian manifolds, Matsumoto[8] and Shaikh [13]was introduced by Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds). Then applications to the general theory of relativitiy and cosmology have been investigated by Shaikh and Baishya [15, 16] Later, it has been carried out by Atcken [2], Prakasha [12], Shaikh and Hui [17], Sreenivasa et al. [22], Shaikh et al. [18] and others.

In modern analysis the geometry of submanifolds has significant applications in applied mathematics and theoretical physics. i.e., the notion of invariant submanifold is used to discuss properties of non-linear autonomous system [5]. Also the notion of geodesics plays an important role in the theory of relativity [9]. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Hence, totally geodesic submanifolds are also very much important in physical sciences. The study of geometry of invariant submanifolds was initiated by Okumara [10]. In general, the geometry of an invariant submanifold inherits almost all properties of the ambient manifold. The invariant submanifolds have been studied by many geometers like Yano and Ishihara [25], Kon [6, 7], Ozgur and Murathan [11], Sular and Ozgur [23], Atcken [2], Anitha and Bagewadi [1], De and Majhi [4], Shaikh et al. [18], Siddesha and Bagewadi [19] - [21] and many others

Motivated by the above studies, in the present paper we consider invariant submanifold of $(LCS)_n$ -manifold satisfying $Q(\sigma, R) = 0,$ $Q(S, \sigma) = 0$ and $Q(\sigma, C) = 0$ where $S, R\ C$ are the Ricci tensor, curvature tensor and concircular curvature tensor respectively and σ is the second fundamental form.

The paper is organized as follows: In section 2, we give some preliminaries which have been used later. In section 3, we recall the notion of $(LCS)_n$ -manifold. In section 4, we recall the definition of invariant submanifolds and some results. Sections 5, 6, 7 deal with the study of invariant submanifolds of $(LCS)_n$ -manifold satisfying $Q(\sigma, R) = 0,$ $Q(S, \sigma) = 0$ and $Q(\sigma, C) = 0$, where S, R, C, are the Ricci tensor, curvature tensor, and concircular curvature tensor respectively.

II. PRELIMINARIES

Let M be a submanifold of a $(LCS)_n$ -manifold $\bar{M}$ with induced metric g. Also let ∇ and $\nabla^{\perp}$ be the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M respectively then the Gauss and Weingarten formulas are given by,

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \qquad (2.1)$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \qquad (2.2)$$

for all $X, Y \in \Gamma(TM)$ and where σ and A_N are second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\bar{M}$, the second fundamental form σ and shape operator A_N are related by,

$$g(A_N X, Y) = g(\sigma(X, Y), N).$$

If $\sigma = 0$ then the manifold is said to be totally geodesic.

Now for a $(0, k)-$tensor T, $\alpha \geq 1$ and a $(0, 2)$-tensor B, $Q(B, T)$ is defined by

$$Q(B,T)(X_1, X_2 \dots X_k; X, Y) = -T((X \wedge_B Y)X_1, X_2, \dots X_k) - \dots - T(X_1, X_2, \dots, (X \wedge_B Y)X_k), \quad (2.3)$$

where $(X \wedge_B Y)$ is defined by

$$(X \wedge_B Y)Z = B(Y, Z)X - B(X, Z)Y. \quad (2.4)$$

A transformation of a n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle ,is called a concircular transformation. The interesting invariant concircular transformation is the curvature tensor C which is defined by

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}\{g(Y, Z)X - g(X, Z)Y\} \quad (2.5)$$

where r is the scalar curvature tensor of the manifold.

III. LCS- MANIFOLD

An n-dimensional Lorentzian manifold M is a smooth connected paracompact hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point the tensor. is a non-degenerate inner product of signature $(-, +, \dots, +)$, where $T_p M$ denotes the tangent vector space of M at p and R is the real number space.

Definition 3.1. *In a Lorentzian manifold (M, g) a vector field P defined by*

$$g(X, P) = A(X),$$

for any $X \in \Gamma(TM)$, is said to be a concircular vector field if

$$(\bar{\nabla}_X A)(Y) = \alpha[[g(X, Y) + \omega(X)A(Y)]$$

where α is a non- zero scalar and ω is a closed 1-form and $\bar{\nabla}$ denotes the operator of covariant differentiation of M with respect to the Lorentzian metric g.

Let M be an n-dimensional Lorentzian manifold admitting a unit time like concircular vector field ξ, called the characteristic vector field of the manifold then we have

$$g(\xi, , \xi) = -1$$

since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X).$$

the equation of the following form hold

$$(\bar{\nabla}_X \eta)(Y) = \alpha[[g(X, Y) + \eta(X)\eta(Y)], \alpha \neq 0$$

for all vector field X, Y,where $\bar{\nabla}$ denotes the operator of covariant differentiation of M with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$(\bar{\nabla}_X \alpha) = X(\alpha) = d\alpha(X) = \rho\eta(X),$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$, Let us take $\phi X = \bar{\nabla}_X \xi$ from which it follows that ϕ symmetric (1.1) tensor and called the structure tensor manifold. thus the Lorentzian manifold M together with unit time like concircular vector field ξ, its associated 1-form η and an (1.1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold) [13]. Especially, if we take $\alpha = 1$ then we can obtain the LP-Sasakian structure of Matsumoto [8]. In a $(LCS)_n$ -manifold $(n > 2)$, the following relation hold [13, 18]:

$$d\eta(X,Y) = g(X,\phi Y),\ \eta(X) = g(X,\xi), \quad (3.1)$$

$$\phi^2(X) = X + \eta(X)\xi,\ \eta(\xi) = -1$$

$$\varphi\xi = 0,\ \eta\cdot\varphi = 0, \quad (3.2)$$

$$g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y)$$

$$g(X,\phi Y) = -g(\phi X,Y), \quad (3.3)$$

$$\tilde{\nabla}_X\xi = \alpha\varphi X, \quad (3.4)$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \quad (3.5)$$

$$R(\xi,Y)Z = (\alpha^2 - \rho)[g(Y,Z)\xi - \eta(Z)Y], \quad (3.6)$$

$$(\nabla_X\phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \quad (3.7)$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \quad (3.8)$$

$$r = (n-1)(\alpha^2 - \rho), \quad (3.9)$$

for all $X,Y,Z \in \Gamma(TM)$, where R is the curvature tensor, S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. From (2.5), we have

$$C(X,Y)\xi = \left(\alpha^2 - \rho - \frac{r}{n(n-1)}\right)[\eta(Y)X - \eta(X)Y]. \quad (3.10)$$

IV. Invariant Submanifold of $(LCS)_n$ Manifold

A submanifold M of a $(LCS)_n$- manifold $\bar{M}$ is said to be invariant if the structure vector field ξ is tangent to M at every point of M and ϕX is tangent to M for any vector field X tangent to M at every point on M, that is, $\phi(TM) \subset TM$ at every point on M. The submanifold M of $(LCS)_n$-- manifold $\bar{M}$ is called totally geodesic if $\sigma(X,Y) = 0$ for any $X,Y \in \Gamma(TM)$.

Proposition 4.1. [18] Let M be an invariant submanifold of a $(LCS)_n$- manifold $\bar{M}$. Then the following equalities hold on M

$$\nabla_X\xi = \alpha\phi X, \quad (4.1)$$

$$\sigma(X,\xi) = 0, \quad (4.2)$$

$$R(\xi,X)Y = (\alpha^2 - \rho) + [g(X,Y)\xi - \eta(Y)X], \quad (4.3)$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \quad (4.4)$$

$$(\nabla_X\phi)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\} \quad (4.5)$$

$$\sigma(X,\phi Y) = \phi\sigma(X,Y), \quad (4.6)$$

for all vector fields X,Y tangent to M. So we have the following:

Theorem 4.2. [18] *An invariant submanifold M of a $(LCS)_n$- manifold $\bar{M}$ is a $(LCS)_n$- manifold.*

V. Invariant Submanifolds of a $(LCS)_n$-Manifolds Satisfying $Q(\sigma,R) = 0$

This section is devoted with the study of invariant submanifolds of $(LCS)_n$ - manifolds satisfying $Q(\sigma,R) = 0$. Therefore by using (2.3) and (2.4) with $B = \sigma, T = R$ we have i.e.,

$$-\sigma(V,X)R(U,Y)Z + \sigma(U,X)R(V,Y)Z - \sigma(V,Y)R(X,U)Z + \sigma(U,Y)R(X,V)Z - \sigma(V,Z)R(X,V)U + \sigma(U,Z)R(X,Y)V = 0. \quad (5.1)$$

Putting $Z = V = \xi$ in (5.1) in view of (4.2), we obtain

$$\sigma(U,X)R(\xi,Y)\xi + \sigma(U,Y)R(X,\xi)\xi = 0. \quad (5.2)$$

Using (4.3) in (5.2) we have

$$(\alpha^2 - \rho)\eta(Y)\sigma(U,X)\xi - (\alpha^2 - \rho)\sigma(U,X)Y + (\alpha^2 - \rho)\sigma(U,Y)X - (\alpha^2 - \rho)\sigma(U,Y)\xi = 0 \quad (5.3)$$

Taking inner product with w yields

$$(\alpha^2 - \rho)\eta(Y)\sigma(U,X)\eta(W) - (\alpha^2 - \rho)\sigma(U,X)g(Y,W) + (\alpha^2 - \rho)\sigma(U,Y)g(X,W) - (\alpha^2 - \rho)\eta(X)\sigma(U,Y)\eta(W) = 0 \quad (5.4)$$

Contracting Y and W we get

$$(\alpha^2 - \rho)\sigma(U,X) - (\alpha^2 - \rho)n\sigma(U,X) + (\alpha^2 - \rho)\sigma(U,X) = 0. \quad (5.5)$$

This implies

$$(\alpha^2 - \rho)(2-n)\sigma(U,X) = 0. \quad (5.6)$$

Hence $\sigma(U,X) = 0$, provided $(\alpha^2 - \rho) = 0$. Thus the manifold is totally geodesic. Conversely,

if $\sigma(X,Y) = 0$, then from (5.1), it follows that $Q(g,R) = 0$. Therefore, in view of the above results we get

Theorem 5.3. An invariant submanifold of a $(LCS)_n$- manifold with $(\alpha^2 - \rho) = 0$ satisfies $Q(\sigma, R) = 0$ if and only if it is totally geodesic.

Take $\alpha = 1$ then $\rho = -(\xi\alpha) = 0$ from (5.6) then $(2 - n)\sigma(U, X) = 0$.

Hence we can state

Corollary 5.1. An invariant submanifold of a LP-Sasakian manifold satisfies $Q(\sigma, R) = 0$ is totally geodesic.

VI. INVARIANT SUBMANIFOLDS OF $(LCS)_n$-MANIFOLD SATISFYING $Q(\sigma, S) = 0$

In this section we study invariant submanifolds of $(LCS)_n$ - manifold satisfying $Q(S, \sigma) = 0$. Therefore by using (2.3) and (2.4) with $B = \sigma, T = R$ we have i.e.,

$$-S(V, X)\sigma(U, Y) + S(U, X)\sigma(V, Y) - S(V, Y)\sigma(X, U) + S(U, Y)\sigma(X, V) = 0. \quad (6.1)$$

Putting $U = Y = \xi$ in (6.1) we obtain

$$S(\xi, \xi)\sigma(X, V) = 0 \quad (6.2)$$

This implies

$$(n - 1)(\alpha^2 - \rho)\sigma(X, V) = 0.$$

It follows that $\sigma(X, V) = 0$, provided $(\alpha^2 - \rho) = 0$. Hence totally geodesic. Conversely, let M be totally geodesic, then from (6.2) we get $Q(S, \sigma) = 0$. Thus we can state the following:

Theorem 6.4. An invariant submanifold of a $(LCS)_n$ – manifold with $(\alpha^2 - \rho) \neq 0$ satisfies $Q(S, \sigma) = 0$ if and only if it is totally geodesic.

Corollary 6.2. An invariant submanifold of a LP-Sasakian manifold satisfies $Q(S, \sigma) = 0$ if and only it is totally geodesic.

VII. INVARIANT SUBMANIFOLDS OF $(LCS)_n$-MANIFOLD SATISFYING $Q(\sigma, C) = 0$

In this section, we consider Invariant submanifolds of $(LCS)_n$-manifold satisfying $Q(\sigma, C) = 0$. Therefore by using (2.3) and (2.4) with $B = \sigma, T = R$ we have i.e.,

$$-\sigma(V, X)C(U, Y)Z + \sigma(U, X)C(V, Y)Z - \sigma(V, Y)C(X, U)Z + \sigma(U, Y)C(X, V)Z - \sigma(V, Z)C(X, V)U + \sigma(U, Z)C(X, Y)V = 0. \quad (7.1)$$

Putting $Z = V = \xi$ in (7.1) and in view of (4.2), we obtain

$$\sigma(U, X)C(\xi, Y)\xi + \sigma(U, Y)C(X, \xi)\xi = 0. \quad (7.2)$$

Using (3.10) in (7.2) we have

$$\left((\alpha^2 - \rho) - \frac{r}{n(n-1)}\right)\sigma(U, X)[\eta(Y)\xi - Y]$$

$$\left((\alpha^2 - \rho) - \frac{r}{n(n-1)}\right)\sigma(U, X)[X - \eta(X)\xi] = 0. \quad (7.3)$$

Taking inner product with W yields

$$\left((\alpha^2 - \rho) - \frac{r}{n(n-1)}\right)\sigma(U, X)$$

$$[\eta(Y)\eta(W) - g(Y, W)]\left((\alpha^2 - \rho) - \frac{r}{n(n-1)}\right)$$

$$\sigma(U, Y)[g(X)W - \eta(X)\eta(W)] = 0. \quad (7.4)$$

Contracting Y and W we get

$$\left((\alpha^2 - \rho) - \frac{r}{n(n-1)}\right)\sigma(U, X)(1 - n)$$

$$+\left((\alpha^2 - \rho) - \frac{r}{n(n-1)}\right)\sigma(U, X) = 0. \quad (7.5)$$

This implies

$$\left[(2 - n)\left((\alpha^2 - \rho) - \frac{r}{n(n-1)}\right)\right]\sigma(U, X) = 0. \quad (7.6)$$

and hence $\sigma(U, X) = 0$, provided $r \neq n(n-1)(\alpha^2 - \rho)$.

Thus the manifold is totally geodesic. Conversely, if $\sigma(X, Y) = 0$, then from (7.1), it follows that $Q(g, C) = 0$. Therefore, in view of the above results we get

Theorem 7.5. *An invariant submanifold of a $(LCS)_n$- manifold with $(n - 2)r\sigma(U, X) = 0$ satisfies $Q(\sigma, C) = 0$ if and only it is totally geodesic.*

Corollary 7.3. *An invariant submanifold of a LP-Sasakian manifold with $r \neq 0$ satisfies $Q(\sigma, C) = 0$ if and only it is totally geodesic.*

From the above Theorems (5.3), (6.2) and (7.5), and corollaries $(5.1)(6.2)(7.3)$ we can state the following

Corollary 7.4. *Let M be an invariant submanifold of (LCS)n- manifold, then following are equivalent*

(1) M is totally geodesic

(2) $Q(\sigma, R) = 0$ and $(\alpha^2 - \rho) \neq 0$,

(3) $Q(\sigma, S) = 0$ and $(\alpha^2 - \rho) \neq 0$,

(4) $Q(\sigma, C) = 0$ and $r \neq n(n-1)(\alpha^2 - \rho)$.

Corollary 7.5. *Let M be an invariant submanifold of LP- sasakian- manifold, then following are equivalent*

(1) *M is totally geodesic*

(2) $(\sigma, R) = 0$,

(3) $Q(\sigma, S) = 0$,

(4) $Q(\sigma, C) = 0$ *and* $r \neq 0$.

REFERENCES

[1] B.S. Anitha and C.S. Bagewadi, *Invariant submanifolds of Kenmotsu manifolds admitting semi-symmetric meetric connection-II*, Acta Universitatis Apulensis, 36 (2013), 129-142.

[2] M. Atceken, *on Geometry of submanifolds of $(LCS)_n$--manifolds*, International Journal of Mathematics and Mathematical Sciences Volume 2012,Article ID 304647,11 Pages. 62.

[3] D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture notes in Math., 509, Springer-Verlag, Berlin (1976).

[4] U.C. De and Pradip Majhi, *On invariant submanifolds of Kenmotsu manifolds*, J. Geom. 106 (2015), 109-122.

[5] Z. Guojing and W. Jianguo, *Invariant sub-manifolds and modes of nonlinear autunomous systems*, Appl. Math. Mech. 19(7) (1998), 687-693.

[6] M. Kon, *Invariant submanifolds of normal contact metric manifolds*, Kodai Math. Sem. Rep., 27 (1973), 330-336.

[7] M. Kon, *Invariant submanifolds in Sasakian manifolds*, Math. Ann., 219 (1976), 277-290. [8] K. Matsumoto, *On Lorentzian almost para contact manifolds*, Bull. Yamagata Univ. Nat. Sci., 12 (1989), 151-156.

[9] K. Matsumoto, I. Mihai and R. Rosca, *ξ-null geodesic gradient vector fields on a Lorentzian para-Sasakian manifold*, J. Korean Math. Soc. 32(1) (1995), 17-31.

[10] M. Okumura, *On contact metric immersion*, Kodai Math. Sem. Rep., 20 (1968), 389-409.

[11] C. Ozgur and C. Murathan, *On invariant submanifolds of Lorenzian para-Sasakian manifolds*, The Arabian J. Sci. Engg., 34, 24 (2009), 177-185.

[12] D.G. Prakasha, *On Ricci η-recurrent $(LCS)_n$--manifolds*, Acta Universitatis Apulensis, 24 (2010), 449- 461.

[13] A.A. Shaikh, *Lorentzian almost paracontact manifolds with a structure of concircular type*, Kyungpook Math. J., 43(2)(2003),305-314.

[14] A.A. Shaikh, *Some Results on $(LCS)_n$-- manifolds*, Korean Math. Soc., 46 (2009) No.3, pp.449-461.

[15] A.A. Shaikh and K.K. Baishya, *On concircular structure spacetimes*, Am. J. Appl. Sci., 3(4) (2006), 1790-1794.

[16] A.A. Shaikh and K.K. Baishya, *On concircular structure spacetimes $- II$*, J. Math. Stat. 1 (2005), 129-132.

[17] A.A. Shaikh and S.K. Hui, *On generalized ϕ $-$ recurrent $(LCS)_n$ $-$ $-$manifolds*
, Proceedings of the AIP conference, 1309 (2010), 419-429.

[18] A.A. Shaikh, Y. Matsuyama, and S.K. Hui, *On invariant submanifolds of $(LCS)_n$ -manifolds*, Journal Of the Egyptian Mathematical Society, 24 (2016), 263-269.

[19] M.S. Siddesha and C.S. Bagewadi, *Totally geodesic submanifolds of (k, μ) $-$contact manifolds*, ISOR Journal of Mathematics Volume 12,(Jul.-Aug.2016),1-6.

[20] M.S. Siddesha and C.S. Bagewadi, *On some classes of invariant submanifolds of (k, µ)-contact manifold*, Journal of Informatics and Mathematical Sciences (Accepted for publication).

[21] M.S. Siddesha and C.S. Bagewadi, *Submanifolds of (k, μ)-contact manifold*, CUBO-A Mathematical Journal (Accepted for publication).

[22] G.T. Sreenivasa, Venkatesha and C.S. Bagewadi, *Some results on $(LCS)_{2n+1}$-manifolds*, Bull. Math. Anal. Appl., 1(3) (2009), 64-70.

[23] S. Sular and C. Ozgr, *On some submanifolds of Kenmotsu manifolds*, Chaos, Solitons and Fractals, 42 (2009), 1990-1995.

[24] L. Verstraelen, *Comments on pseudosymmetry in the sense of Ryszard Deszcz. In: Geometry and Topology of submanifolds*, World Scientific Publishing, River Edge, vol. VI, (1994), 199-209.

[25] K. Yano and S. Ishihara, *Invariant submanifolds of almost contact manifold*, Kodai Math. Sem. Rep., 21 (1969), 350-364.