Quasi-static Von-Karman evolution and Numerical approach

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Abstract

In this paper we consider the vibration of nonlinear deformation of elastic shallow shell. This is a parabolic problem of Von-Karman evolution without rotational inertia, in quasi-static form. The aim of this article is to finding a condition verified by the internal and external loads in up to have a uniqueness weak solution.
For illustrate our theoretical results we use the method of finite difference known that by alternating direction implicit schemes (ADI).

Keywords: *Elastic shallow shell, Quasi-static Von-Karman, Finite difference method, ADI methods.*

1. Introduction

In [3] I.Chueshov and I.Lasiecka present the problem of quasi-static Von-Karman evolution and establish the existence of weak solution, but the uniqueness is not known. However the existence and uniqueness dose hold for strong solution. This model in the quasi-static form of clamped boundary conditions describe the case when the inertia forces are small in compression with the resisting forces of the medium ($\mu = 0$). We well known the quasi-static Von-Karman evolution without rotational inertia ($\alpha = 0$), for vertical displacement $u(x, y, t)$ and the Airy stress $\phi(x, y, t)$ has the following form [3] :

$$(P_0)\begin{cases} u_t + \Delta^2 u - [\phi + F_0, u + \theta] + L(u) = p & in \quad \omega \times [0, T], \\ u_{|t=0} = u_0 & in \quad \omega, \\ u = \partial_\nu u = 0 & on \quad \Gamma \times [0, T], \\ \Delta^2 \phi + [u, u + 2\theta] = 0 & in \quad \omega \times [0, T], \\ \phi = 0, \partial_\nu \phi = 0 & on \quad \Gamma \times [0, T]. \end{cases}$$

Where ω is the middle surface of the shell, u_0 is initial datum and $[.,.]$ is the Monge-Ampère operator [10]. The shell is subjected to the internal force F_0 and external force p, but $\theta(x, y)$ [2] is the mapping measuring the deviation of the middle surface of the reference configuration of the shell from a plane, moreover L is a linear operator source and characterize the non conservative potentially loads to the system.
The aim of this article is to finding a condition verified by the external and internal loads and the linear bounded operator L [3], in up to have a weakly uniqueness solution of the Von-Karman evolutions, without rotational inertia with clamped boundary conditions, in quasi-static form, by another approach distinct from the preceding presented in [3], one which yields immediately a simple numerical approach of finite difference method to the considered problem.
This paper is organized as follows. In section 2 we present the some theoretical results for established a uniqueness weak solution. But in the third section we use the noncoupled approach of 13-point [9] and the alternating direction implicit schemes [11] for approached this solution.

2. Dynamic quasi-static Von-Karman equations

In this paper, ω denotes a nonempty bounded open domain in IR^2, with regular boundary $\Gamma = \partial\omega$.
Let us consider the following problem:[3]
Fund $(u, \phi) \in (L^2([0, T], H_0^2(\omega)))^2$ such that

$$(P_0)\begin{cases} u_t + \Delta^2 u - [\phi + F_0, u + \theta] + L(u) = p & in \quad \omega \times [0, T], \\ u_{|t=0} = u_0 & in \quad \omega, \\ u = \partial_\nu u = 0 & on \quad \Gamma \times [0, T], \\ \Delta^2 \phi + [u, u + 2\theta] = 0 & in \quad \omega \times [0, T], \\ \phi = 0, \partial_\nu \phi = 0 & on \quad \Gamma \times [0, T]. \end{cases}$$

Where $T \succ 0$ is a real number, $u_t = \frac{\partial u}{\partial t}$ and
$[\phi, u] = \partial_{11}\phi\partial_{22}u + \partial_{11}u\partial_{22}\phi - 2\partial_{12}\phi\partial_{12}u$.
Let $p \geq 1$ and $m \in IN^*$, we put by:
$|u|_p = (\int_\omega |u|^p)^{1/p}$, $\|u\| = |\Delta u|_2$, $\|u\|_{m,\omega}$ the classical norm in $H^m(\omega)$ and
$W(0, T) = \left\{u/u \in L^2([0, T], H_0^2(\omega)), u_t \in L^2([0, T], L^2(\omega))\right\}$
is a complete Hilbert space with associated norm
$\|.\|_{W(0,T)} = (\|.\|^2_{L^2([0,T],H_0^2(\omega))} + |.|^2_{L^2([0,T],L^2(\omega))})^{1/2}$.

Theorem 2.1 [7, 8] *Let $f \in L^2(\omega)$, then the next problem*

$$(Q)\begin{cases} \Delta^2 v = f & in \quad \omega, \\ v = 0 & on \quad \Gamma, \\ \partial_\nu v = 0 & on \quad \Gamma. \end{cases}$$

Has one and only one solution v in $H_0^2(\omega) \cap H^4(\omega)$ satisfying that $\|v\| \leq c_0 |f|_2$. Where $c_0 \succ 0$ is a constant which depends only of $mes(\omega)$.

Remark 2.1 *If $f \in L^2([0,T], L^2(\omega))$, the uniqueness solution of the problem (Q) is in $L^2([0,T], H_0^2(\omega) \cap H^4(\omega))$.*

Theorem 2.2 [6] *Let $0 \prec T \leq +\infty$ and f in $L^2([0,T], L^2(\omega))$. The Dirichlet problem for linear fourth order parabolic equation:*

$$u_t + \Delta^2 u = f \quad in \quad \omega \times [0,T]$$

with initial datum $u_0 \in H_0^2(\omega)$ admits a unique weak solution in the space
$C([0,T], H_0^2(\omega)) \cap L^2([0,T], H^4(\omega)) \cap H^1([0,T], L^2(\omega))$.
The corresponding problem with initial datum u_0 in $H_0^1(\omega) \cap H^2(\omega)$ admits a unique weak solution in the following space
$C([0,T], H^2(\omega) \cap H_0^1(\omega)) \cap L^2([0,T], H^4(\omega)) \cap H^1([0,T], L^2(\omega))$.
Furthermore, both cases admit the estimate.
$\sup\limits_{0\leq t\leq T} \|u\|^2 + \int_0^T \|u\|^2 + \int_0^T |u_t|_2^2 \leq c(\|u_0\|^2 + \int_0^T |f|_2^2)$.

We will study the problem (P_0) by considering the following iterative problem :
Let $n \geq 2$ and $0 \neq u^1(x,y) \in H_0^2(\omega)$ is given. In the firstly we find $\phi_{n-1} \in H_0^2(\omega)$ as a solution to the problem $\Delta^2 \phi_{n-1} = -[u_{n-1}, u_{n-1} + 2\theta]$ and $u_n \in L^2([0,T], H_0^2(\omega))$ is constructed by the following problem :

$$(P_n)\begin{cases} (u_n)_t + \Delta^2 u_n = F_1(u_{n-1}, \phi_{n-1}) + p & in \quad \omega \times [0,T], \\ u_n = \partial_\nu u_n = 0 & on \quad \Gamma \times [0,T], \\ (u_n)_{|t=0} = u_0 & in \quad \omega. \end{cases}$$

Where $\widetilde{F}(u_{n-1}, \phi_{n-1}) = (F_1, F_2) = ([\,\phi_{n-1} + F_0, u_{n-1} + \theta\,] - L(u_{n-1}), -[\,u_{n-1}, u_{n-1} + 2\theta\,])$.

Remark 2.2 *By virtues of the theorem 2.1 and theorem 2.2 . If $\forall n \geq 0$, (u_n, ϕ_{n-1}) is a solution of the problem (P_n), this solution has the regularity :*
$(u_n, \phi_{n-1}) \in L^2([0,T], H^4(\omega) \cap H_0^2(\omega)) \times H^4(\omega) \cap H_0^2(\omega)$.

Theorem 2.3 *Let $c \succ 0$, $\widetilde{u} = (u, \phi)$ and $\widetilde{v} = (v, \psi)$ in $L^2([0,T], H^4(\omega) \cap H_0^2(\omega)) \times H^4(\omega) \cap H_0^2(\omega)$ such that $\forall 0 \leq t \leq T$, $\|\widetilde{u}\|_{H_0^2(\omega)\times H_0^2(\omega)} \leq c$ and $\|\widetilde{v}\|_{H_0^2(\omega)\times H_0^2(\omega)} \leq c$. If $\|\theta\|_{2,\omega}$, $\|F_0\|_{2,\omega}$ are small and $\|L\| \prec 1$, then there exists $0 \prec c_1 \prec 1$ such that*
$\left\|\widetilde{F}(\widetilde{u}) - \widetilde{F}(\widetilde{v})\right\|_{(L^2(\omega))^2} \leq c_1 \|\widetilde{u} - \widetilde{v}\|_{H_0^2(\omega)\times H_0^2(\omega)}$.

Proof Let $\widetilde{u} = (u, \psi)$ and $\widetilde{v} = (v, \varphi)$ in $L^2([0,T], H^4(\omega) \cap H_0^2(\omega)) \times H^4(\omega) \cap H_0^2(\omega)$ such that $\|\widetilde{u}\|_{H_0^2(\omega)\times H_0^2(\omega)} \leq c$ and $\|\widetilde{v}\|_{H_0^2(\omega)\times H_0^2(\omega)} \leq c$, we have

$$\begin{aligned} \left\|\widetilde{F}(\widetilde{u}) - \widetilde{F}(\widetilde{v})\right\|_{(L^2(\omega))^2} \leq & \; |[\psi, u] + [\psi - \varphi, \theta] + [u - v, F_0] - [\phi, v]|_2 \\ & + |-[u,u] - [u - v, 2\theta] + [v,v]|_2 + |L(u-v)|_2, \\ \leq & \; |[\psi, u] - [\phi, v]|_2 + |[\psi - \varphi, \theta]|_2 + |[u,u] - [v,v]|_2 \\ & + |[u - v, 2\theta]|_2 + \|L\|\,\|u - v\| + |[u - v, F_0]|_2. \end{aligned}$$

It follows that

$$\begin{aligned} |[\psi, u] - [\phi, v]|_2 &= |[\psi - \varphi, u] - [\phi, u - v]|_2 \\ &\leq |[\psi - \varphi, u]|_2 + |[\phi, u - v]|_2 \end{aligned}$$

Using the injection $H^2(\omega) \hookrightarrow C(\overline{\omega})$ we have

$$\begin{aligned} |[\psi - \varphi, u]|_2 \; & \leq (\textstyle\int_\omega |\partial_{11}(\psi - \varphi)|^2 |\partial_{22} u|^2)^{1/2} + (\int_\omega |\partial_{22}(\psi - \varphi)|^2 |\partial_{11} u|^2)^{1/2} \\ & + 2(\textstyle\int_\omega |\partial_{12}(\psi - \varphi)|^2 |\partial_{12} u|^2)^{1/2}, \\ & \leq \|\partial_{11}(\psi - \varphi)\|_{+\infty} |\partial_{11} u|_2 + \|\partial_{22}(\psi - \varphi)\|_{+\infty} |\partial_{22} u|_2 \\ & + 2\|\partial_{12}(\psi - \varphi)\|_{+\infty} |\partial_{12} u|_2, \\ & \leq c_2(\|\psi - \varphi\|\,\|u\| + \|\psi - \varphi\|\,\|u\| + 2\|\psi - \varphi\|\,\|u\|) \\ & \leq 4cc_2 \|\psi - \varphi\|. \end{aligned}$$

By an analogous method we have
$|[\varphi, u - v]|_2 \leq 4cc_2 \|u - v\|$,
$|[\psi - \varphi, \theta]|_2 \leq 4c_2 \|\theta\|_{2,\omega} \|\psi - \varphi\|$,
$|[u - v, 2\theta]|_2 \leq 8c_2 \|\theta\|_{2,\omega} \|u - v\|$,
$|[u,u] - [v,v]|_2 \leq 8cc_2 \|u - v\|$.
$|[u - v, F_0]|_2 \leq 4c_2 \|F_0\|_{2,\omega} \|u - v\|$.
It become that

$$\begin{aligned} \left\|\widetilde{F}(\widetilde{u}) - \widetilde{F}(\widetilde{v})\right\|_{(L^2(\omega))^2} \leq & \; c_2(12c + 8\|\theta\|_{2,\omega} + \|L\| \\ & + 4\|F_0\|_{2,\omega}) \|\widetilde{u} - \widetilde{v}\|_{H_0^2(\omega)\times H_0^2(\omega)}. \end{aligned}$$

We put by $c_1 = 12c + 8\|\theta\|_{2,\omega} + \|L\| + 4\|F_0\|_{2,\omega}$, if we choose $0 \prec c \prec \frac{1 - \|L\|}{12}$, $0 \prec c_1 c_2 \prec 1$ and $8\|\theta\|_{2,\omega} + 4\|F_0\|_{2,\omega} \prec 1 - 12c - \|L\|$, we have $\left\|\widetilde{F}(\widetilde{u}) - \widetilde{F}(\widetilde{v})\right\|_{(L^2(\omega))^2} \leq c_1(\|\widetilde{u} - \widetilde{v}\|_{H_0^2(\omega)\times H_0^2(\omega)})$ and $0 \prec c_1 \prec 1$. Then the desired result is obtained.

Remark 2.3 *by virtue of the theorem 2.3 there exist a constant $0 \prec c_1 \prec 1$ such that*

$$\begin{aligned} |F_1(u) - F_1(v)|_2 &\leq \left\|\widetilde{F}(\widetilde{u}) - \widetilde{F}(\widetilde{v})\right\|_{(L^2(\omega))^2} \\ &\leq c_1(\|\widetilde{u} - \widetilde{v}\|_{H_0^2(\omega)\times H_0^2(\omega)}), \end{aligned}$$

and by analogous method from the theorem 2.3 we can proved that. $|F_2(u) - F_2(v)|_2 \leq c_1 \|u - v\|$.

Proposition 2.1 *let u, v in $H_0^2(\omega)$ and θ in $H^2(\omega)$, of small norm. If ϕ and ψ are two solutions of the following Dirichlet problem:*
$\Delta^2 \phi = -[u + 2\theta, u]$ *and* $\Delta^2 \psi = -[v + 2\theta, v]$.
Then, there exist $0 \prec c_1 \prec 1$, such that
$|[u, \phi] - [v, \psi]|_2 \leq c_1 \|u - v\|$.

Proof Let $c \succ 0$, $\|u\| \leq c$ and $\|v\| \leq c$. In [3] we have

$$\begin{aligned} |[u, \phi] - [v, \psi]|_2 &\leq c_0(\|u\|^2 + \|v\|^2 + \|\theta\|_{2,\omega}^2)\,\|u - v\| \\ &\leq c_0(2c^2 + \|\theta\|_{2,\omega}^2)\,\|u - v\|. \end{aligned}$$

If we choose c sufficiently small, $c_1 = 2c^2 + \|\theta\|_{2,\omega}^2 \prec 1$ and $0 \prec c_1 c_0 \prec 1$, we conclude that
$|[u, \phi] - [v, \psi]|_2 \leq c_1 \|u - v\|$ and $0 \prec c_1 \prec 1$.

Proposition 2.2 *Let $u \in H_0^2(\omega)$ and ϕ be a uniqueness solution of the following problem:*

$$\begin{cases} \Delta^2 \phi + [u, u + 2\theta] = 0 & in \quad \omega \times [0,T], \\ \phi = 0, \partial_\nu \phi = 0 & on \quad \Gamma \times [0,T]. \end{cases}$$

Then, there exist a constant $K \succ 0$ such that
$([\,\phi + F_0, u + \theta\,], u)_{L^2(\omega), L^2(\omega)} \leq -\|\phi\|^2 + K$.
Where $(.,.)_{L^2(\omega), L^2(\omega)}$, is the duality crochet in $L^2(\omega)$.

Proof : Let $u \in H_0^2(\omega)$, then
$([\,\phi + F_0, u+\theta\,], u)_{L^2(\omega),L^2(\omega)} = ([\,u, u+2\theta\,], \phi)_{L^2(\omega),L^2(\omega)} - ([\,u,\phi\,],\theta)_{L^2(\omega),L^2(\omega)} + ([\,u,u+\theta\,], F_0)_{L^2(\omega),L^2(\omega)} . = -\int_\omega \Delta^2\phi\phi - \int_\omega \theta\,[u,\phi] + \int_\omega F_0\,[u+\theta,u]$.
By using the Green formula and the injection $H^2(\omega) \hookrightarrow C(\overline{\omega})$ we have
$$([\,\phi+F_0,u+\theta\,],u)_{L^2(\omega),L^2(\omega)} \le -\|\phi\|^2 + \|\theta\|_\infty\,|[\,u,\phi\,]|_1 + \|F_0\|_\infty\,|[\,u+\theta,u\,]|_1.$$
According to the theorem 2.3, there exist a constant $K \succ 0$ which depend only of $\|\theta\|_{2,\omega}$, $\|F_0\|_{2,\omega}$, $\|u\|$ such that $([\,\phi+F_0,u+\theta\,],u)_{L^2(\omega),L^2(\omega)} \le -\|\phi\|^2 + K$.

Theorem 2.4 *Let $p(x,y) \in L^2(\omega)$ and $u_0 \in H_0^2(\omega)$. If $\|\theta\|_{2,\omega}$, $\|F_0\|_{2,\omega}$, $|p|_2$, $\|u_0\|^2$ are small and $\|L\| \prec 1$, then the problem (P_0) has one and only one weak solution (u,ϕ) in the following space : $(C([0,T], H_0^2(\omega)) \cap L^2([0,T], H_0^2(\omega))) \times H_0^2(\omega)$ and $u_t \in L^2([0,T], L^2(\omega))$. Such that for any $0 \prec t \prec T$ this solution verifies the inequality*
$|u|_2^2 + \int_0^t(\|u\|^2 + \|\phi\|^2) \le |u_0|_2^2 + Kt.$
Moreover there exist a constant $w \succ 0$, such that
$|u|_2^2 \le |u_0|_2^2\, e^{-wt} + \frac{K}{w}$.
Where, $K \succ 0$ is a constant which depend only of $\|\theta\|_{2,\omega}$ and $\|F_0\|_{2,\omega}$.

Proof Let consider the problem (P_n) with $u^1(x,y) \neq 0$ as not depend of t.
We will prove that for all $n \ge 2$ and $\forall 0 \le t \le T$, $\|u_n\|^2 \le \|u^1\|$, $\|\phi_{n-1}\|^2 \le \|u^1\|$ and $\|u_n\|^2_{W(0,T)} \le \|u^1\|$.
Suppose that for $k = 2,...,n$ and $\forall 0 \le t \le T$, $\|u_k\|^2 \le \|u^1\|$, $\|\phi_{k-1}\|^2 \le \|u^1\|$ and $\|u_k\|^2_{W(0,T)} \le \|u^1\|$.
According to the theorem 2.1 and remark 2.3 we have $\|\phi_n\|^2 \le c_0\,|[u_n+2\theta,u_n]|^2 \le c_0\,|F(u_n)|_2^2 \le c_0c_1\,\|u_n\|^2$.
Since (u_{n+1},ϕ_n) is a solution of the problem (P_{n+1}) by virtue of the theorem 2.2 yields that there exists $c_0 \succ 0$ such that $\forall 0 \le t \le T$
$$\begin{aligned}
&\sup_{0\le t\le T} \|u_{n+1}\|^2 + \int_0^T(\|u_{n+1}\|^2 + |(u_{n+1})_t|_2^2) \\
&\le c_0(\|u_0\|^2 + \int_0^T(\|\widetilde{F}(u_n,\phi_n)\|^2_{(L^2(\omega))^2} + |p|_2^2) \\
&\le c_0(\|u_0\|^2 + \int_0^T c_1^2(\|(u_n,\phi_n)\|^2_{H_0^2(\omega)\times H_0^2(\omega)} + (|p|_2)^2) \\
&\le c_0(\|u_0\|^2 + \int_0^T c_1^2(4\|u_n\|^2 + 4\|\phi_n\|^2 + (|p|_2)^2) \\
&\le c_0(\|u_0\|^2 + \int_0^T c_1^2(4\|u_n\|^2 + 4c_1^2c_0^2\|u_n\|^2 + (|p|_2)^2).
\end{aligned}$$
If we choose $c \succ 0$ sufficiently small, then $0 \prec c_1 \prec 1$, $0 \prec c_0c_1 \prec 1$, $0 \prec 8c_0^2c_1T \prec 1$ and
$\|u_0\|^2 + |p|_2^2 \le \frac{(1-8c_0^2c_1^2T)}{c_0}\,\|u^1\|$, then
$$\begin{aligned}
&\sup_{0\le t\le T} \|u_{n+1}\|^2 + \int_0^T(\|u_{n+1}\|^2 + |(u_{n+1})_t|_2^2) \\
&\le c_0(\|u_0\|^2 + 8c_1^2c_0^2T(\|u^1\| + |p|_2^2)),
\end{aligned}$$
It follows that $\|u_{n+1}\|^2 \le \|u^1\|$, $\|\phi_n\|^2 \le \|u^1\|$ and $\|u_{n+1}\|^2_{W(0,T)} \le \|u^1\|$.
Hence for all $n \ge 2$ and $\forall 0 \le t \le T$, $\|u_n\|^2 \le \|u^1\|$, $\|\phi_n\|^2 \le \|u^1\|$ and $\|u_n\|^2_{W(0,T)} \le \|u^1\|$.
Let $m \prec n$, (u_n,ϕ_{n-1}) (resp, (u_m,ϕ_{m-1})) is a solution of the problem (P_n) (resp, (P_m)), then $(u_n - u_m)$ is a solution of the following problem :
$$\begin{cases}
(u_n)_t - (u_m)_t + \Delta^2(u_n-u_m) = F_1(u_{n-1},\phi_{n-1}) - F_1(u_{m-1},\phi_{m-1}) \\
u_n - u_m = 0 \\
\partial_\nu(u_n-u_m) = 0 \\
(u_n-u_m)_{|t=0} = 0\ ,
\end{cases}$$
Using theorem 2.2 again, we have
$$\begin{aligned}
&\sup_{0\le t\le T} \|u_n-u_m\|^2 + \int_0^T(\|u_n-u_m\|^2 + |(u_n)_t-(u_m)_t|_2^2) \\
&\le c_0\int_0^T |F_1(u_{n-1},\phi_{n-1}) - F_1(u_{m-1},\phi_{m-1})|_2^2\,, \\
&\le c_0c_1^2\int_0^T(\|(u_{n-1}-u_{m-1}),(\phi_{n-1}-\phi_{m-1})\|^2_{H_0^2(\omega)\times H_0^2(\omega)}\,, \\
&\le c_0c_1^2\int_0^T(4\|u_{n-1}-u_{m-1}\|^2 + 4\|\phi_{n-1}-\phi_{m-1}\|^2).
\end{aligned}$$
Moreover $(\phi_{n-1}-\phi_{m-1})$ is a solution of the next problem :
$$\begin{cases}
\Delta^2(\phi_{n-1}-\phi_{m-1}) = -[u_{n-1}+2\theta,u_{n-1}] + [u_{m-1}+2\theta,u_{m-1}] \\
\phi_{n-1}-\phi_{m-1} = 0 \text{ and } \partial_\nu(\phi_{n-1}-\phi_{m-1}) = 0
\end{cases}$$
Afterword theorem 2.1 and remark 2.3 we have $\|\phi_{n-1}-\phi_{m-1}\| \le c_0c_1\,\|u_{n-1}-u_{m-1}\|$.
It become that
$$\begin{aligned}
&\sup_{0\le t\le T} \|u_n-u_m\|^2 + \int_0^T(\|u_n-u_m\|^2 + |(u_n)_t-(u_m)_t|_2^2) \\
&\le c_0c_1^2\int_0^T(4\|u_{n-1}-u_{m-1}\|^2 + 4(c_0c_1)^2\|u_{n-1}-u_{m-1}\|^2), \\
&\le 8c_0^2c_1^2\int_0^T(\|u_{n-1}-u_{m-1}\|^2), \\
&\le (8c_0^2c_1^2)^{m-2}\int_0^T ... \int_0^T(\|u_{n-m+2}-u^1\|^2), \\
&\le (8c_0^2c_1^2)^{m-2}\int_0^T ... \int_0^T \sum_{k=0}^{n-m+1}(8c_0^2c_1^2)^k \int_0^T ... \int_0^T \|(u_2-u^1)\|^2), \\
&\le (8c_0^2c_1^2T)^{m-2}\sum_{k=0}^{n-m+1}(8c_0^2c_1^2T)^k(\|u^1\| + \|u^1\|^2).
\end{aligned}$$
We conclude that
$$\begin{aligned}
\|u_n-u_m\|^2_{W(0,T)} &= \int_0^T \|u_n-u_m\|^2 + |(u_n)_t-(u_m)_t|_2^2 \\
&\le (8c_0^2c_1^2T)^{m-2}\sum_{k=0}^{n-m+1}(8c_0^2c_1^2T)^k(\|u^1\| + \|u^1\|^2),
\end{aligned}$$
$$\sup_{0\le t\le T}\|u_n-u_m\|^2 \le (8c_0^2c_1^2T)^{m-2}\sum_{k=0}^{n-m+1}(8c_0^2c_1^2T)^k(\|u^1\| + \|u^1\|^2),$$
and we have
$\|\phi_n-\phi_m\| \le c_0\,|F_2(u_n)-F_2(u_m)| \le c_0c_1\,\|u_n-u_m\|$.
Therefore c is small, the sequence $(u_n,\phi_{n-1})_{n\ge2}$ is a Cauchy sequence in $H_0^2(\omega)\times H_0^2(\omega)$ and $(u_n)_{n\ge2}$ is also a Cauchy sequence in $W(0,T)$ and $C([0,T], H_0^2(\omega))$.
Moreover $(u_n,\phi_{n-1})_{n\ge0}$ converge to (u,ϕ) in $(H_0^2(\omega))^2$, u_n converge to u in $C([0,T], H_0^2(\omega))$ and $(u_n)_t$ converge to u_t in $L^2([0,T], L^2(\omega))$.
$\Delta^2(u_n,\phi_{n-1})$ that weakly converge to $\Delta^2(u,\phi)$ in $(H_0^2(\omega))^2$ and afterword proposition 2.1 we have, $\widetilde{F}(u_{n-1},\phi_{n-1}) + (p,0)$ converge to $\widetilde{F}(u,\phi) + (p,0)$ in $L^2(\omega)$.
Since the operators "trace" and "∂_ν" are continuous and $\forall n \ge 2$, $(u_n,\phi_{n-1})_\Gamma = 0$, $\partial_\nu(u_n,\phi_{n-1})_\Gamma = 0$ then, $(u,\phi)_\Gamma = 0$ and $\partial_\nu(u,\phi)_\Gamma = 0$.
According to theorem 2.2 we have $\forall n \ge 2$, u_n is in $C([0,T], H_0^2(\omega))$ and $(u_n)_{|t=0} = u_0$, this implies that $(u)_{|t=0} = u_0$.
It becomes that (u,ϕ) is a weak solution of the quasi-static Von-Karman evolution.
For the uniqueness, we suppose that the problem (P_0) has two solutions (u^1,ϕ_1) and (u^2,ϕ_2) in $L^2([0,T], H_0^2(\omega)) \times H_0^2(\omega))$ such that $\|u^1\| \le c$, $\|u^2\| \le c$, $\|\phi_1\| \le c$ and $\|\phi_2\| \le c$.
With c is sufficiently small.
We have

$$(P_3)\begin{cases} (u^1)_t - (u^2)_t + \Delta^2(u^1 - u^2) = F_1(u^1,\phi_1) - F_1(u^2,\phi_2) \\ \Delta^2(\phi_1-\phi_2) = -\left[u^1, u^1+2\theta\right] + \left[u^2, u^2+2\theta\right] \\ u^1 - u^2 = 0 \qquad \partial_\nu(u^1-u^2) = 0 \\ \phi_1-\phi_2 = 0, \partial_\nu(\phi_1-\phi_2) = 0 \\ u^1(x,y,0) - u^2(x,y,0) = 0, \end{cases}$$

then $(u^1 - u^2, \phi_1 - \phi_2)$ is a solution of the problem (P_3), Theorem 2.2 and theorem 2.1 given that there exists $c_0 \succ 0$ such that

$$\left\|u^1-u^2\right\|^2 \le c_0(\int_0^T \left|F_1(u^1,\phi_1) - F_1(u^2,\phi_2)\right|_2^2$$
$$\le c_0c_1(\left\|u^1-u^2\right\|^2)$$

and $\|\phi_1-\phi_2\| \le c_0 \left\|u^1-u^2\right\|$

c is small thus $0 \prec c_0c_1 \prec 1$, then $u_1 = u_2$ and $\phi_1 = \phi_2$. Lastly the dynamic quasi-static Von-Karman equations without rotational inertia admits a unique weak solution (u,ϕ) in $C([0,T], H_0^2(\omega)) \cap L^2([0,T], H_0^2(\omega)) \times H_0^2(\omega)$ and $u_t \in L^2([0,T], L^2(\omega))$.

Now we will show that

$$|u|_2^2 + \int_0^t(\|u\|^2 + \|\phi\|^2) \le |u_0|_2^2 + Kt.$$

We have u is weak solution of the problem (P_0), satisfy that

$$(u_t,u)_{L^2(\omega),L^2(\omega)} + (\Delta^2u,u)_{L^2(\omega),L^2(\omega)} = (F_2(u,\phi),u)_{L^2(\omega),L^2(\omega)}.$$

Use the proposition 2.2 to deduce that

$$\frac{d}{dt}|u|_2^2 + \|u\|^2 \le -\|\phi\|^2 + K,$$

then, for all $0 \le t \le T$ we find that

$$\int_0^t \frac{d}{dt}|u|_2^2 + \int_0^t \|u\|^2 \le -\int_0^t \|\phi\|^2 + \int_0^t K \text{ hence,}$$
$$|u|_2^2 - |u_0|_2^2 + \int_0^t(\|u\|^2 + \|\phi\|^2) \le +Kt,$$

finally we deduce that

$$|u|_2^2 + \int_0^t(\|u\|^2 + \|\phi\|^2) \le |u_0|_2^2 + Kt.$$

Afterword the inequality of Poincare, there exist a constant $w_0 \succ 0$, such that $\forall u \in H_0^2(\omega)$, $|u|_2^2 \le w_0\|u\|^2$, moreover we have, $\|\phi\| \le c_0\|u\|$ then, there exist a constant $w \succ 0$ such that

$$\frac{d}{dt}|u|_2^2 + w|u|_2^2 \le K,$$

using the lemme of Gronwall, we find the desired inequality: $|u|_2^2 \le |u_0|_2^2 e^{-wt} + \frac{K}{w}$.

3. Numerical application

In this section let ω be the square $]\,0,1\,[\times]\,0,1\,[$ in IR^2 and $T \succ 0$. For approached the weak uniqueness solution of the quasi-statc Von-Karman evolution without rotational inertia, we utilize the following iterative method:

$$(*)_{n\ge2}\begin{cases} v_1(x,y) \text{ is given in } H_0^2(\omega) \\ \Delta^2\phi_{n-1} = -\left[v_{n-1}, v_{n-1}+2\theta\right] \\ \phi_{n-1} = 0, \partial_\nu\phi_{n-1} = 0 \\ (v_n)_t + \Delta^2 v_n = \left[\phi_{n-1}+F_0, v_{n-1}+\theta\right] - L(v_{n-1}) + p \\ v_n = \partial_\nu v_n = 0 \\ (v_n)_{|t=0} = u_0 \end{cases}$$

For approached v_n over ω for all $0 \le t \le T$ and $n \ge 2$ if v_{n-1} is known, we have need to calculate firstly ϕ_{n-1} and we find v_n over $\omega \times [0,T]$.

Then we have transformed the above problem to the numerical resolution in tow steps:

First step : we use the numerical procedure of 13-point formula of finite difference developed by M.Gubta in [9]. For illustrate a uniqueness solution of the next biharmonic problem:

$$\begin{cases} \Delta^2 v = f_1 & in \quad \omega, \\ v = g_1 & on \quad \Gamma, \\ \partial_\nu v = g_2 & on \quad \Gamma. \end{cases}$$

Second step : We present the alternating direction scheme developed by T.P.Witelski and M.Bowen in [11], to the following parabolic problem :

$$(**)\begin{cases} u_t + \Delta^2 u = f & in \quad \omega\times[0,T], \\ u_{|t=0} = u_0 & in \quad \omega, \\ u = \partial_\nu u = 0 & on \quad \Gamma\times[0,T]. \end{cases}$$

3.1. Noncoupled Approach

In [9], M.Gupta present the numerical analysis of finite-difference method for solving the Biharmonic equation. This method has known that of noncoupled method of 13-point. Moreover Glowinski and Pironneau [7] made the observation that the 13-point finite difference scheme combined with a quadratic extrapolation formula near the boundary is equivalent to mixed finite element method with piecewise linear elements.

3.1.1. Discrete formulation of 13-point

In order to solve the problem (P) of Biharmonic equation numerically, we introduce a uniform mesh of width h. Let ω_h be the set of all mesh points inside ω with internal points $x_i = ih$, $y_j = jh$, $i,j = 1,...N-1$, $h = \frac{1}{N+1}$, $\overline{\omega}_h$ be the set of boundary mesh points and v_h represent the finite-difference approximation of v.

Lemma 3.1 [9] *The 13-point approximation of the Biharmonic equation for approaching the uniqueness solution v of the problem (P) is defined by:*

$$(1)\begin{cases} L_hv_{ij} = & h^{-4}\left[v_{ij-2} + v_{ij+2} + v_{i-2j} + v_{i+2j} - 8(v_{ij-1} + v_{ij+1})\right. \\ & +8(v_{i-1j} + v_{i+1j}) + 2(v_{i-1j-1} + v_{i-1j+1} + v_{i+1j-1}) \\ & \left.+2v_{i+1j+1} - 20v_{ij}\right] = f_1(x_i,y_j) \ , \ for\ i,j = 1,....N-1 \end{cases}$$

where $v_{ij} = v(x_i,y_j)$.

When the mesh point (x_i,y_j) is adjacent to the boundary $\overline{\omega}_h$, then the undefined values of v_h are conventionally calculated by the following approximation of $\partial_\nu v$ defined by [9]:

$$v_{i-2j} = \tfrac{1}{2}v_{i+1j} - v_{ij} + \tfrac{3}{2}v_{i-1j} - h(\partial_x v)_{i-1j}$$
$$v_{ij-2} = \tfrac{1}{2}v_{ij+1} - v_{ij} + \tfrac{3}{2}v_{ij-1} - h(\partial_y v)_{ij-1}$$
$$v_{i+2j} = \tfrac{1}{2}v_{i+1j} - v_{ij} + \tfrac{3}{2}v_{i-1j} - h(\partial_x v)_{i+1j}$$
$$v_{ij+2} = \tfrac{1}{2}v_{ij+1} - v_{ij} + \tfrac{3}{2}v_{ij-1} - h(\partial_y v)_{ij+1}$$

Remark 3.1 *In [9] M.Gubta generalized the approximation of the $\partial_\nu v$ known that by the (p,q) formula or the two-point formula. In the next Lemma 3.1 the approximation of $\partial_\nu v$ correspond at the (2,0) formula.*

3.1.2. Matrix system of scheme (1)

Let $V = (v_{11}, v_{12}, ..., v_{1N-1}, v_{21}, ...v_{2N-1}, ..., v_{N-1N-1})$ be a vector of unknown values of the approached solution v_h, by using the 13-point finite difference method, the discreet problem :

$$\begin{aligned} L_h v_{ij} = \quad & h^{-4}[v_{ij-2}+v_{ij+2}+v_{i-2j}+v_{i+2j}-8(v_{ij-1}+v_{ij+1}) \\ & +8(v_{i-1j}+v_{i+1j})+2(v_{i-1j-1}+v_{i-1j+1}+v_{i+1j-1}) \\ & +2v_{i+1j+1}-20v_{ij}] = f_1(x_i,y_j)\ ,\ \ for \text{ i,j =1,....N-1} \end{aligned}$$

is equivalent to the linear system $AV = \widetilde{F}$, where A is a matrix system of scheme (1) of order $(N-1)^2$ and $\widetilde{F}$ a known vector depend only of body forces f_1 and lateral forces g_0, g_1.
Such that

$$A = \begin{pmatrix} a_1 & a_2 & I & 0 & ... & & & 0 \\ b_1 & b_2 & b_1 & I & 0... & & & 0 \\ I & b_1 & b_2 & b_1 & I & ... & & 0 \\ 0 & & ... & ... & & & & 0 \\ 0 & & ...0 & I & b_1 & b_2 & b_1 & I \\ 0 & & & ...0 & I & b_1 & b_2 & b_1 \\ 0 & & ... & ... & 0 & I & a_2 & a_1 \end{pmatrix}$$

Where I is the identity matrix, $A_1 = (a_1, a_2)$ and $B = (b_1, b_2, b_1)$ such that

$$a_1 = \begin{pmatrix} 22 & \frac{-17}{2} & 1 & 0 & ... & & & & 0 \\ -8 & 21 & -8 & 1 & 0 & ... & & & 0 \\ 1 & -8 & 21 & -8 & 1 & 0 & ... & & 0 \\ 0 & 1 & ... & ... & ... & ... & ... & & \\ 0 & ... & ... & ... & ... & ... & ... & ... & \\ 0 & ... & ... & ... & ... & ... & ... & ... & 0 \\ 0 & ... & ... & ... & ... & ... & ... & 1 & \\ 0 & & ... & 0 & 1 & -8 & 21 & -8 & 1 \\ 0 & & & ... & 0 & 1 & -8 & 21 & -8 \\ 0 & & & & ... & 0 & 1 & \frac{-17}{2} & 22 \end{pmatrix}$$

and

$$a_2 = \begin{pmatrix} \frac{-17}{2} & 2 & 0 & ... & ... & 0 \\ 2 & \frac{-17}{2} & 2 & 0 & ... & 0 \\ 0 & ... & ... & ... & ... & 0 \\ 0 & ... & ... & ... & ... & 0 \\ 0 & ... & ... & ... & ... & 0 \\ 0 & ... & ... & 2 & \frac{-17}{2} & 2 \\ 0 & ... & ... & ... & 2 & \frac{-17}{2} \end{pmatrix}$$

$$b_1 = \begin{pmatrix} -8 & 2 & 0 & ... & ... & 0 \\ 2 & -8 & 2 & 0 & ... & 0 \\ 0 & ... & ... & ... & ... & 0 \\ 0 & ... & ... & ... & ... & 0 \\ 0 & ... & ... & ... & ... & 0 \\ 0 & ... & ... & 2 & -8 & 2 \\ 0 & ... & ... & ... & 2 & -8 \end{pmatrix}$$

$$b_2 = \begin{pmatrix} 21 & \frac{-17}{2} & 1 & 0 & ... & & & & 0 \\ -8 & 20 & -8 & 1 & 0 & ... & & & 0 \\ 1 & -8 & 20 & -8 & 1 & 0 & ... & & 0 \\ 0 & 1 & ... & ... & ... & ... & ... & & 0 \\ 0 & ... & ... & ... & ... & ... & ... & ... & 0 \\ 0 & ... & ... & ... & ... & ... & ... & ... & 0 \\ 0 & ... & ... & ... & ... & ... & ... & 1 & 0 \\ 0 & & ... & 0 & 1 & -8 & 20 & -8 & 1 \\ 0 & & & ... & 0 & 1 & -8 & 20 & -8 \\ 0 & & & & ... & 0 & 1 & \frac{-17}{2} & 21 \end{pmatrix}$$

Theorem 3.1 [9] *The scheme (1) of 13-point is convergent and the error is of h^2 order.*

3.2. Numerical solution of parabolic problem

In [11], T.P.Witelski and M.Bowen present a new finite difference approximation to the last problem (∗∗), known that of alternating direction implicit schemes (ADI) and study the stability and convergence. Moreover the authors generalize the some results of the (ADI) scheme in the case of linear problem to the nonlinear equations.

3.2.1. Discrete formulation of finite difference method

In order to solve the problem (∗∗) of parabolic equation numerically, we introduce a uniform mesh presented in the last subsection 3.1.1 and we introduce the next typical notation for difference operators :

$w(i\Delta x, j\Delta y, n\Delta t) = w_{ijn}$

$\Delta_t w_{ijn} = \partial_t w_{ijn} = \frac{w_{ijn+1} - w_{ijn}}{(\Delta t)}$

$\Delta_x^2 w_{ijn} = \partial_x^4 w_{ijn} = \frac{w_{i+2jn} - 4w_{i+1jn} + 6w_{ijn} - 4w_{i-1jn} + w_{i-2jn}}{(\Delta x)^4}$.

$\Delta_x w_{ijn} = \partial_x^2 w_{ijn} = \frac{w_{i+1jn} - 2w_{ijn} + w_{i-1jn}}{(\Delta x)^2}$.

Now we approximate the problem (∗∗), by the following finite difference (ADI) system presented by T.P.Witelski and M.Bowen in [11]:

$$(2)\begin{cases} L_x w^* = -(\Delta t)\Delta^2 w_{ijn} + f_{ijn} \\ L_y v^* = w^* \\ w_{ijn+1} = w_{ijn} + v^* \quad for \quad ij = 1, ..., N-1 \\ Boundary \ \ conditions \\ w_{0jn} = w_{Njn} = w_{i0n} = w_{iNn} = 0 \ \ for \ \ i = 0, ...N, \\ j = 0, ...N \ and \ 0 \le n\Delta t \le T \\ w_{ij0} = (u_0)_{ij0} \quad for \quad ij = 0, ...N \end{cases}$$

Where w^* and v^* represent an intermediate results obtained from solving the first and second equations, but $L_x = I + \theta(\Delta t)\Delta_x^2$, $L_y = I + \theta(\Delta t)\Delta_y^2$ are two operators and $0 \le \theta \le 1$.

3.2.2. Matrix system of scheme (2)

Let $W^n = (w_{11n}, w_{12n}, ...w_{1N-1n}, w_{21n}, ...w_{2N-1n}, ...w_{N-1N-1n})$ be a vector of unknown values of the approached weakly uniqueness solution w_h of the problem (∗∗), by using the next (ADI) scheme (2) of finite difference approximation to the parabolic problem :

$$(2)\begin{cases} L_x w^* = -(\Delta t)\Delta^2 w_{ijn} + f_{ijn} \\ L_y v^* = w^* \\ w_{ijn+1} = w_{ijn} + v^* \qquad\qquad for \quad ij = 1, ..., N-1 \end{cases}$$

This scheme (2) presented under matrix form, is equivalent to the following linear system :

$BW^* = AW^n + F^n$,

$CV^* = W^*$,

$W^{n+1} = W^n + V^*$.

Where A, $B = I + \theta(\Delta t)B_1$ and $C = I + \theta(\Delta t)C_1$ are tree matrix of order $(N-1)^2$ and F is the known vector depend only of body forces and the boundary conditions. Such that

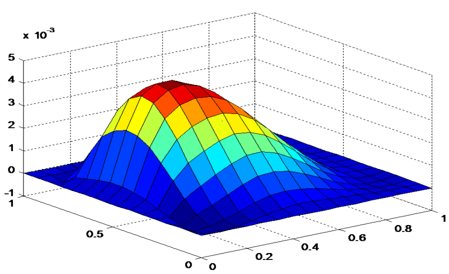
$$B_1 = \begin{pmatrix} 7I & -4I & I & ... & & & & 0 \\ -4I & 6I & -4I & I & ... & & & 0 \\ I & -4I & 6I & -4I & 0 & ... & & 0 \\ 0 & I & ... & ... & ... & ... & & 0 \\ 0 & ... & ... & ... & ... & ... & ... & 0 \\ 0 & ... & ... & ... & ... & ... & I & 0 \\ 0 & & ... & 0 & -4I & 6I & -4I & I \\ 0 & & & ... & I & -4I & 6I & -4I \\ 0 & & & & 0 & I & -4I & 7I \end{pmatrix}$$

$$C^1_{ii} = \begin{pmatrix} 7 & -4 & 1 & ... & & & & & 0 \\ -4 & 6 & -4 & 1 & 0 & ... & & & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & ... & & 0 \\ 0 & 1 & ... & ... & ... & ... & ... & & 0 \\ 0 & ... & ... & ... & ... & ... & ... & ... & 0 \\ 0 & ... & ... & ... & ... & ... & ... & 1 & 0 \\ 0 & & ... & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & & & ... & 0 & 1 & -4 & 6 & -4 \\ 0 & & & & ... & 0 & 1 & -4 & 7 \end{pmatrix}.$$

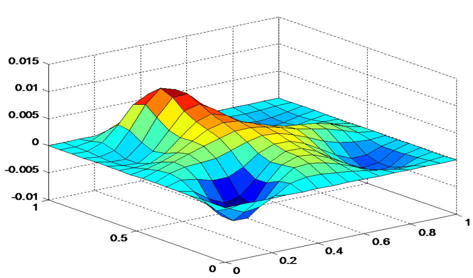
Where $C_1 = (C^1_{ij})_{1 \leq ij \leq N-1}$ is matrix of block diagonal matrix and A is the matrix of scheme (1).

Example 1. We consider in this example the following analytical external forces p, internal forces F_0, source L and the mapping θ.

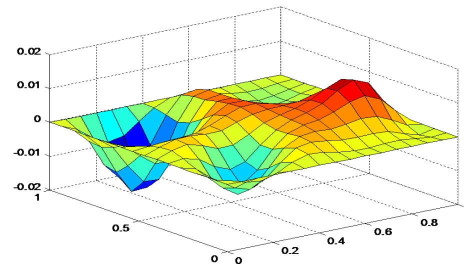
$$F_0(x,y) = 0.8xe^{-x^2-y^2} \quad ,$$
$$L(u) = 0.8x(e^{-x^2} - e^{-y^2})u \quad ,$$
$$\theta(x,y) = -0.5xy(x-1)(y-1)e^{-x^2-y^2} \quad ,$$
$$p(x,y) = sin^2(\pi x)cos^2(\pi x) \quad .$$



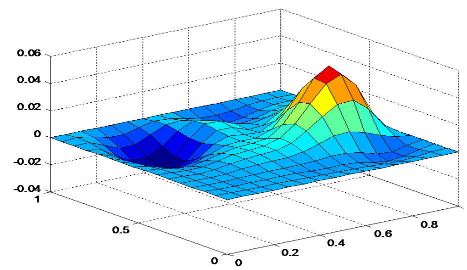
$Transversal\ displacement\ of\ plate,\ t_1 = 0.08s$



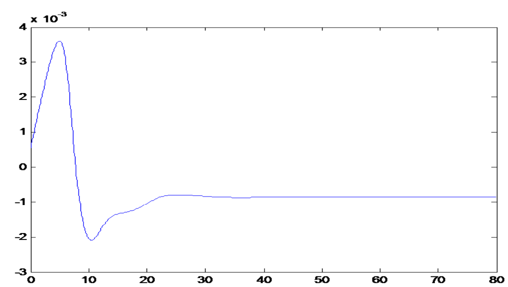
$Transversal\ displacement\ of\ plate,\ t_{30} = 2.4s$



$Transversal\ displacement\ of\ plate,\ t_{80} = 6.4s$



$Transversal\ displacement\ of\ plate,\ t_{1000} = 80s$



$displacement\ in\ the\ point\ (3,7)\ for,\ t_{1000} = 80s$

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