Algorithm on relation between Gamma function and Sine/Cosine function, Exponential Function

Karan Jain
Student of Mech. Engg at N.S.I.T , Dwarka

Algorithm

Equation A

$$\frac{(m)!\cos\frac{\pi m}{2}}{b^m} = \int_0^\infty \cos(bx^{\frac{1}{m}})dx$$

Where m $\mathcal{E}$ n

- *let's assume 'k' to be a variable and k $\mathcal{E}$ n i.e. 'k' belongs to asset of natural numbers*

Results1: if m= (4k)

$\int_0^\infty \cos(bx^{1/4k})dx = \frac{(4k)!}{b^{4k}}$

Where k ∈ N

Result2: if m= (4k-2)

$\int_0^\infty \cos(bx^{\frac{1}{(4k-2)}})dx = \frac{-(4k-2)!}{b^{4k-2}}$

Where k ∈ N

***Result 3: if m= (2k-1)** then;*

$\int_0^\infty \cos(bx^{\frac{1}{2k-1}})dx = 0$

Where k ∈ N

Equation B

$$\frac{(m)!\sin\frac{\pi m}{2}}{b^m} = \int_0^\infty \sin(bx^{\frac{1}{m}})dx$$

Where m $\mathcal{E}$ n

Lets assumes the same variable 'k' following the same condition, that is m $\mathcal{E}$ n

Result1: If m=2k

$\int_0^\infty \sin(bx^{\frac{1}{(2k)}})dx = 0$

Where k ∈ N

Result 2**: **if m=(4k-3)

$$\int_0^\infty \sin(bx^{\frac{1}{(4k-3)}})dx = \frac{(4k-3)!}{b^{4k-3}}$$

Where k ∈ N

Result 3: if m= (4k-1)

$$\int_0^\infty \sin(bx^{\frac{1}{(4k-1)}})dx = \frac{-(4k-1)!}{b^{4k-1}}$$

Where k ∈ N

The result written is derived in the following pages starting from the basic definition of gamma function.

Deriving a relation between gamma function trigonometry function (sine or cos) using complex number approach.

To start from basic definition of gamma function

$\Gamma(t) = \int_0^\infty u^{t-1}e^{-u}\,du$

Where t $\notin Z^-$

Now make the following substitution:

$u = sx^n$

Where s is the complex number of the (a+ib)

Differentiating both sides

$$du = (s)(n)x^{n-1}\,dx$$

n $\mathcal{E}$ $R - \{0\}$ i.e. n cannot be equal to 0.

After making the substitution:

$$\Gamma(t) = \int_0^\infty [(sx^n)^{t-1}](e^{-sx^n})(s)(n)\,x^{(n-1)}dx$$

$$\Gamma(t) = \int_0^\infty s^{(t-1+1)}\,x^{(nt-n+n-1)}e^{-sx^n}ndx$$

$$\Gamma(t) = \int_0^\infty (ns^t)\,x^{nt-1}\,e^{-sx^n}dx$$

$\frac{\Gamma(t)}{n(s^t)} = \int_0^\infty x^{(nt-1)} e^{-sx^n} dx$ -------------------------(1)

Where $t \notin Z^-$ **&** $n \neq 0$

Since we assumed 's' to be a complex number of the form (a+ib), we can rewrite Equation(1) by replacing 's' with its complex conjugate, $\bar{s}$*.* $\bar{s}$ *is of the form (a - ib).*

Replacing 's' by $\bar{s}$

$\frac{\Gamma(t)}{n(\bar{s})^t} = \int_0^\infty x^{(nt-1)} e^{-\bar{s}x^n} dx$------------------------ (2)

Where $t \notin Z^-$ **&** $n \neq 0$

- **Equation 1**

$$\frac{\Gamma(t)}{n(s^t)} = \int_0^\infty x^{(nt-1)} e^{-sx^n} dx$$

Where $t \notin Z^-$ **&** $n \neq 0$

- **Equation 2**

$$\frac{\Gamma(t)}{n(\bar{s}^t)} = \int_0^\infty x^{(nt-1)} e^{-\bar{s}x^n} dx$$

Where $t \notin Z^-$ **&** $n \neq 0$
Where s is the complex number of the (a+ib)

$$s = a + ib$$

$s = |s|e^{i\alpha}$

where $|s| = \sqrt{a^2 + b^2}$ $\tan\alpha = \frac{b}{a}$

$s^t = |s|^t e^{it\alpha}$ -------------------------(3)

$\bar{s} = a - ib$ $\bar{s} = |s|^t e^{-i\alpha}$

$(\bar{s})^t = (|s|)^t e^{-it\alpha}$ ----------------------(4)

- Substituting the results of (3) and (4) in equations (1) and (2) respectively.

-Substituting the following values

- $s = a + ib$
- $\bar{s} = a - ib$

Modified equation (1)

$\frac{\Gamma(t)}{n|s|^t e^{it\alpha}} = \int_0^\infty x^{(nt-1)} e^{-(a+ib)x^n} dx$

Where $t \notin Z^-$ **&** $n \neq 0$

Modified equation (2)

$\frac{\Gamma(t)}{n|s|^t e^{-it\alpha}} = \int_0^\infty x^{(nt-1)} e^{-(a-ib)x^n} dx$

Where $t \notin Z^-$ **&** $n \neq 0$

Case(i) :
Adding equation (1) and (2)

- *Case(ii) :*

Subtracting equation (1) from (2)

Simplifying case (i)

- $\frac{\Gamma(t)}{n|s|^t e^{t(i\alpha)}} + \frac{\Gamma(t)}{n|s|^t e^{t(-i\alpha)}} = \int_0^\infty (x^{(nt-1)} * [e^{-(a+ib)x^n} + e^{-(a-ib)x^n}]) dx$

- $\frac{\Gamma(t)}{n|s|^t}(e^{-i\alpha} + e^{i\alpha}) = \int_0^\infty (x^{(nt-1)} e^{-ax^n}[e^{-ibx^n} + e^{ibx^n}]) dx$

Applying Euler's formula, which is given by:-

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

on LHS and RHS

- $\frac{\Gamma(t)\cos(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)} e^{-ax^n} \cos(bx^n)\, dx$

$$\frac{\Gamma(t)\cos(t\alpha)}{n|s|^t} =$$

$\int_0^\infty x^{(nt-1)} e^{-ax^n} \cos(bx^n)\, dx$------------ **equation(5)**

Where $t \notin Z^-$ **&** $n \neq 0$

Simplifying case (ii)

- $\frac{\Gamma(t)}{n|s|^t e^{t(i+\alpha)}} - \frac{\Gamma(t)}{n|s|^t e^{t(-i+\alpha)}} = \int_0^\infty (x^{(nt-1)} [e^{-(a+ib)x^n} - e^{-(a-ib)x^n}]) dx$

- $\frac{\Gamma(t)}{n|s|^t}(e^{t(-i+\alpha)} - e^{t(i+\alpha)}) = \int_0^\infty (x^{(nt-1)} e^{-ax^n}[e^{-ibx^n} - e^{ibx^n}]) dx$

Apply Euler's formula which is given by:-

$\sin y = \frac{e^{iy} - e^{-iy}}{2}$

on LHS and RHS

- $\frac{\Gamma(t)\, s\,in(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)} e^{-ax^n} s\,in(bx^n)\,dx$
- $\frac{\boldsymbol{\Gamma(t)\ sin(t\alpha)}}{\boldsymbol{n|s|^t}} = \int_0^\infty \boldsymbol{x^{(nt-1)} e^{-ax^n} s\,in(bx^n)\,dx}$ **-------------(6)**

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Equation (5)

$$\frac{\boldsymbol{\Gamma(t)\cos(t\alpha)}}{\boldsymbol{n|s|^t}} = \int_0^\infty \boldsymbol{x^{(nt-1)} e^{-ax^n} \cos(bx^n)\,dx}$$

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Equation (6)

$$\frac{\boldsymbol{\Gamma(t)\sin(t\alpha)}}{\boldsymbol{n|s|^t}} = \int_0^\infty \boldsymbol{x^{(nt-1)} e^{-ax^n} s\,in(bx^n)\,dx}$$

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Where $s = a + ib$

$$|s| = \sqrt{a^2+b^2}$$
$$\alpha = tan^{-1}\frac{b}{a}$$

Simplyfying equation (5)

$$\frac{\Gamma(t)\cos(\alpha t)}{n|s|^t} = \int_0^\infty x^{(nt-1)} e^{-ax^n} \cos(bx^n)\,dx$$

Now simplifying some variables by assigning them values

Let : a=0, $n = \frac{1}{t}$

Where t≠ 0 & t $\notin Z^-$

Since a=0; $|s| = \sqrt{b^2}$*=b. But*

$\tan\alpha = \frac{b}{a}$ *is not defined implying* $\boldsymbol{\alpha = (n\pi + \pi/2)}$ *because* $\boldsymbol{tan\frac{\pi}{2}}$ *is not defined as well. Considering only the principal argument*

- $\alpha = \frac{\pi}{2}$

Thus equation 5 reduces as

$$\frac{\Gamma(t)\cos[(\frac{\pi}{2})t]}{\frac{1}{t}|b|^t} = \int_0^\infty x^{(nt-1)} e^{(0x^n)} \cos(bx^{\frac{1}{t}})\,dx$$

$$\frac{\boldsymbol{\Gamma(t)\ t\cos[\frac{\pi t}{2}]}}{\boldsymbol{b^t}} = \int_0^\infty \boldsymbol{\cos(bx^{(t)})\,dx}$$

-------------equation (7)

Where $t \notin Z^-$ & $t \neq 0$

Simplifying Equation (6)

$$\frac{\Gamma(t)\sin(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)} e^{-ax^n} s\,in(bx^n)\,dx$$

Now simplifying some variables by assigning them values

Let a tend to zero; a→0;

$$lim_{h \to 0}^{\Delta}(a = h)$$
$$n = \left(\frac{1}{t}\right), Where\ t \neq\ 0\ \&\ t\ \notin\ Z^-$$

Now as a→0 ; $|s| = \sqrt{b^2}$*=b and*

$\tan\alpha \to \infty$

$=> \ \alpha = \frac{\pi}{2}$. *As we equate* $n\ = \left(\frac{1}{t}\right)$ *where t≠ 0*

$=> (nt-1) = 0\ i.e. exponential\ power\ of\ x$*=0*

These simplification reduces equation 6 to

$$\frac{t\Gamma(t)\sin(\frac{\pi t}{2})}{b^t} = \int_0^\infty s\,in(bx^{\frac{1}{t}})\ dx$$

--------------equation(8)

Where t≠ 0 & t $\notin Z^-$

Equations derived are:

- ***Equation 7***

$$\frac{\boldsymbol{t\Gamma(t)\ \cos[\frac{\pi t}{2}]}}{\boldsymbol{b^t}} = \int_0^\infty \boldsymbol{\cos(bx^{\frac{1}{t}})\,dx}$$

- ***Equation 8***

$$\frac{\boldsymbol{t\Gamma(t)\sin(\frac{\pi t}{2})}}{\boldsymbol{b^t}} = \int_0^\infty \boldsymbol{s\,in(bx^{\frac{1}{t}})\ dx}$$

***Where t≠ 0 & t* $\notin Z^-$**

Equation 7

$$\frac{\boldsymbol{t\,\Gamma(t)\cos(\frac{\pi t}{2})}}{\boldsymbol{(b)^t}} = \int_0^\infty \boldsymbol{\cos(bx^{\frac{1}{t}})\,dx}$$

***Where t≠ 0 & t* $\notin Z^-$**

Now let us assume 'm' to be a variable belonging to the set of natural numbers i.e. $m \in N$

Case (I): t ∈ (m)

This assumption will reduce the above equation to:

$$m\,\Gamma(m)\cos(\frac{\pi m}{2}) = \int_0^\infty \cos(bx^{\frac{1}{m}})dx$$

Since m is a positive natural number the following relation:

$$\Gamma(n) = (n-1)\Gamma(n-1)$$
$$\Gamma(n) = (n-1)!$$

This will reduce the above equation to: −

$$=\blacktriangleright \frac{m(m-1)!\cos(\frac{\pi m}{2})}{(b)^m}$$
$$=\int_0^\infty \cos\left(bx^{\frac{1}{m}}\right)dx$$

$$=\blacktriangleright \frac{m!\ \cos(\pi m/2)}{(b)^m}=\int_0^\infty \cos\left(bx^{\frac{1}{m}}\right)dx$$

-------***equation (7.1)***

$$where\ m \in N$$

Case(II): t ∈(-m)

This will reduce the above equation 5 to: −

$$\frac{(-m)\Gamma(-m)\cos(\frac{-\pi m}{2})}{(b)^{-m}}=\int_0^\infty \cos\left(bx^{\frac{-1}{m}}\right)dx$$

Now applying the following relation:

$$\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin(\pi n)}$$

Replacing n→ (1+n)

$$\Gamma(1+n)\,\Gamma(-n)=\frac{\pi}{\sin(\pi+\pi n)}$$

$$=\blacktriangleright\ \ \Gamma(-n)=\frac{-\pi}{(1+n)\sin(\pi n)}$$

Replacing the above situation in the formula:

$$=\blacktriangleright \frac{(-m)(-\pi)\cos(\pi m/2)}{(1+m)\sin(\pi m)\,(b)^{-m}}$$
$$=\int_0^\infty \cos\left(bx^{\frac{-1}{m}}\right)dx$$

$$=\blacktriangleright \frac{m\pi\cos(\pi m/2)(b)^m}{(m)!\sin(\pi m)}=\int_0^\infty \cos\left(bx^{\frac{-1}{m}}\right)dx$$

---------------------*equation (7.2)*

Now the above relation has the constraint : $m \in N$. *Since this is in contradiction what we started with i.e.* $m \in (-N)$, *thus case II is nullified*

Case (III) : $\boldsymbol{t \in R^+ - N}$ *i.e. t belongs to a positive number, excluding all natural numbers.* ***Let 'n' be a variable belonging to the class of*** $(\boldsymbol{R^+ - N})$.

Let $\boldsymbol{t \in (n)}$***and t≠ 0 & t ∉Z*** $^-$

This substitution changes Equation (7) to

$$\frac{n\Gamma(n)\cos\frac{\pi n}{2}}{(b^n)}=\int_0^\infty \cos\left(bx^{\frac{1}{n}}\right)dx$$----------

------*equation (7.3)*

$$where\ n\ \in\ (R^+ - W)$$

Case(IV): $\boldsymbol{t \in -(R^+ - N)}$ ***i.e.***

t belongs to a class of negative numbers not subsuming any negative integer.

$$\frac{(-n)\Gamma(-n)\cos(\frac{-\pi n}{2})}{(b^n)}=\int_0^\infty \cos\left(bx^{\frac{1}{n}}\right)dx$$

---------------------*equation (7.4)*

$$where\ n\ \in\ (R^+ - W)$$

Since the above equation (7.4) *deals with a set of a numbers similar To that in an equation (7.2) and both these equations cannot be Solved further ,thus they won't be consider further.*

Equation 8

$$\boldsymbol{\frac{t\Gamma(t)\sin(\frac{\pi t}{2})}{(b^t)}=\int_0^\infty \sin\left(bx^{\frac{1}{t}}\right)dx}$$

Where t≠ 0 & t ∉Z $^-$

(i) Now applying similar cases which were applied in

Equation 7

(ii) Using same variables without changing the set they belonged to in cases of ***equation 7*** *keeping in mind that sin(-x)= -sin(x) which is in contrary to the equation followed by the*

Cosine function: cos(-x)= cos(x)

(i) t≠ 0 & t ∉Z $^-$

Case (I): t ∈m

Applying the same derivations that were applied in equation (7) to get the same result in sine function .

$$=\blacktriangleright \frac{m!\sin(\pi m/2)}{(b)^m}$$
$$=\int_0^\infty \sin\left(bx^{\frac{1}{m}}\right)dx--$$
$$--equation\ 8.1$$

$$where\ m\ \in N$$

CASE (II): t ∈(-m)

Considering the above scenario ,case II will be nullified as happened in equation 7 (case II)

CASE (III): t ∈(+n) and t≠0

This substitution will provide a similar result as that equation 7, (case III)

$$\frac{n\Gamma(n)\sin(\frac{\pi n}{2})}{b^n}=\int_0^\infty \sin(bx^{\frac{1}{n}})\ \ dx-$$ ----------

equation 8.3

$$where\ n\ \in\ (R^+ - W)$$

CASE (IV): t ∈(-n) and t≠0

Similar to equation 7 (case IV)

$$\frac{(-n)\Gamma(-n)\sin\left(\frac{-\pi n}{2}\right)}{b^{-n}} = \int_0^\infty \sin\left(bx^{\frac{-1}{n}}\right)\, dx$$

-----------*equation 8.4*

$$where\ n\ \in\ (R^+ - W)$$

since the equation 8.4 cannot be reduced to any more simpler form, equation 8.4 will not be considered. Similar is the case for equation 8.3

Final Results

Equation (7.1)

$$\frac{m!\cos(\pi m/2)}{(b)^m} = \int_0^\infty \cos\left(bx^{\frac{+1}{m}}\right)dx$$

$$WHERE\ m \in N$$

Now lets assume 'k' to be a natural number i.e. $k \in N$

CASE(I) : m=(4k)

$$= \blacktriangleright\ \frac{(4k)!\ \cos(4k\pi/2)}{(b)^{4k}} = \int_0^\infty \cos\left(bx^{\frac{+1}{4k}}\right)dx$$

$$= \blacktriangleright \frac{(4k)!}{(b)^{4k}} = \int_0^\infty \cos\left(bx^{\frac{+1}{4k}}\right)dx$$ **------------result (1)**

Where k ∈ N

CASE(II) : m=(4k-2)

$$= \blacktriangleright\ \frac{(4k-2)!\ \cos[(4k-2)\pi/2]}{b^{(4k-2)}} = \int_0^\infty \cos\left(bx^{\frac{+1}{4k-2}}\right)dx$$

$$= \blacktriangleright\ \frac{(4k-2)!\ \cos[(2k-1)\pi]}{b^{(4k-2)}} = \int_0^\infty \cos\left(bx^{\frac{+1}{4k-2}}\right)dx$$

$$= \blacktriangleright \frac{-[(4k-2)!]}{b^{(4k-2)}} = \int_0^\infty \cos\left(bx^{\frac{+1}{4k-2}}\right)dx$$

--------result (2)

Where k ∈ N

CASE III : m=(4k-3) or m=(4k-1) i.e. that means: m=(2k-1)

$$= \blacktriangleright\ \frac{(2k-1)!\ \cos[(2k-1)\pi/2]}{b^{(2k-1)}} = \int_0^\infty \cos\left(bx^{\frac{+1}{2k-1}}\right)dx$$

$$= \blacktriangleright\ 0 = \int_0^\infty \cos\left(bx^{\frac{+1}{2k-1}}\right)dx ---- -result\ (3)$$

Where k ∈ N

Equation 8.1

$$\frac{m!\sin(\pi m/2)}{(b)^m} = \int_0^\infty \sin\left(bx^{\frac{1}{m}}\right)dx$$

$$WHERE\ m \in N$$

Now lets assume 'k' to ba a natural number i.e. $k \in N$

CASE(I) : m=(2k)

$$= \blacktriangleright\ \frac{(2k)!\ \sin(2k\pi/2)}{(b)^{2k}} = \int_0^\infty \sin\left(bx^{\frac{+1}{2k}}\right)dx$$

$$= \blacktriangleright 0 = \int_0^\infty \sin\left(bx^{\frac{+1}{2k}}\right)dx$$

------------result (1)

Where k ∈ N

CASE (II): m=(4k-3)

$$= \blacktriangleright\ \frac{(4k-3)!\ \sin[(4k-3)\pi/2]}{b^{(4k-3)}} = \int_0^\infty \sin\left(bx^{\frac{+1}{4k-3}}\right)dx$$

$$= \blacktriangleright \frac{(4k-3)!}{b^{(4k-3)}} = \int_0^\infty \sin\left(bx^{\frac{+1}{4k-3}}\right)dx ---- -result\ (2)$$

Where k ∈ N

CASE (III) : m=(4k-1)

$$= \blacktriangleright\ \frac{(4k-1)!\ \sin[(4k-1)\pi/2]}{b^{(4k-)}} = \int_0^\infty \sin\left(bx^{\frac{+1}{4k-1}}\right)dx$$

$$= \blacktriangleright \frac{-[(4k-1)!\ ]}{b^{(4k-1)}} = \int_0^\infty \sin\left(bx^{\frac{+1}{4k-1}}\right)dx ---- -result\ (3)$$

Where k ∈ N