Some Commutativity Results For Periodic Rings

[1] Y. Madana Mohana Reddy, [2] Dr. G. Shobhalatha, [3] Dr. D.V. Rami Reddy
[1] *Scholar In Rayalaseema University, Kurnool*
[2] *Head Of The Dept In S.K.University, Anantapuramu*
[3] *Dept Of Mathematics, Avr & Svr Engineering College, Nandayal*

Abstract:
Herstein $[38]$ *proved that a ring R which satisfies the identity* $(xy)^n = x^n y^n$, *where n is a fixed positive integer greater than 1, must have nil commutator ideal. Abu-Khuzam* $[1]$ *provethat if n is a fixed positive integer greater than 1 and R is an n (n-1) – torsion free ring with unity such that* $(xy)^n = x^n y^n$ *for all x,y in R, then R is commutative. In* $[32]$ *Gupta proved that if R is a semi prime ring satisfying* $(xy)^2 = x^2 y^2 \in Z(R)$ *for all x,y in R, where Z(R) in the center of R, then R is commutative. In* $[3]$ *it is proved that a semi prime ring R such that for each x in R there exists a positive integer* $n = n(x) > 1$ *such that* $(xy)^n - x^n y^n \in Z(R)$ *and* $(x^2 y)^n - x^{2n} y^n \in Z(R)$ *for all y in r then R is commutative. In this direction we prove that if R is an n (n+1)-torsion free periodic ring such that* $(xy)^n - y^n x^n \in Z(R)$ *and* $(xy)^{n+1} - y^{n+1} x^{n+1} \in Z(R)$ *or* $(xy)^{n+1} - x^{n+1} y^{n+1} \in Z(R)$ *and if the set of nilpotent elements of R is commutative. Then R is commutative.*

We know that R is periodic if for every x in R, there exist distinct positive integers m and n such that $x^m = x^n$. *By a theorem of Chancron* $[22]$, *R is periodic if and only if for each x in R, there exists a positive integer k = k(x) and a polynomial* $f(\lambda) = f_x(\lambda)$ *with integer coefficients such that* $x^k = x^{k+1} f(x)$. *Throughout this section R is an associative ring, Z(R) denotes the center of R and N denotes the set of nilpotent elements of R.*

Keywords*: center, periodic ring, direct sum, left and right ideals.*

1. Introduction:

During the last seven or e9ight decades abstract algebra has been developing very rapidly. In algebra the theory of rings serves as the building blocks for all branches of mathematics. In ring theory, the study of both associative and non-associative rings has evoked great interest and assumed importees.

The results on non – associative rings in which one does assume a type of partial associative have been scattered though out the literature. Many mathematicians of recent years studies a certain special properties. Their general non – associative rings. Among those mathematicians Max Zorn, A.A. Albert, N.Jocobson, R.D.Schafer, Erwin Kleinfeld, R.L.Sansoucie, A.H.Boers and Armin They are the ones whose contributions to this filed are outstanding.

2. Preliminaries:

Associative Ring: An associative ring R, sometimes called a ring in short, is an algebraic system with two binary operations addition '+' and multiplication '.' Such that

(i) The elements of R form an abelian group under addition and a semi group under multiplication,
(ii) Multiplication is distributive on the right as well as on the left over addition i,e., (x+y)z = xz+yz, z(x+y)=zx+zy for all x, y, z in R.

Non – associative Ring: A non – associative ring R is an additive abelian group in which multiplication is defined, which is distributive over addition on the left as well as on the right, i.e., (x+y)z = xz+yz, z(x+y)=zx + zy, for all x, y, z in R.

A non-associative ring differs from an associative ring in that the full associative law of multiplication is no longer assumed to be associative, i.e., it is not necessarily associative. Strictly speaking the associative law of multiplication has not been done away with, it has merely weakened.

The well known examples of non-associative rings are alternative rings, Lie rings and Jordan rings. In 1930 Artin and Max Zorn defined alternative rings.

Alternative Ring: An alternative ring R is a ring in which (xx)y = x(xy), y(xx) =(yx)x for all x, y in R. These equations are known as the left and right alternative laws respectively.

Lie Ring: A Lie ring R is a ring in which the multiplication is anti-commutative, i.e., $x^2 = 0$ (implying xy=-yx)and the Jacobi identity (xy)z + (yz)x + (yz)x +(zx)y = 0 for all x,y,z in R is satisfied.

Jordan Ring: A Jordan Ring R is a ring in which products are commutative, i.e., xy=yx and satisfy the Jordan identify $(xy)x^2 = x(yx^2)$ for all x, y in R.

Associator: The associator (x. y, z) is defined by (x, y, z) = (xy)z – x(yz) for all x, y, z in a ring.

This plays a key role in the study of non-associative rings. It can be viewed as a measure of the non-associativity of a ring. This definition is due to Maxzorn wherein he proved that a finite alternative division ring is associative.

In terms of associators, a ring R is called left alternative if (x,x,y)=0, right alternative if (y, x, x)=0 for all x, y in R and alternative if both the conditions hold.

Commutator: The commutator $[x, y]$ is defined by $[x, y]$ = xy – yx for all x, y in a ring.

Commutative Ring: If the multiplication in a ring R is such that xy=yx for all x, y in R then we call R a commutative ring.

A non-commutative ring differs from commuatative ring in that the multiplication is not assumed to be commutative. i.e., we do not assume xy = yx for all x, y in R as an axiom. However, it does not mean that there always exist elements x,y in R such that $xy \neq yx$.

The ring of 2x2 matrices over rationals and the ring of real quaternions due to Hamilton are the examples of non-commuatative rings.

Periodic element: An element $x \in R$ is called a periodic element if there exists distinct $m, n \in z^+$ such that $x^n = x^m$

Potent element: An element x of R is called potent if $x^n = x$ for some positive integer $n = n(x) > 1.$

Assosymmetric Ring: An Assosymmetric ring R is one in which (x, y, z) = (P(x), P(y) P(z)), where P is any permutation of x, y, z in R.

Standard Ring: A ring R is defined to be standard if it satisfies the following two identities:

(i) (wx, y, z) + (xz, y, w) + (wz, y, x) = 0
(ii) (x, y, z) + (z, x, y) – (x, z, y) =0, for all w, x, y and z in R.

Accessible Ring: A ring R is called accessible in case it satisfies the identities:

(i) (x, y, z) + (z, x, y) – (x, z, y) =0

(ii) ((w, x), y, z) = 0, for all w, x, y and z in R.

Periodic Ring: A ring R is called a periodic ring if for every x in R, there exists distinct positive integers m=m(x), n=n(x) such that $x^m = x^n$. Due to Chacron R is periodic if and only if for each $x \in R$, there exists a positive integers k=k(x) and a polynomial $f(\lambda) = f_x(\lambda)$ with integer co-efficients such that $x^k = x^{k+1} f(x)$.

s-Unital Ring: A ring R is called a left (respectively right) s – unital ring if $x \in Rx$ (respectively $x \in xR$) for each $x \in R$. Further R is called s-unital if it is both left as well as right s-unital, i.e., if $x \in xR \cap Rx$, for each $x \in R$.

Weakly Periodic Ring: A ring R is called a weakly periodic ring if every element of R is expressible as a sum of a nilpotent element and a potent element of R, R=N+P, where N is the set of nilpotent elements of R and P is the set of potent elements of R. It is well-known that if R is periodic, then it is weakly periodic.

Quasi – Periodic Ring: A ring R is called quasi-periodic if for each $x \in R$ there exist integers n, m with n>m$\geq$1 such that $x^n = kx^m$ for some integer k.

Generalised Quasi – Periodic Ring: A ring R is called generalized quasi – periodic if for each $x \in R$ there exist distinct positive integers m, n and non-zero integers r, s with (r,s) = 1 for which $rx^m = sx^n$.

Prime Ring: A ring R is called a primo ring if whenever A and B are ideals of R such that AB = 0, then either A=0 or B=0.

Semi Prime Ring: A ring R is semi prime if for any ideal A of R, A^2 = 0 implies A = 0. These rings are also referred to as rings free from trivial ideals.

Simple Ring: A ring R is said to be simple if whenever A is an ideal of R, then either A = R or A = 0.

Semi – Simple Ring: A ring is semi simple in case the radical. (i.e., the maximal ideal consisting of all nilpotent elements) is the zero ideal.

Obviously a simple ring is prime, which in turn is free from trivial ideals.

Primitive Ring: A ring R is defined as primitive in case it possesses a regular maximal right ideal, which contains no two-sided ideal of the ring other than the zero ideal.

Division Ring: A ring R is said to be a division ring if its non-zero elements form a group with respect to multiplication.

Flexible Ring: If in a ring R, the identity (x, y, x) = 0 i.e., (xy)x=x(yx) for all x, y in R holds then R is called flexible.

Alternative, commutative, anti-commutative and there by Jordan and Lie rings are flexible.

Nilpotent Ring: A ring is called nilpotent if there is a fixed positive integer t such that every product involving t elements is zero.

Torsion-free Ring: A ring R is said to be m-torsion free if mx=0 implies x=0 for all x in R.

Reduced Ring: A ring R is called reduced if N= {0}, where N is the set of nilpotent elements of R.

Center: In a ring R, the center denoted by Z(R) is the set of all elements $x \in R$ such that xy = yx for all $y \in R$.

Derivation: A derivation of a ring R is an additive group homomorphism $d : R \to R$ satisfying $d(r_1 r_2) = (dr_1) r_2 + r_1 dr_2$.

It is important to note that this definition does not depend on the associative of multiplication and in fact, we shall have occasion to deal with derivations of non-associative algebras.

3. Main results:

Lemma 1: If $\left[x,\left[x,y\right]\right]=0$, then $\left[x^{k},y\right]=kx^{k-1}\left[x,y\right]$ for all integers $k\geq 1.$

Proof: The identity $\left[x^{k},y\right]=k\,x^{k-1}\left[x,y\right]$ is true for integer k=1.

Suppose we assume that $\left[x^{k},y\right]=k\,x^{k-1}\left[x,y\right]$

Now, $\left[x^{k+1},y\right]=\left[x^{k}x,y\right]=x^{k}\left[x,y\right]+\left[x^{k},y\right]x$

$$=x^{k}\left[x,y\right]+k\,x^{k-1}\left[x,y\right]x \quad by\left(4.1.1\right)$$

$$=x^{k}\left[x,y\right]+k\;x^{k}\left[x,y\right],\text{ since }\left[x,\left(x,y\right)\right]=0$$

$$=\left(k+1\right).x^{k}\left[x,y\right]\text{ for all k>1.}$$

i.e., $\left[x^{k+1},y\right]=\left(k+1\right)x^{k}\left[x,y\right]$

Therefore by Induction for all integers $k\geq 1,\left[x^{k},y\right]=k\,x^{k-1}\left[x,y\right]$

Lemma 2: If R is a periodic ring, then R has each of the following properties:

a) For each $x\in R$, some power of x is idempotent.
b) For each $x\in R$, there exists an integer n(x)>1 such that $x-x^{n(x)}$ is nilpotent.
c) If f: $R\rightarrow R^{*}$ is an epimorphism, then f(N) coincides with the set of nilpotent elements of R^{*}
d) If N is central, then R is eommutative.

Proof: a) If $x^{n}=x^{m}$ with n>m, then $x^{j+k(n-m)}=x^{j}$ for each positive integer k and each $j\geq m.$ Thus we assume $n-m+1\geq m.$ It follows that $x^{n-m+1}=\left(x^{n-m+1}\right)^{n-m+1}$ and hence $\left(x^{n-m+1}\right)^{n-m}$ is idempotent.

b) Let $x^{n}=x^{m},n>m>1.$

Then $x^{m-1}\left(x-x^{n-m+1}\right)=0=x^{m-2}x\left(x-x^{n-m+1}\right)=x^{m-2}x^{n-m+1}\left(x-x^{n-m+1}\right).$ Therefore $x^{m-2}\left(x-x^{n-m+1}\right)^{2}=0$ and the result follows by the obvious induction.

c) From lemma 1$\left[10\right]$ we know that if I is an ideal of R and x+1 is a nonzero nilpotent element of R^{*} then R contains a nil potent element u such that x$\equiv$u (mod I).

So if $f:R\rightarrow R^{*}$ is an epimorphism, then f(N) co-incides with the set of nilpotent elements of R^{*}.

d) Let N denote the set of nilpotent elements, the usual argument for commutative rings shows that N is an ideal. Moreover for $x \in R$ and e an idempotent R, both ex-exe and xe-exe are in R. Hence commutative with e. Thus idempotents in R central.

We see that homomorphic images inherit the hypothesis on R. Consequently we need consider only the case of subdirectly irreducible R. Under this assumption, part (a) of the lemma 2 shows that R is either nil and hence commutative or R has a unique non-zero central idempotent necessarily a multiplicative identity element 1.

Considering this latter possibility we see from (a) of lemma 2 that each element of R is either nilpotent or invertible. Thus the set D of zero divisors is equal to N and hence is a central ideal. Moreover by lemma 4.1.2 of (b) $\overline{R}$ =R/D has the $a^n = a$ property of Jacobson, hence R, is commutative and its additive group is a torsion group. Thus if a,b$\in$R/D the sub ring of generated by $\overline{a} = a + D \ and \ \overline{b} = b + D$ is a finite field, which has cyclic multiplicative group. There must therefore exist g$\in$R and $d_1, d_2 \in D$ such that $a = g^i + d_1$ and $b = g^j + d_2$ for some positive integers i,j. If follows that a and b must commute and our proof is complete.

4. References:

[1] Abu-Khuzam.H *A commutativity theorem for rings*, Math. Japonica.., 25(1980), 593-595

[2] Abu-Khuzam.H *A commutativity theorem for periodic rings*, Math. Japonica.., 32No.1(1987) 1-3

[3] Bell H.E *On some commutativity theorems of Herstein*, Arch. Math.., 24(1973), 34-38

[4] Chacron, M. *On a theorem of Herstein*, Canad. J. Math.., 21(1969), 1348-1353

[5] Gupta, V. *Some remarks on the commutativity of rings*, Acta.Math.Acad.Sei.Hung, 36(1980), 232-236.

[6] Herstein, I.N. *A generalization of a theorem of Jacobson*, Amer. J.Math.., 73(1951), 756-762

[7] Herstein, I.N. *Powermaps in rings*, Michigan Math.J., 8(1960), 29-32

[8] Johnsen, E.C, Out calt, D.C, and Yaqub, A. *An elementary commutativity theorem for rings*, The Amer. Math. Monthly.., 75(3) (1988), 288-289.