# Eulerian integral associated with product of two multivariable 

## Gautam's A-functions and classes of polynomials

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## ABSTRACT

The present paper is evaluated a new Eulerian integral associated with the product of two multivariable A-functions defined by Gautam et al [1] a generalized Lauricella function, a classes of multivariable polynomials with general arguments. We will study the case concerning the multivariable H-function defined by Srivastava et al [7] and Srivastava-Daoust polynomial [4].

Keywords: Eulerian integral, multivariable A-function, generalized Lauricella function of several variables, multivariable H -function, generalized hypergeometric function, class of polynomials

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## 1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable A-functions defined by Gautam et al [1] , and classes of polynomials with general arguments.

The A-function is defined and represented in the following manner.
$A\left(z_{1}, \cdots, z_{r}\right)=A_{p, q: p_{1}, q_{1} ; \cdots ; p_{r}, q_{r}}^{m, n: m_{1}, n_{1} ; \cdots ; m_{r}, n_{r}}\left(\begin{array}{c|c}\mathrm{z}_{1} \\ \cdot & \left(\mathrm{a}_{j} ; A_{j}^{(1)}, \cdots, A_{j}^{(r)}\right)_{1, p}: \\ \cdot & \\ \cdot & \left(\mathrm{b}_{j} ; B_{j}^{(1)}, \cdots, B_{j}^{(r)}\right)_{1, q}: \\ \mathrm{z}_{r} & \end{array}\right.$
$\left.\begin{array}{c}\left(\mathrm{c}_{j}^{(1)}, C_{j}^{(1)}\right)_{1, p_{1}} ; \cdots ;\left(c_{j}^{(r)}, C_{j}^{(r)}\right)_{1, p_{r}} \\ \left(\mathrm{~d}_{j}^{(1)}, D_{j}^{(1)}\right)_{1, q_{1}} ; \cdots ;\left(d_{j}^{(r)}, D_{j}^{(r)}\right)_{1, q_{r}}\end{array}\right)$

$$
\begin{equation*}
=\frac{1}{(2 \pi \omega)^{r}} \int_{L_{1}} \cdots \int_{L_{r}} \phi\left(s_{1}, \cdots, s_{r}\right) \prod_{i=1}^{r} \theta_{i}\left(s_{i}\right) z_{i}^{s_{i}} \mathrm{~d} s_{1} \cdots \mathrm{~d} s_{r} \tag{1.2}
\end{equation*}
$$

where $\phi\left(s_{1}, \cdots, s_{r}\right), \theta_{i}\left(s_{i}\right), i=1, \cdots, r$ are given by :
$\phi\left(s_{1}, \cdots, s_{r}\right)=\frac{\prod_{j=1}^{m} \Gamma\left(b_{j}-\sum_{i=1}^{r} B_{j}^{(i)} s_{i}\right) \prod_{j=1}^{n} \Gamma\left(1-a_{j}+\sum_{i=1}^{r} A_{j}^{(i)} s_{j}\right)}{\prod_{j=n+1}^{p} \Gamma\left(a_{j}-\sum_{i=1}^{r} A_{j}^{(i)} s_{j}\right) \prod_{j=m+1}^{q} \Gamma\left(1-b_{j}+\sum_{i=1}^{r} B_{j}^{(i)} s_{j}\right)}$
$\theta_{i}\left(s_{i}\right)=\frac{\prod_{j=1}^{n_{i}} \Gamma\left(1-c_{j}^{(i)}+C_{j}^{(i)} s_{i}\right) \prod_{j=1}^{m_{i}} \Gamma\left(d_{j}^{(i)}-D_{j}^{(i)} s_{i}\right)}{\prod_{j=n_{i}+1}^{p_{i}} \Gamma\left(c_{j}^{(i)}-C_{j}^{(i)} s_{i}\right) \prod_{j=m_{i}+1}^{q_{i}} \Gamma\left(1-d_{j}^{(i)}+D_{j}^{(i)} s_{i}\right)}$

Here $m, n, p, m_{i}, n_{i}, p_{i}, c_{i} \in \mathbb{N}^{*} ; i=1, \cdots, r ; a_{j}, b_{j}, c_{j}^{(i)}, d_{j}^{(i)}, A_{j}^{(i)}, B_{j}^{(i)}, C_{j}^{(i)}, D_{j}^{(i)} \in \mathbb{C}$
The multiple integral defining the A-function of r variables converges absolutely if :
$\left|\arg \left(\Omega_{i}\right) z_{k}\right|<\frac{1}{2} \eta_{k} \pi, \xi^{*}=0, \eta_{i}>0$
$\Omega_{i}=\prod_{j=1}^{p}\left\{A_{j}^{(i)}\right\}^{A_{j}^{(i)}} \prod_{j=1}^{q}\left\{B_{j}^{(i)}\right\}^{-B_{j}^{(i)}} \prod_{j=1}^{q_{i}}\left\{D_{j}^{(i)}\right\}^{D_{j}^{(i)}} \prod_{j=1}^{p_{i}}\left\{C_{j}^{(i)}\right\}^{-C_{j}^{(i)}} ; i=1, \cdots, r$
$\xi_{i}^{*}=\operatorname{Im}\left(\sum_{j=1}^{p} A_{j}^{(i)}-\sum_{j=1}^{q} B_{j}^{(i)}+\sum_{j=1}^{q_{i}} D_{j}^{(i)}-\sum_{j=1}^{p_{i}} C_{j}^{(i)}\right) ; i=1, \cdots, r$
$\eta_{i}=\operatorname{Re}\left(\sum_{j=1}^{n} A_{j}^{(i)}-\sum_{j=n+1}^{p} A_{j}^{(i)}+\sum_{j=1}^{m} B_{j}^{(i)}-\sum_{j=m+1}^{q} B_{j}^{(i)}+\sum_{j=1}^{m_{i}} D_{j}^{(i)}-\sum_{j=m_{i}+1}^{q_{i}} D_{j}^{(i)}+\sum_{j=1}^{n_{i}} C_{j}^{(i)}-\sum_{j=n_{i}+1}^{p_{i}} C_{j}^{(i)}\right)$
$i=1, \cdots, r$
Consider the second multivariable A-function.
$A\left(z_{1}^{\prime}, \cdots, z_{s}^{\prime}\right)=A_{p^{\prime}, q^{\prime}: p_{1}^{\prime}, q_{1}^{\prime} ; \cdots ; p_{r}^{\prime}, q_{r}^{\prime}}^{m^{\prime}, n_{r}^{\prime}: m_{r}^{\prime}, n_{1}^{\prime} ; \cdots ; m_{r}^{\prime}, n_{r}^{\prime}}\left(\left.\begin{array}{c}\mathrm{z}_{1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_{r}\end{array} \right\rvert\,\left(\mathrm{a}_{j} ; A_{j}^{\prime(1)}, \cdots, A_{j}^{\prime(s)}\right)_{1, p^{\prime}}: ~\left(\mathrm{~b}_{j} ; B_{j}^{\prime(1)}, \cdots, B_{j}^{\prime(s)}\right)_{1, q^{\prime}}:\right.$
$\left.\begin{array}{l}\left(\mathrm{c}^{\prime(1)}, C_{j}^{\prime(1)}\right)_{1, p_{1}^{\prime}} ; \cdots ;\left({c_{j}^{\prime}}^{(s)}, C_{j}^{\prime(s)}\right)_{1, p_{s}^{\prime}} \\ \left(\mathrm{d}^{\prime(1)}, D_{j}^{\prime(1)}\right)_{1, q_{1}^{\prime}} ; \cdots ;\left(d_{j}^{\prime(s)}, D_{j}^{\prime(s)}\right)_{1, q_{s}^{\prime}}\end{array}\right)$

$$
\begin{equation*}
=\frac{1}{(2 \pi \omega)^{s}} \int_{L_{1}^{\prime}} \cdots \int_{L_{s}^{\prime}} \phi^{\prime}\left(t_{1}, \cdots, t_{s}\right) \prod_{i=1}^{s} \theta_{i}^{\prime}\left(t_{i}\right) z_{i}^{s_{i}} \mathrm{~d} s_{1} \cdots \mathrm{~d} s_{r} \tag{1.10}
\end{equation*}
$$

where $\phi^{\prime}\left(t_{1}, \cdots, t_{s}\right), \theta_{i}^{\prime}\left(t_{i}\right), i=1, \cdots, s$ are given by :
$\left.\phi^{\prime}\left(t_{1}, \cdots, t_{s}\right)=\frac{\prod_{j=1}^{m^{\prime}} \Gamma\left(b_{j}^{\prime}-\sum_{i=1}^{s} B_{j}^{\prime(i)} t_{i}\right) \prod_{j=1}^{n^{\prime}} \Gamma\left(1-a_{j}^{\prime}+\sum_{i=1}^{s} A_{j}^{\prime(i)} t_{j}\right)}{\prod_{j=n^{\prime}+1}^{p^{\prime}} \Gamma\left(a_{j}^{\prime}-\sum_{i=1}^{s} A_{j}^{\prime}(i)\right.} t_{j}\right) \prod_{j=m^{\prime}+1}^{q^{\prime}} \Gamma\left(1-b_{j}^{\prime}+\sum_{i=1}^{s} B_{j}^{\prime(i)} t_{j}\right) \quad$,
$\theta_{i}^{\prime}\left(t_{i}\right)=\frac{\prod_{j=1}^{n_{i}^{\prime}} \Gamma\left(1-c_{j}^{\prime(i)}+C_{j}^{\prime(i)} t_{i}\right) \prod_{j=1}^{m_{i}^{\prime}} \Gamma\left(d_{j}^{\prime}{ }^{(i)}-D_{j}^{\prime(i)} t_{i}\right)}{\prod_{j=n_{i}^{\prime}+1}^{p_{i}^{\prime}} \Gamma\left(c_{j}^{\prime(i)}-C_{j}^{\prime(i)} t_{i}\right) \prod_{j=m_{i}^{\prime}+1}^{q_{i}^{\prime}} \Gamma\left(1-{d_{j}^{\prime}(i)}^{\prime}+D_{j}^{\prime(i)} t_{i}\right)}$

Here $m^{\prime}, n^{\prime}, p^{\prime}, m_{i}^{\prime}, n_{i}^{\prime}, p_{i}^{\prime}, c_{i}^{\prime} \in \mathbb{N}^{*} ; i=1, \cdots, r ; a_{j}^{\prime}, b_{j}^{\prime}, c_{j}^{\prime(i)}, d_{j}^{\prime(i)}, A_{j}^{\prime(i)}, B_{j}^{\prime(i)}, C_{j}^{\prime(i)}, D_{j}^{\prime(i)} \in \mathbb{C}$
The multiple integral defining the A -function of r variables converges absolutely if :
$\left|\arg \left(\Omega_{i}^{\prime}\right) z_{k}^{\prime}\right|<\frac{1}{2} \eta_{k}^{\prime} \pi, \xi^{*}=0, \eta_{i}^{\prime}>0$

$$
\begin{align*}
& \Omega_{i}^{\prime}=\prod_{j=1}^{p^{\prime}}\left\{A_{j}^{\prime(i)}\right\}^{A_{j}^{\prime(i)}} \prod_{j=1}^{q^{\prime}}\left\{B_{j}^{\prime(i)}\right\}^{-B_{j}^{\prime(i)}} \prod_{j=1}^{q_{i}^{\prime}}\left\{D_{j}^{\prime(i)}\right\}^{D_{j}^{\prime(i)}} \prod_{j=1}^{p_{i}^{\prime}}\left\{C_{j}^{\prime(i)}\right\}^{-C_{j}^{\prime(i)}} ; i=1, \cdots, s  \tag{1.14}\\
& \xi_{i}^{\prime *}=\operatorname{Im}\left(\sum_{j=1}^{p^{\prime}} A_{j}^{\prime(i)}-\sum_{j=1}^{q^{\prime}}{\left.B_{j}^{\prime(i)}+\sum_{j=1}^{q_{i}^{\prime}} D_{j}^{\prime(i)}-\sum_{j=1}^{p_{i}^{\prime}} C_{j}^{\prime(i)}\right) ; i=1, \cdots, s}_{\eta_{i}^{\prime}=\operatorname{Re}\left(\sum_{j=1}^{n^{\prime}} A_{j}^{\prime(i)}-\sum_{j=n^{\prime}+1}^{p^{\prime}} A_{j}^{\prime(i)}+\sum_{j=1}^{m^{\prime}} B_{j}^{\prime(i)}-\sum_{j=m^{\prime}+1}^{q^{\prime}} B_{j}^{\prime(i)}+\sum_{j=1}^{m_{i}^{\prime}} D_{j}^{\prime(i)}-\sum_{j=m_{i}^{\prime}+1}^{q_{i}^{\prime}} D_{j}^{\prime(i)}+\sum_{j=1}^{n_{i}^{\prime}} C_{j}^{\prime(i)}-\sum_{j=n_{i}^{\prime}+1}^{p_{i}^{\prime}} C_{j}^{\prime(i)}\right)}^{i=1, \cdots, s}\right. \tag{1.15}
\end{align*}
$$

The generalized polynomials of multivariables defined by Srivastava [3], is given in the following manner :

$$
\begin{equation*}
S_{N_{1}, \cdots, N_{v}}^{\mathfrak{M}_{1}, \cdots, \mathfrak{M}_{\mathfrak{v}}}\left[y_{1}, \cdots, y_{v}\right]=\sum_{K_{1}=0}^{\left[N_{1} / \mathfrak{M}_{1}\right]} \cdots \sum_{K_{v}=0}^{\left[N_{v} / \mathfrak{M}_{\mathfrak{v}}\right]} \frac{\left(-N_{1}\right)_{\mathfrak{M}_{1} K_{1}}}{K_{1}!} \cdots \frac{\left(-N_{v}\right)_{\mathfrak{M}_{\mathfrak{v}} K_{v}}}{K_{v}!} \tag{1.17}
\end{equation*}
$$

$A\left[N_{1}, K_{1} ; \cdots ; N_{v}, K_{v}\right] y_{1}^{K_{1}} \cdots y_{v}^{K_{v}}$
where $\mathfrak{M}_{1}, \cdots, \mathfrak{M}_{\mathfrak{v}}$ are arbitrary positive integers and the coefficients $A\left[N_{1}, K_{1} ; \cdots ; N_{v}, K_{v}\right]$ are arbitrary constants, real or complex.

Srivastava and Garg [5] introduced and defined a general class of multivariable polynomials as follows
$S_{L}^{h_{1}, \cdots, h_{u}}\left[z_{1}, \cdots, z_{u}\right]=\sum_{R_{1}, \cdots, R_{u}=0}^{h_{1} R_{1}+\cdots h_{u} R_{u} \leqslant L}(-L)_{h_{1} R_{1}+\cdots+h_{u} R_{u}} B\left(E ; R_{1}, \cdots, R_{u}\right) \frac{z_{1}^{R_{1}} \cdots z_{u}^{R_{u}}}{R_{1}!\cdots R_{u}!}$
The coefficients are $B\left[E ; R_{1}, \ldots, R_{v}\right]$ arbitrary constants, real or complex.
We will note $a_{v}=\frac{\left(-N_{1}\right)_{\mathfrak{M}_{1} K_{1}}}{K_{1}!} \cdots \frac{\left(-N_{v}\right)_{\mathfrak{M}_{\mathfrak{v}} K_{v}}}{K_{v}!} A\left[N_{1}, K_{1} ; \cdots ; N_{v}, K_{v}\right]$ and
$b_{u}=\frac{(-E)_{F_{1} L_{1}+\cdots+F_{u} L_{u} B\left(E ; L_{1}, \cdots, L_{u}\right)}^{L_{1}!\cdots L_{u}!}}{\text { ! }}$

## 2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [6 ,page 39 eq .30]
$\frac{\prod_{j=1}^{P} \Gamma\left(A_{j}\right)}{\prod_{j=1}^{Q} \Gamma\left(B_{j}\right)}{ }_{P} F_{Q}\left[\left(A_{P}\right) ;\left(B_{Q}\right) ;-\left(x_{1}+\cdots+x_{r}\right)\right]$
$=\frac{1}{(2 \pi \omega)^{r}} \int_{L_{1}} \cdots \int_{L_{r}} \frac{\prod_{j=1}^{P} \Gamma\left(A_{j}+s_{1}+\cdots+s_{r}\right)}{\prod_{j=1}^{Q} \Gamma\left(B_{j}+s_{1}+\cdots+s_{r}\right)} \Gamma\left(-s_{1}\right) \cdots \Gamma\left(-s_{r}\right) x_{1}^{s_{1}} \cdots x_{r}^{s_{r}} \mathrm{~d} s_{1} \cdots \mathrm{~d} s_{r}$
where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma\left(A_{j}+s_{1}+\cdots+s_{r}\right)$ are separated from those of $\Gamma\left(-s_{j}\right), j=1, \cdots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma\left(-s_{j}\right), j=1, \cdots, r$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$
\begin{align*}
& \int_{a}^{b}(t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{l}\left[1-\tau_{j}(t-a)^{h_{i}}\right]^{-\lambda_{j}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{\sigma_{j}} \mathrm{~d} t=(b-a)^{\alpha+\beta-1} B(\alpha, \beta) \prod_{j=1}^{k}\left(a f_{j}+g_{j}\right)^{\sigma_{j}} \\
& F_{1: 0, \cdots, 0 ; 0, \cdots, 0}^{1: 1, \cdots, 1 ; 1 \cdots, 1}\left(\begin{array}{c}
\left(\alpha: h_{1}, \cdots, h_{l}, 1, \cdots, 1\right):\left(\lambda_{1}: 1\right), \cdots,\left(\lambda_{l}: 1\right) ;\left(-\sigma_{1}: 1\right), \cdots,\left(-\sigma_{k}: 1\right) \\
\cdots \\
\left(\alpha+\beta: h_{1}, \cdots, h_{l}, 1, \cdots, 1\right):-, \cdots,-;-, \cdots,- \\
; \tau_{1}(b-a)^{h_{1}}, \cdots, \tau_{l}(b-a)^{h_{l}},-\frac{(b-a) f_{1}}{a f_{1}+g_{1}}, \cdots,-\frac{(b-a) f_{k}}{a f_{k}+g_{k}}
\end{array}\right)
\end{align*}
$$

where $a, b \in \mathbb{R}(a<b), \alpha, \beta, f_{i}, g_{i}, \sigma_{i}, \tau_{j}, h_{j} \in \mathbb{C}, \lambda_{j} \in \mathbb{R}^{+}(i=1, \cdots, k ; j=1, \cdots, l)$
$\min (\operatorname{Re}(\alpha), \operatorname{Re}(\beta))>0, \max _{1 \leqslant j \leqslant l}\left\{\left|\tau_{j}(b-a)^{h_{j}}\right|\right\}<1, \max _{1 \leqslant j \leqslant k}\left\{\left|\frac{(b-a) f_{i}}{a f_{i}+g_{i}}\right|\right\}<1$,
and $F_{1: 0, \cdots, 0 ; 0, \cdots, 0}^{1: 1, \cdots, 1}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[6,page 454] given by :
$F_{1: 0, \cdots, 0 ; 0, \cdots, 0}^{1: 1, \cdots, 1 ; 1 \cdots, 1}$$\left(\begin{array}{c}\left(\alpha: h_{1}, \cdots, h_{l}, 1, \cdots, 1\right):\left(\lambda_{1}: 1\right), \cdots,\left(\lambda_{l}: 1\right) ;\left(-\sigma_{1}: 1\right), \cdots,\left(-\sigma_{k}: 1\right) \\ \cdots \\ \left(\alpha+\beta: h_{1}, \cdots, h_{l}, 1, \cdots, 1\right):-, \cdots,-;-, \cdots,-\end{array}\right.$
$\left.; \tau_{1}(b-a)^{h_{1}}, \cdots, \tau_{l}(b-a)^{h_{l}},-\frac{(b-a) f_{1}}{a f_{1}+g_{1}}, \cdots,-\frac{(b-a) f_{k}}{a f_{k}+g_{k}}\right)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \prod_{j=1}^{l} \Gamma\left(\lambda_{j}\right) \prod_{j=1}^{k} \Gamma\left(-\sigma_{j}\right)}$
$\frac{1}{(2 \pi \omega)^{l+k}} \int_{L_{1}} \cdots \int_{L_{l+k}} \frac{\Gamma\left(\alpha+\sum_{j=1}^{l} h_{j} s_{j}+\sum_{j=1}^{k} s_{l+j}\right)}{\Gamma\left(\alpha+\beta+\sum_{j=1}^{l} h_{j} s_{j}+\sum_{j=1}^{k} s_{l+j}\right)} \prod_{j=1}^{l} \Gamma\left(\lambda_{j}+s_{j}\right) \prod_{j=1}^{k} \Gamma\left(-\sigma_{j}+s_{l+j}\right)$
$\prod_{j=1}^{l+k} \Gamma\left(-s_{j}\right) z_{1}^{s_{1}} \cdots z_{l}^{s_{l}} z_{l+1}^{s_{l+1}} \cdots, z_{l+k}^{s_{l+k}} \mathrm{~d} s_{1} \cdots \mathrm{~d} s_{l+k}$
Here the contour $L_{j}^{\prime} s$ are defined by $L_{j}=L_{w \zeta_{j} \infty}\left(\operatorname{Re}\left(\zeta_{j}\right)=v_{j}^{\prime \prime}\right)$ starting at the point $v_{j}^{\prime \prime}-\omega \infty$ and terminating at the point $v_{j}^{\prime \prime}+\omega \infty$ with $v_{j}^{\prime \prime} \in \mathbb{R}(j=1, \cdots, l)$ and each of the remaining contour $L_{l+1}, \cdots, L_{l+k}$ run from $-\omega \infty$ to $\omega \infty$
(2.2) can be easily established by expanding $\prod_{j=1}^{l}\left[1-\tau_{j}(t-a)^{h_{i}}\right]^{-\lambda_{j}}$ by means of the formula :
$(1-z)^{-\alpha}=\sum_{r=0}^{\infty} \frac{(\alpha)_{r}}{r!} z^{r}(|z|<1)$
integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [6, page 454].

In this section, we note :

$$
\begin{align*}
& \theta_{i}=\prod_{j=1}^{l}\left[1-\tau_{j}(t-a)^{h_{i}}\right]^{-\zeta_{j}^{(i)}}, \zeta_{j}^{(i)}>0(i=1, \cdots, r) ; \theta_{i}^{\prime}=\prod_{j=1}^{l}\left[1-\tau_{j}(t-a)^{h_{i}}\right]^{-\zeta_{j}^{\prime(i)}}, \zeta_{j}^{\prime(i)}>0(i=1, \cdots, s) \\
& \theta_{i}^{\prime \prime}=\prod_{j=1}^{l}\left[1-\tau_{j}(t-a)^{h_{i}}\right]^{-\zeta_{j}^{\prime \prime(i)}}, \zeta_{j}^{\prime \prime(i)}>0(i=1, \cdots, u) \\
& \theta_{i}^{\prime \prime \prime}=\prod_{j=1}^{l}\left[1-\tau_{j}(t-a)^{h_{i}}\right]^{-\zeta_{j}^{\prime \prime \prime}(i)}, \zeta_{j}^{\prime \prime \prime(i)}>0(i=1, \cdots, v)  \tag{3.1}\\
& X=m_{1}, n_{1} ; \cdots ; m_{r}, n_{r} ; m_{1}^{\prime}, n_{1}^{\prime} ; \cdots ; m_{s}^{\prime}, n_{s}^{\prime} ; 1,0 ; \cdots ; 1,0 ; 1,0 ; \cdots ; 1,0  \tag{3.2}\\
& Y=p_{1}, q_{1} ; \cdots ; p_{r}, q_{r} ; p_{1}^{\prime}, q_{1}^{\prime} ; \cdots ; p_{s}^{\prime}, q_{s}^{\prime} ; 0,1 ; \cdots ; 0,1 ; 0,1 ; \cdots ; 0,1  \tag{3.3}\\
& A=\left(a_{j} ; A_{j}^{(1)}, \cdots, A_{j}^{(r)}, 0, \cdots, 0,0, \cdots, 0,0, \cdots, 0\right)_{1, p}  \tag{3.4}\\
& B=\left(b_{j} ; B_{j}^{(1)}, \cdots, B_{j}^{(r)}, 0 \cdots, 0,0 \cdots, 0,0 \cdots, 0\right)_{1, q}  \tag{3.5}\\
& A^{\prime}=\left(a_{j}^{\prime} ; 0, \cdots, 0, A_{j}^{\prime(1)}, \cdots, A_{j}^{\prime(s)}, 0, \cdots, 0,0, \cdots, 0\right)_{1, p^{\prime}}  \tag{3.6}\\
& B^{\prime}=\left(b_{j}^{\prime} ; 0, \cdots, 0, B_{j}^{\prime(1)}, \cdots, B_{j}^{\prime(s)}, 0, \cdots, 0,0, \cdots, 0\right)_{1, q^{\prime}}  \tag{3.7}\\
& \mathrm{C}=\left(\mathrm{c}_{j}^{(1)}, C_{j}^{(1)}\right)_{1, p_{1}} ; \cdots ;\left(c_{j}^{(r)}, C_{j}^{(r)}\right)_{1, p_{r}} ;\left(c_{j}^{\prime}(1), C_{j}^{\prime(1)}\right)_{1, p_{1}^{\prime}} ; \cdots ;\left(c_{j}^{\prime(r)}, C_{j}^{\prime(s)}\right)_{1, p_{s}^{\prime}} \\
& (1,0) ; \cdots ;(1,0) ;(1,0) ; \cdots ;(1,0) \tag{3.8}
\end{align*}
$$

$D=\left(\mathrm{d}_{j}^{(1)}, D_{j}^{(1)}\right)_{1, q_{1}} ; \cdots ;\left(d_{j}^{(r)}, D_{j}^{(r)}\right)_{1, q_{r}} ;\left(\mathrm{d}_{j}^{,(1)}, D_{j}^{\prime(1)}\right)_{1, q_{1}^{\prime}} ; \cdots ;\left(d_{j}^{\prime(s)}, D_{j}^{\prime(s)}\right)_{1, q_{s}^{\prime}} ;$
$(0,1) ; \cdots ;(0,1) ;(0,1) ; \cdots ;(0,1)$
$X=m_{1}, n_{1} ; \cdots ; m_{r}, n_{r} ; m_{1}^{\prime}, n_{1}^{\prime} ; \cdots ; m_{s}^{\prime}, n_{s}^{\prime} ; 1,0 ; \cdots ; 1,0 ; 1,0 ; \cdots ; 1,0$
$Y=p_{1}, q_{1} ; \cdots ; p_{r}, q_{r} ; p_{1}^{\prime}, q_{1}^{\prime} ; \cdots ; p_{s}^{\prime}, q_{s}^{\prime} ; 0,1 ; \cdots ; 0,1 ; 0,1 ; \cdots ; 0,1$
$A=\left(a_{j} ; A_{j}^{(1)}, \cdots, A_{j}^{(r)}, 0, \cdots, 0,0, \cdots, 0,0, \cdots, 0\right)_{1, p}$
$B=\left(b_{j} ; B_{j}^{(1)}, \cdots, B_{j}^{(r)}, 0 \cdots, 0,0 \cdots, 0,0 \cdots, 0\right)_{1, q}$
$A^{\prime}=\left(a_{j}^{\prime} ; 0, \cdots, 0, A_{j}^{\prime(1)}, \cdots, A_{j}^{\prime(s)}, 0, \cdots, 0,0, \cdots, 0\right)_{1, p^{\prime}}$
$B^{\prime}=\left(b_{j}^{\prime} ; 0, \cdots, 0, B_{j}^{\prime(1)}, \cdots, B_{j}^{\prime(s)}, 0, \cdots, 0,0, \cdots, 0\right)_{1, q^{\prime}}$

$$
\begin{align*}
& \mathrm{C}=\left(c_{j}^{(1)}, C_{j}^{(1)}\right)_{1, p_{1}} ; \cdots ;\left(c_{j}^{(r)}, C_{j}^{(r)}\right)_{1, p_{r}} ;\left(c_{j}^{\prime(1)}, C_{j}^{\prime(1)}\right)_{1, p_{1}^{\prime}} ; \cdots ;\left(c_{j}^{\prime(r)}, C_{j}^{\prime(s)}\right)_{1, p_{s}^{\prime}} \\
& (1,0) ; \cdots ;(1,0) ;(1,0) ; \cdots ;(1,0) \tag{3.8}
\end{align*}
$$

$$
\begin{align*}
& D=\left(\mathrm{d}_{j}^{(1)}, D_{j}^{(1)}\right)_{1, q_{1}} ; \cdots ;\left(d_{j}^{(r)}, D_{j}^{(r)}\right)_{1, q_{r}} ;\left(\mathrm{d}_{j}^{,(1)}, D_{j}^{\prime(1)}\right)_{1, q_{1}^{\prime}} ; \cdots ;\left(d_{j}^{\prime(s)}, D_{j}^{\prime(s)}\right)_{1, q_{s}^{\prime}} ; \\
& (0,1) ; \cdots ;(0,1) ;(0,1) ; \cdots ;(0,1) \tag{3.9}
\end{align*}
$$

$K_{1}=\left(1-\alpha-\sum_{i=1}^{u} R_{i} a_{i}-\sum_{i=1}^{v} K_{i} a_{i}^{\prime} ; \mu_{1}, \cdots, \mu_{r}, \mu_{1}^{\prime}, \cdots, \mu_{s}^{\prime}, h_{1}, \cdots, h_{l}, 1, \cdots, 1\right)$
$K_{2}=\left(1-\beta-\sum_{i=1}^{u} R_{i} b_{i}-\sum_{i=1}^{v} K_{i} b_{i}^{\prime} ; \rho_{1}, \cdots, \rho_{r}, \rho_{1}^{\prime}, \cdots, \rho_{s}^{\prime}, 0, \cdots, 0,0 \cdots, 0\right)$
$K_{j}=\left[1-\lambda_{j}-\sum_{i=1}^{u} R_{i} \zeta_{j}^{\prime \prime(i)}-\sum_{i=1}^{v} K_{i} \zeta_{j}^{\prime \prime \prime(i)} ; \zeta_{j}^{(1)}, \cdots, \zeta_{j}^{(r)}, \zeta_{j}^{\prime(1)} \cdots, \zeta_{j}^{\prime(s)}\right.$,
$0, \cdots, 1, \cdots, 0,0 \cdots, 0]_{1, l}$
$K_{j}^{\prime}=\left[1+\sigma_{j}-\sum_{i=1}^{u} R_{i} \lambda_{j}^{\prime \prime(i)}-\sum_{i=1}^{v} K_{i} \lambda_{j}^{\prime \prime \prime(i)} ; \lambda_{j}^{(1)}, \cdots, \lambda_{j}^{(r)}, \lambda_{j}^{\prime(1)} \cdots, \lambda_{j}^{\prime(s)}\right.$,
$0, \cdots, 0,0 \cdots, \underset{j}{1, \cdots, 0]_{1, k}}$
$L_{1}=\left(1-\alpha-\beta-\sum_{i=1}^{u} R_{i}\left(a_{i}+b_{i}\right)-\sum_{i=1}^{v}\left(a_{i}^{\prime}+b_{i}^{\prime}\right) K_{i} ; \mu_{1}+\rho_{1}, \cdots, \mu_{r}+\rho_{r}, \mu_{1}^{\prime}+\rho_{1}^{\prime}, \cdots, \mu_{r}^{\prime}+\rho_{r}^{\prime}\right.$, $\left.h_{1}, \cdots, h_{l}, 1, \cdots, 1\right)$
$L_{j}=\left[1-\lambda_{j}-\sum_{i=1}^{u} R_{i} \zeta_{j}^{\prime \prime(i)}-\sum_{i=1}^{s} \zeta_{j}^{\prime \prime \prime(i)} K_{i} ; \zeta_{j}^{(1)}, \cdots, \zeta_{j}^{(r)}, \zeta_{j}^{\prime(1)} \cdots, \zeta_{j}^{\prime(s)}, 0, \cdots, 0,0 \cdots, 0\right]_{1, l(3,15}$
$L_{j}^{\prime}=\left[1+\sigma_{j}-\sum_{i=1}^{u} R_{i} \lambda_{j}^{\prime \prime(i)}-\sum_{i=1}^{v} \lambda_{j}^{\prime \prime \prime(i)} K_{i} ; \lambda_{j}^{(1)}, \cdots, \lambda_{j}^{(r)}, \lambda_{j}^{\prime(1)} \cdots, \lambda_{j}^{\prime(s)}, 0, \cdots, 0,0, \cdots, 0\right]_{1, k}$
$P_{1}=(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{h}\left(a f_{j}+g_{j}\right)^{\sigma_{j}}\right\}$
$B_{u, v}=(b-a)^{\sum_{i=1}^{v} K_{i}\left(a_{i}^{\prime}+b_{i}^{\prime}\right)+\sum_{i=1}^{u}\left(a_{i}+b_{i}\right) R_{i}}\left\{\prod_{j=1}^{h}\left(a f_{j}+g_{j}\right)^{-\sum_{i=1}^{v} K_{i} \lambda_{i}^{\prime \prime \prime}-\sum_{i=1}^{u} \lambda_{i}^{\prime \prime} R_{i}}\right\}$
$V_{1}=V ; V^{\prime} ; 1,0 ; \cdots ; 1,0 ; 1,0 ; \cdots ; 1,0 ; W_{1}=W ; W^{\prime} ; 0,1 ; \cdots ; 0,1 ; 0,1 ; \cdots ; 0,1$

$$
\begin{equation*}
C_{1}=C ; C^{\prime} ;(1,0), \cdots,(1,0) ;(1,0), \cdots,(1,0) ; D_{1}=D ; D^{\prime} ;(0,1), \cdots,(0,1) ;(0,1), \cdots,(0,1) \tag{3.20}
\end{equation*}
$$

We have the following integral :

$$
\begin{gathered}
\int_{a}^{b}(t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{l}\left[1-\tau_{j}(t-a)^{h_{i}}\right]^{-\lambda_{j}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{\sigma_{j}} \\
S_{L}^{h_{1}, \cdots, h_{u}}\left(\begin{array}{c}
\mathrm{z}_{1}^{\prime \prime} \theta_{1}^{\prime \prime}(t-a)^{a_{1}}(b-t)^{b_{1}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime(1)}} \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{z}_{u}^{\prime \prime} \theta_{u}^{\prime \prime}(t-a)^{a_{u}}(b-t)^{b_{u}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime( }(u)}
\end{array}\right) \\
S_{N_{1}, \cdots, N_{v}}^{\mathfrak{M}_{1}, \cdots, \mathfrak{M}_{v}}\left(\begin{array}{c}
\mathrm{z}_{1}^{\prime \prime \prime} \theta_{1}^{\prime \prime \prime}(t-a)^{a_{1}^{\prime}}(b-t)^{b_{1}^{\prime}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime \prime}(1)} \\
\cdot \\
\cdot \\
\mathrm{z}_{v}^{\prime \prime \prime} \theta_{v}^{\prime \prime \prime}(t-a)^{a_{v}^{\prime}}(b-t)^{b_{v}^{\prime}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime \prime}(v)}
\end{array}\right)
\end{gathered}
$$

$$
A\left(\begin{array}{c}
\mathrm{z}_{1} \theta_{1}(t-a)^{\mu_{1}}(b-t)^{\rho_{1}} \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\left.\mathrm{z}_{r} \theta_{r}(t-a)^{\mu_{r}}(b-t)^{\rho_{r}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{(1)}}\right)^{-\lambda_{j}^{(r)}}
\end{array}\right)
$$

$$
A\left(\begin{array}{c}
\mathrm{z}_{1}^{\prime} \theta_{1}^{\prime}(t-a)^{\mu_{1}^{\prime}}(b-t)^{\rho_{1}^{\prime}} \\
\cdot \\
\cdot \\
\cdot \\
\left.\mathrm{z}_{s}^{\prime} \theta_{s}^{\prime}(t-a)^{\mu_{s}^{\prime}}(b-t)_{j}^{\rho_{s}^{\prime}} t+g_{j}\right)^{-\lambda_{j}^{\prime(1)}} \\
j=1
\end{array}\right) \mathrm{d} t
$$

$$
=P_{1} \sum_{K_{1}=0}^{\left[N_{1} / \mathfrak{M}_{1}\right]} \cdots \sum_{K_{v}=0}^{\left[N_{v} / \mathfrak{M}_{\mathfrak{v}}\right] h_{1} R_{1}+\cdots h_{u} R_{u} \leqslant L} \sum_{R_{1}, \cdots, R_{u}=0} a_{v} b_{u} B_{u, v} \prod_{k=1}^{u} z^{\prime \prime R_{k}} \prod_{k=1}^{v} z^{\prime \prime \prime K_{k}}
$$

| $A_{p+p^{\prime}+l+k+2, q+q^{\prime}+l+k+1 ; Y}^{m+m^{\prime}, n+n^{\prime}+l+k+2 ; X}$ | $\left(\begin{array}{c} \frac{z_{1}(b-a)^{\mu_{1}+\rho_{1}}}{\prod_{j=1}^{k}\left(a f_{j}+g_{j}\right)^{\lambda_{j}^{(1)}}} \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \frac{z_{r}(b-a)^{\mu_{r}+\rho_{r}}}{\prod_{j=1}^{k}\left(a f_{j}+g_{j}\right)^{\lambda_{j}^{(r)}}} \\ \frac{z_{1}^{\prime}(b-a)^{\mu_{1}^{\prime}+\rho_{1}^{\prime}}}{\prod_{j=1}^{k}\left(a f_{j}+g_{j}\right)^{\lambda_{j}^{\prime(1)}}} \\ \cdot \cdot \\ \cdot \cdot \cdot \\ \frac{z_{s}^{\prime}(b-a)^{\mu_{s}^{\prime}+\rho_{s}^{\prime}}}{\prod_{j=1}^{k}\left(a f_{j}+g_{j}\right)^{\lambda_{j}^{\prime(s)}}}  \tag{3.21}\\ \tau_{1}(b-a)^{h_{1}} \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \tau_{l}(b-a)^{h_{l}} \\ \frac{(b-a) f_{1}}{a f_{1}+g_{1}} \\ \cdot \cdot \\ \cdot \cdot \cdot \\ a-a) f_{k} \\ a+f_{k} \end{array}\right.$ | $\mathfrak{A}, \mathrm{K}_{1}, K_{2}, K_{j}, K_{j}^{\prime}: C$ $\mathfrak{B}, \mathrm{L}_{1}, L_{j}, L_{j}^{\prime}: D$ |
| :---: | :---: | :---: |

where $\mathfrak{A}, \mathfrak{B}, C, D, X, K_{1}, K_{2}, K_{j}, K_{j}^{\prime}, L_{1}, L_{j}, L_{j}^{\prime}, P_{1}, P_{u}, B_{u}$ and $\mathfrak{B}_{1}$ are defined above.
Provided that
(A) $a, b \in \mathbb{R}(a<b) ; \mu_{i}, \mu_{u}^{\prime}, \rho_{i}, \rho_{u}^{\prime}, \lambda_{j}^{(i)}, \lambda_{j}^{\prime(u)}, h_{v} \in \mathbb{R}^{+}, f_{i}, g_{j}, \tau_{v}, \sigma_{j}, \lambda_{v} \in \mathbb{C}(i=1, \cdots, r ; j=1, \cdots ; k$;
$u=1, \cdots, s ; v=1, \cdots, l), a_{i}, b_{i}, \lambda_{j}^{\prime \prime(i)}, \zeta_{j}^{\prime \prime(i)} \in \mathbb{R}^{+},(i=1, \cdots, u ; j=1, \cdots, k)$
$a_{i}^{\prime}, b_{i}^{\prime}, \lambda_{j}^{\prime \prime \prime(i)}, \zeta_{j}^{\prime \prime \prime}(i) \in \mathbb{R}^{+},(i=1, \cdots, v ; j=1, \cdots, k)$
(B) $m, n, p, m_{i}, n_{i}, p_{i}, c_{i} \in \mathbb{N}^{*} ; i=1, \cdots, r ; a_{j}, b_{j}, c_{j}^{(i)}, d_{j}^{(i)}, A_{j}^{(i)}, B_{j}^{(i)}, C_{j}^{(i)}, D_{j}^{(i)} \in \mathbb{C}$
$m^{\prime}, n^{\prime}, p^{\prime}, m_{i}^{\prime}, n_{i}^{\prime}, p_{i}^{\prime}, c_{i}^{\prime} \in \mathbb{N}^{*} ; i=1, \cdots, r ; a_{j}^{\prime}, b_{j}^{\prime},,_{j}^{\prime(i)}, d_{j}^{\prime(i)}, A_{j}^{\prime(i)}, B_{j}^{\prime(i)}, C_{j}^{\prime(i)}, D_{j}^{\prime(i)} \in \mathbb{C}$
(C) $\max _{1 \leqslant j \leqslant k}\left\{\left|\frac{(b-a) f_{i}}{a f_{i}+g_{i}}\right|\right\}<1$
(D) $\operatorname{Re}\left[\alpha+\sum_{j=1}^{r} \mu_{j} \min _{1 \leqslant k \leqslant m_{i}} \frac{d_{k}^{(j)}}{D_{k}^{(j)}}+\sum_{j=1}^{s} \mu_{i}^{\prime} \min _{1 \leqslant k \leqslant m_{i}^{\prime}} \frac{d_{k}^{(j)}}{D_{k}^{\prime(j)}}\right]>0$
$R e\left[\beta+\sum_{j=1}^{r} \rho_{j} \min _{1 \leqslant k \leqslant m_{i}} \frac{D_{k}^{(j)}}{D_{k}^{(j)}}+\sum_{j=1}^{s} \rho_{j}^{\prime} \min _{1 \leqslant k \leqslant m_{i}^{\prime}} \frac{d_{k}^{\prime(j)}}{D_{k}^{\prime(j)}}\right]>0$
(E) $R e\left(\alpha+\sum_{i=1}^{v} K_{i} a_{i}^{\prime}+\sum_{i=1}^{u} R_{i} a_{i}+\sum_{i=1}^{r} \mu_{i} s_{i}+\sum_{i=1}^{s} t_{i} \mu_{i}^{\prime}\right)>0$
$\operatorname{Re}\left(\beta+\sum_{i=1}^{v} K_{i} b_{i}^{\prime}+\sum_{i=1}^{u} R_{i} b_{i}+\sum_{i=1}^{r} v_{i} s_{i}+\sum_{i=1}^{s} t_{i} \rho_{i}^{\prime}\right)>0$
$\operatorname{Re}\left(\lambda_{j}+\sum_{i=1}^{v} K_{i} \lambda_{j}^{\prime \prime \prime(i)}+\sum_{i=1}^{u} R_{i} \lambda_{j}^{\prime \prime(i)}+\sum_{i=1}^{r} s_{i} \zeta_{j}^{(i)}+\sum_{i=1}^{s} t_{i} \zeta_{j}^{\prime(i)}\right)>0(j=1, \cdots, l) ;$
$\operatorname{Re}\left(-\sigma_{j}+\sum_{i=1}^{v} K_{i} \lambda^{\prime \prime \prime(i)}+\sum_{i=1}^{u} R_{i} \lambda_{j}^{\prime \prime(i)}+\sum_{i=1}^{r} s_{i} \lambda_{j}^{(i)}+\sum_{i=1}^{s} t_{i} \lambda_{j}^{\prime(i)}\right)>0(j=1, \cdots, k)$
(F) $\left|\arg \left(\Omega_{i}\right) z_{k}\right|<\frac{1}{2} \eta_{i} \pi, \xi^{*}=0, \eta_{i}>0$
$\Omega_{i}=\prod_{j=1}^{p}\left\{A_{j}^{(i)}\right\}^{A_{j}^{(i)}} \prod_{j=1}^{q}\left\{B_{j}^{(i)}\right\}^{-B_{j}^{(i)}} \prod_{j=1}^{q_{i}}\left\{D_{j}^{(i)}\right\}^{D_{j}^{(i)}} \prod_{j=1}^{p_{i}}\left\{C_{j}^{(i)}\right\}^{-C_{j}^{(i)}} ; i=1, \cdots, r$
$\xi_{i}^{*}=\operatorname{Im}\left(\sum_{j=1}^{p} A_{j}^{(i)}-\sum_{j=1}^{q} B_{j}^{(i)}+\sum_{j=1}^{q_{i}} D_{j}^{(i)}-\sum_{j=1}^{p_{i}} C_{j}^{(i)}\right) ; i=1, \cdots, r$
$\eta_{i}=\operatorname{Re}\left(\sum_{j=1}^{n} A_{j}^{(i)}-\sum_{j=n+1}^{p} A_{j}^{(i)}+\sum_{j=1}^{m} B_{j}^{(i)}-\sum_{j=m+1}^{q} B_{j}^{(i)}+\sum_{j=1}^{m_{i}} D_{j}^{(i)}-\sum_{j=m_{i}+1}^{q_{i}} D_{j}^{(i)}+\sum_{j=1}^{n_{i}} C_{j}^{(i)}-\sum_{j=n_{i}+1}^{p_{i}} C_{j}^{(i)}\right)$
$-\mu_{i}-\rho_{i}-\sum_{l=1}^{k} \lambda_{j}^{(i)}-\sum_{l=1}^{l} \zeta_{j}^{(i)}>0 ; i=1, \cdots, r$
$\left|\arg \left(\Omega_{i}^{\prime}\right) z_{k}^{\prime}\right|<\frac{1}{2} \eta_{i}^{\prime} \pi, \xi^{\prime *}=0, \eta_{i}^{\prime}>0$
$\Omega_{i}^{\prime}=\prod_{j=1}^{p^{\prime}}\left\{A_{j}^{\prime(i)}\right\}^{A_{j}^{\prime(i)}} \prod_{j=1}^{q^{\prime}}\left\{B_{j}^{\prime(i)}\right\}^{-B_{j}^{\prime(i)}} \prod_{j=1}^{q_{i}^{\prime}}\left\{D_{j}^{\prime(i)}\right\}^{D_{j}^{(i)}} \prod_{j=1}^{p_{i}^{\prime}}\left\{C_{j}^{\prime(i)}\right\}^{-C_{j}^{\prime(i)}} ; i=1, \cdots, s$
$\xi_{i}^{\prime *}=\operatorname{Im}\left(\sum_{j=1}^{p^{\prime}} A_{j}^{\prime(i)}-\sum_{j=1}^{q^{\prime}}{B_{j}^{\prime(i)}}+\sum_{j=1}^{q_{i}^{\prime}} D_{j}^{\prime(i)}-\sum_{j=1}^{p_{i}^{\prime}} C_{j}^{\prime(i)}\right) ; i=1, \cdots, s$
$\eta_{i}^{\prime}=\operatorname{Re}\left(\sum_{j=1}^{n^{\prime}} A_{j}^{\prime(i)}-\sum_{j=n^{\prime}+1}^{p^{\prime}} A_{j}^{\prime(i)}+\sum_{j=1}^{m^{\prime}} B_{j}^{\prime(i)}-\sum_{j=m^{\prime}+1}^{q^{\prime}} B_{j}^{\prime(i)}+\sum_{j=1}^{m_{i}^{\prime}} D_{j}^{\prime(i)}-\sum_{j=m_{i}^{\prime}+1}^{q_{i}^{\prime}} D_{j}^{\prime(i)}+\sum_{j=1}^{n_{i}^{\prime}} C_{j}^{\prime(i)}-\sum_{j=n_{i}^{\prime}+1}^{p_{i}^{\prime}} C_{j}^{\prime(i)}\right)$
$-\mu_{i}^{\prime}-\rho_{i}^{\prime}-\sum_{l=1}^{k} \lambda_{j}^{\prime(i)}-\sum_{l=1}^{l}{\zeta_{j}^{\prime}}^{(i)}>0 ; i=1, \cdots, s$
(H) $\left|\arg \left(z_{i} \prod_{j=1}^{l}\left[1-\tau_{j}(t-a)^{h_{i}}\right]^{-\zeta_{j}^{(i)}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{(i)}}\right)\right|<\frac{1}{2} \eta_{i} \pi(a \leqslant t \leqslant b ; i=1, \cdots, r)$
$\left|\arg \left(z_{i}^{\prime} \prod_{j=1}^{l}\left[1-\tau_{j}^{\prime}(t-a)^{h_{i}^{\prime}}\right]^{-\zeta_{j}^{\prime(i)}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime(i)}}\right)\right|<\frac{1}{2} \eta_{i}^{\prime} \pi(a \leqslant t \leqslant b ; i=1, \cdots, s)$

## Proof

To prove (3.21), first expressing a class of multivariable polynomials $S_{N_{1}, \cdots, N_{v}}^{\mathfrak{M}_{1}, \cdots} \mathfrak{M}_{\mathfrak{v}}$ [.] defined by Srivastava [3] in serie with the help of (1.17), a class of multivariable polynomials $S_{L}^{h_{1}, \cdots, h_{u}}$ [.] defined by Srivastava et al [5] in serie with the help of (1.18) and we interchange the order of summations and x-integral (which is permissible under the conditions stated). Expressing the A-functions of r-variables and s-variables defined by Gautam et al [1] in terms of Mellin-Barnes type contour integral with the help of (1.2) and (1.10) respectively and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Now collect the power of $\left[1-\tau_{j}(t-a)^{h_{i}}\right]$ with $(i=1, \cdots, r ; j=1, \cdots, l)$ and collect the power of $\left(f_{j} t+g_{j}\right)$ with $j=1, \cdots, k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral in multivariable A-function defined by Gautam et al [1], we obtain the equation (3.21).

## Remarks

If a) $\rho_{1}=\cdots, \rho_{r}=\rho_{1}^{\prime}=\cdots, \rho_{s}^{\prime}=0$; b) $\mu_{1}=\cdots, \mu_{r}=\mu_{1}^{\prime}=\cdots, \mu_{s}^{\prime}=0$, we obtain the similar formulas that (3.21) with the corresponding simplifications.

## 4. Particular cases

a)If $A_{j}^{(i)}, B_{j}^{(i)}, C_{j}^{(i)}, D_{j}^{(i)} \in \mathbb{R}, m=0$ and $A_{j}^{\prime(i)}, B_{j}^{\prime(i)}, C_{j}^{\prime(i)}, D_{j}^{\prime(i)} \in \mathbb{R}$ and $m^{\prime}=0$, the multivariable A-functions reduces to multivariable H -functions defined by Srivastava et al [7], we obtain the following result.

$$
\begin{gathered}
\int_{a}^{b}(t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{l}\left[1-\tau_{j}(t-a)^{h_{i}}\right]^{-\lambda_{j}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{\sigma_{j}} \\
S_{L}^{h_{1}, \cdots, h_{u}}\left(\begin{array}{c}
\mathrm{z}_{1}^{\prime \prime} \theta_{1}^{\prime \prime}(t-a)^{a_{1}}(b-t)^{b_{1}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime(1)}} \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{z}_{u}^{\prime \prime} \theta_{u}^{\prime \prime}(t-a)^{a_{u}}(b-t)^{b_{u}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime(u)}}
\end{array}\right) \\
S_{N_{1}, \cdots, N_{v}}^{\mathfrak{M}_{1}, \cdots, \mathfrak{M}_{v}}\left(\begin{array}{c}
\mathrm{z}_{1}^{\prime \prime \prime} \theta_{1}^{\prime \prime \prime}(t-a)^{a_{1}^{\prime}}(b-t)^{b_{1}^{\prime}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime \prime(1)}} \\
\cdot \\
\cdot \\
\mathrm{z}_{v}^{\prime \prime \prime} \theta_{v}^{\prime \prime \prime}(t-a)^{a_{v}^{\prime}}(b-t)^{b_{v}^{\prime}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime \prime}(v)}
\end{array}\right)
\end{gathered}
$$

$$
H\left(\begin{array}{c}
\mathrm{z}_{1} \theta_{1}(t-a)^{\mu_{1}}(b-t)^{\rho_{1}} \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\left.\mathrm{z}_{r} \theta_{r}(t-a)^{\mu_{r}}(b-t)^{\rho_{r}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{(1)}}\right)^{-\lambda_{j}^{(r)}}
\end{array}\right)
$$

$$
H\left(\begin{array}{c}
\mathrm{z}_{1}^{\prime} \theta_{1}^{\prime}(t-a)^{\mu_{1}^{\prime}}(b-t)^{\rho_{1}^{\prime}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime(1)}} \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{z}_{s}^{\prime} \theta_{s}^{\prime}(t-a)^{\mu_{s}^{\prime}}(b-t)^{\rho_{s}^{\prime}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime(s)}}
\end{array}\right) \mathrm{d} t
$$

$$
=P_{1} \sum_{K_{1}=0}^{\left[N_{1} / \mathfrak{M}_{1}\right]} \cdots \sum_{K_{v}=0}^{\left[N_{v} / \mathfrak{M}_{v}\right] h_{1} R_{1}+\cdots h_{u} R_{u} \leqslant L} \sum_{R_{1}, \cdots, R_{u}=0}^{u} a_{v} b_{u} B_{u, v} \prod_{k=1}^{u} z^{\prime \prime R_{k}} \prod_{k=1}^{v} z^{\prime \prime \prime K_{k}}
$$

| $H_{p+p^{\prime}+l+k+2, q+q^{\prime}+l+k+1 ; ~}^{0, n+n^{\prime}+l+k+2 ; X}$ | $\left(\begin{array}{c} \frac{z_{1}(b-a)^{\mu_{1}+\rho_{1}}}{\prod_{j=1}^{k}\left(a f_{j}+g_{j}\right)^{\lambda_{j}^{(1)}}} \\ \cdots \cdot \\ \cdot \cdot \\ \frac{z_{r}(b-a)^{\mu_{r}+\rho_{r}}}{\prod_{j=1}^{k}\left(a f_{j}+g_{j}\right)^{\lambda_{j}^{(r)}}} \\ \frac{z_{1}^{\prime}(b-a)^{\mu_{1}^{\prime}+\rho_{1}^{\prime}}}{\prod_{j=1}^{k}\left(a f_{j}+g_{j}\right)^{\lambda_{j}^{\prime(1)}}}  \tag{4.1}\\ \cdots \cdot \\ \cdot \cdot \\ \frac{z_{s}^{\prime}(b-a)^{\mu_{s}^{\prime}+\rho_{s}^{\prime}}}{\prod_{j=1}^{k}\left(a f_{j}+g_{j}\right)_{j}^{\lambda_{j}^{\prime(s)}}} \\ \tau_{1}(b-a)^{h_{1}} \\ \cdot \cdot \\ \cdot \cdot \\ \tau_{l}(b-a)^{h_{l}} \\ \frac{(b-a) f_{1}}{a f_{1}+g_{1}} \\ \cdot \cdot \\ \cdot \cdot \\ \frac{(b-a) f_{k}}{a f_{k}+g_{k}} \end{array}\right.$ | $\left\|\begin{array}{c} \mathfrak{A}, \mathrm{K}_{1}, K_{2}, K_{j}, K_{j}^{\prime}: C \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}\right\|$ |
| :---: | :---: | :---: |

under the same notations and validity conditions that (3.21) with $A_{j}^{(i)}, B_{j}^{(i)}, C_{j}^{(i)}, D_{j}^{(i)} \in \mathbb{R} \quad, m=0 \quad$ and $A_{j}^{\prime}{ }^{(i)}, B_{j}^{\prime(i)}, C_{j}^{\prime(i)}, D_{j}^{\prime(i)} \in \mathbb{R}$ and $m^{\prime}=0$

then the general class of multivariable polynomial $S_{L}^{h_{1}, \cdots, h_{u}}\left[z_{1}, \cdots, z_{u}\right]$ reduces to generalized Lauricella function defined by Srivastava et al [4]. We have

$$
\begin{aligned}
& \int_{a}^{b}(t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{l}\left[1-\tau_{j}(t-a)^{h_{i}}\right]^{-\lambda_{j}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{\sigma_{j}} \\
& F_{\bar{C}: D^{\prime} ; \cdots ; D^{(u)}}^{1+\bar{A} ; B^{\prime} ; \cdots ; B^{(u)}}\left(\begin{array}{c}
\mathrm{z}_{1}^{\prime \prime} \theta_{1}^{\prime \prime}(t-a)^{a_{1}}(b-t)^{b_{1}} \\
\cdot \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime(1)}} \\
\cdot \\
\cdot \\
\mathrm{z}_{u}^{\prime \prime} \theta_{u}^{\prime \prime}(t-a)^{a_{u}}(b-t)^{b_{u}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime(u)}}
\end{array}\right. \\
& \left.\left[(-\mathrm{L}) ; \mathrm{R}_{1}, \cdots, R_{u}\right]\left[(a) ; \theta^{\prime}, \cdots, \theta^{(u)}\right]:\left[\left(b^{\prime}\right) ; \phi^{\prime}\right] ; \cdots ;\left[\left(b^{(u)}\right) ; \phi^{(u)}\right]\right) \\
& {\left[(c) ; \psi^{\prime}, \cdots, \psi^{(u)}\right]:\left[\left(d^{\prime}\right) ; \delta^{\prime}\right] ; \cdots ;\left[\left(d^{(u)}\right) ; \delta^{(u)}\right]} \\
& S_{N_{1}, \cdots, N_{v}}^{\mathfrak{M}_{1}, \cdots, \mathfrak{M}_{v}}\left(\begin{array}{c}
\mathrm{z}_{1}^{\prime \prime \prime} \theta_{1}^{\prime \prime \prime}(t-a)^{a_{1}^{\prime}}(b-t)^{b_{1}^{\prime}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime \prime(1)}} \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{z}_{v}^{\prime \prime \prime} \theta_{v}^{\prime \prime \prime}(t-a)^{a_{v}^{\prime}}(b-t)^{b_{v}^{\prime}} \\
\prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime \prime \prime}(v)}
\end{array}\right) \\
& A\left(\begin{array}{c}
\mathrm{z}_{1} \theta_{1}(t-a)^{\mu_{1}}(b-t)^{\rho_{1}} \\
\cdot \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{(1)}} \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{z}_{r} \theta_{r}(t-a)^{\mu_{r}}(b-t)^{\rho_{r}} \\
\prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{(r)}}
\end{array}\right) \\
& A\left(\begin{array}{c}
\mathrm{z}_{1}^{\prime} \theta_{1}^{\prime}(t-a)^{\mu_{1}^{\prime}}(b-t)^{\rho_{1}^{\prime}} \prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime(1)}} \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{z}_{s}^{\prime} \theta_{s}^{\prime}(t-a)^{\mu_{s}^{\prime}}(b-t)^{\rho_{s}^{\prime}} \\
\prod_{j=1}^{k}\left(f_{j} t+g_{j}\right)^{-\lambda_{j}^{\prime(s)}}
\end{array}\right) \mathrm{d} t \\
& =P_{1} \sum_{K_{1}=0}^{\left[N_{1} / \mathfrak{M}_{1}\right]} \cdots \sum_{K_{v}=0}^{\left[N_{v} / \mathfrak{M}_{\mathfrak{v}}\right] h_{1} R_{1}+\cdots h_{u} R_{u} \leqslant L} \sum_{R_{1}, \cdots, R_{u}=0} a_{v} b_{u}^{\prime} B_{u, v} \prod_{k=1}^{u} z^{\prime \prime R_{k}} \prod_{k=1}^{v} z^{\prime \prime \prime K_{k}}
\end{aligned}
$$


under the same conditions and notations that (3.13)
where $b_{u}^{\prime}=\frac{(-L)_{h_{1} R_{1}+\cdots+h_{u} R_{u} B\left(E ; R_{1}, \cdots, R_{u}\right)}^{R_{1}!\cdots R_{u}!}, B\left[E ; R_{1}, \ldots, R_{v}\right] \text { is defined by }}{}$

## Remark:

By the following similar procedure, the results of this document can be extented the classes of multivariable polynomials defined by Srivastava et al [5] and Srivastava [3].

## 5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable A-functions defined by Gautam et al [1] and the classes of multivariable polynomials defined by Srivastava [3] and Srivastava et al [5] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

## REFERENCES

[1] Gautam B.P., Asgar A.S. and Goyal A.N. On the multivariable A-function. Vijnana Parishas Anusandhan Patrika Vol

29(4) 1986, page 67-81.
[2] Saigo M. and Saxena R.K. Unified fractional integral formulas for the multivariable H-function I. J.Fractional Calculus 15 (1999), page 91-107.
[3] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), page183-191.
[4] Srivastava H.M. and Daoust M.C. Certain generalized Neumann expansions associated with Kampé de Fériet function. Nederl. Akad. Wetensch. Proc. Ser A72 = Indag Math 31(1969) page 449-457.
[5] Srivastava H.M. And Garg M. Some integral involving a general class of polynomials and multivariable H-function. Rev. Roumaine Phys. 32(1987), page 685-692.
[6] Srivastava H.M. and Karlsson P.W. Multiple Gaussian Hypergeometric series. Ellis.Horwood. Limited. New-York, Chichester. Brisbane. Toronto , 1985.
[7] H.M. Srivastava And R.Panda. Some expansion theorems and generating relations for the H -function of several complex variables. Comment. Math. Univ. St. Paul. 24(1975), p.119-137.

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