

Eulerian integral associated with product of two multivariable

A-functions and the classes of polynomials

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ABSTRACT

The present paper is evaluated a new Eulerian integral associated with the product of two multivariable A-functions defined by Gautam et al [1] a generalized Lauricella function and a classes of multivariable polynomials with general arguments . We will study the case concerning the multivariable H-function defined by Srivastava et al [7] and Srivastava-Daoust polynomial [4].

Keywords: Eulerian integral, multivariable A-function, generalized Lauricella function of several variables, multivariable H-function, generalized hypergeometric function, class of polynomials

2010 Mathematics Subject Classification :33C05, 33C60

1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable a-functions defined by Gautam et al [1] , and the classes of polynomials with general arguments.

The A-function is defined and represented in the following manner.

$$A(z_1, \dots, z_r) = A_{p,q;p_1,q_1;\dots;p_r,q_r}^{m,n;m_1,n_1;\dots;m_r,n_r} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{l} (a_j; A_j^{(1)}, \dots, A_j^{(r)})_{1,p} : \\ \\ (b_j; B_j^{(1)}, \dots, B_j^{(r)})_{1,q} : \end{array} \right. \\ \left. \begin{array}{l} (c_j^{(1)}, C_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)}, C_j^{(r)})_{1,p_r} \\ \\ (d_j^{(1)}, D_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)}, D_j^{(r)})_{1,q_r} \end{array} \right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi(s_1, \dots, s_r) \prod_{i=1}^r \theta_i(s_i) z_i^{s_i} ds_1 \dots ds_r \tag{1.2}$$

where $\phi(s_1, \dots, s_r), \theta_i(s_i), i = 1, \dots, r$ are given by :

$$\phi(s_1, \dots, s_r) = \frac{\prod_{j=1}^m \Gamma(b_j - \sum_{i=1}^r B_j^{(i)} s_i) \prod_{j=1}^n \Gamma(1 - a_j + \sum_{i=1}^r A_j^{(i)} s_j)}{\prod_{j=n+1}^p \Gamma(a_j - \sum_{i=1}^r A_j^{(i)} s_j) \prod_{j=m+1}^q \Gamma(1 - b_j + \sum_{i=1}^r B_j^{(i)} s_j)} \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + C_j^{(i)} s_i) \prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - D_j^{(i)} s_i)}{\prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - C_j^{(i)} s_i) \prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + D_j^{(i)} s_i)} \tag{1.4}$$

Here $m, n, p, m_i, n_i, p_i, c_i \in \mathbb{N}^*; i = 1, \dots, r; a_j, b_j, c_j^{(i)}, d_j^{(i)}, A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i)z_k| < \frac{1}{2}\eta_k\pi, \xi^* = 0, \eta_i > 0 \tag{1.5}$$

$$\Omega_i = \prod_{j=1}^p \{A_j^{(i)}\} A_j^{(i)} \prod_{j=1}^q \{B_j^{(i)}\}^{-B_j^{(i)}} \prod_{j=1}^{q_i} \{D_j^{(i)}\} D_j^{(i)} \prod_{j=1}^{p_i} \{C_j^{(i)}\}^{-C_j^{(i)}}; i = 1, \dots, r \tag{1.6}$$

$$\xi_i^* = Im\left(\sum_{j=1}^p A_j^{(i)} - \sum_{j=1}^q B_j^{(i)} + \sum_{j=1}^{q_i} D_j^{(i)} - \sum_{j=1}^{p_i} C_j^{(i)}\right); i = 1, \dots, r \tag{1.7}$$

$$\eta_i = Re\left(\sum_{j=1}^n A_j^{(i)} - \sum_{j=n+1}^p A_j^{(i)} + \sum_{j=1}^m B_j^{(i)} - \sum_{j=m+1}^q B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\right) \tag{1.8}$$

$i = 1, \dots, r$

Consider the second multivariable A-function.

$$A(z'_1, \dots, z'_s) = A_{p',q':p'_1,q'_1;\dots;p'_r,q'_r}^{m',n':m'_1,n'_1;\dots;m'_r,n'_r} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \middle| \begin{matrix} (a'_j; A'_j(1), \dots, A'_j(s))_{1,p'} : \\ \\ \\ (b'_j; B'_j(1), \dots, B'_j(s))_{1,q'} : \end{matrix} \right)$$

$$\left((c'_j(1), C'_j(1))_{1,p'_1}; \dots; (c'_j(s), C'_j(s))_{1,p'_s} \right)$$

$$\left((d'_j(1), D'_j(1))_{1,q'_1}; \dots; (d'_j(s), D'_j(s))_{1,q'_s} \right) \tag{1.9}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L'_1} \dots \int_{L'_s} \phi'(t_1, \dots, t_s) \prod_{i=1}^s \theta'_i(t_i) z_i^{s_i} ds_1 \dots ds_r \tag{1.10}$$

where $\phi'(t_1, \dots, t_s), \theta'_i(t_i), i = 1, \dots, s$ are given by :

$$\phi'(t_1, \dots, t_s) = \frac{\prod_{j=1}^{m'} \Gamma(b'_j - \sum_{i=1}^s B'_j(i)t_i) \prod_{j=1}^{n'} \Gamma(1 - a'_j + \sum_{i=1}^s A'_j(i)t_j)}{\prod_{j=n'+1}^{p'} \Gamma(a'_j - \sum_{i=1}^s A'_j(i)t_j) \prod_{j=m'+1}^{q'} \Gamma(1 - b'_j + \sum_{i=1}^s B'_j(i)t_j)} \tag{1.11}$$

$$\theta'_i(t_i) = \frac{\prod_{j=1}^{n'_i} \Gamma(1 - c'_j(i) + C'_j(i)t_i) \prod_{j=1}^{m'_i} \Gamma(d'_j(i) - D'_j(i)t_i)}{\prod_{j=n'_i+1}^{p'_i} \Gamma(c'_j(i) - C'_j(i)t_i) \prod_{j=m'_i+1}^{q'_i} \Gamma(1 - d'_j(i) + D'_j(i)t_i)} \tag{1.12}$$

Here $m', n', p', m'_i, n'_i, p'_i, c'_i \in \mathbb{N}^*; i = 1, \dots, r; a'_j, b'_j, c'_j(i), d'_j(i), A'_j(i), B'_j(i), C'_j(i), D'_j(i) \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega'_i)z'_k| < \frac{1}{2}\eta'_k\pi, \xi'^* = 0, \eta'_i > 0 \tag{1.13}$$

$$\Omega'_i = \prod_{j=1}^{p'} \{A'_j(i)\} A'_j(i) \prod_{j=1}^{q'} \{B'_j(i)\}^{-B'_j(i)} \prod_{j=1}^{q'_i} \{D'_j(i)\} D'_j(i) \prod_{j=1}^{p'_i} \{C'_j(i)\}^{-C'_j(i)}; i = 1, \dots, s \tag{1.14}$$

$$\xi_i^* = Im\left(\sum_{j=1}^{p'} A_j^{(i)} - \sum_{j=1}^{q'} B_j^{(i)} + \sum_{j=1}^{q'_i} D_j^{(i)} - \sum_{j=1}^{p'_i} C_j^{(i)}\right); i = 1, \dots, s \tag{1.15}$$

$$\eta_i' = Re\left(\sum_{j=1}^{n'} A_j^{(i)} - \sum_{j=n'+1}^{p'} A_j^{(i)} + \sum_{j=1}^{m'} B_j^{(i)} - \sum_{j=m'+1}^{q'} B_j^{(i)} + \sum_{j=1}^{m'_i} D_j^{(i)} - \sum_{j=m'_i+1}^{q'_i} D_j^{(i)} + \sum_{j=1}^{n'_i} C_j^{(i)} - \sum_{j=n'_i+1}^{p'_i} C_j^{(i)}\right)$$

$$i = 1, \dots, s \tag{1.16}$$

The generalized polynomials of multivariables defined by Srivastava [3], is given in the following manner :

$$S_{N_1, \dots, N_v}^{\mathfrak{M}_1, \dots, \mathfrak{M}_v} [y_1, \dots, y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \dots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \dots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!}$$

$$A[N_1, K_1; \dots; N_v, K_v] y_1^{K_1} \dots y_v^{K_v} \tag{1.17}$$

where $\mathfrak{M}_1, \dots, \mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \dots; N_v, K_v]$ are arbitrary constants, real or complex.

Srivastava and Garg [5] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1, \dots, h_u} [z_1, \dots, z_u] = \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} (-L)_{h_1 R_1 + \dots + h_u R_u} B(E; R_1, \dots, R_u) \frac{z_1^{R_1} \dots z_u^{R_u}}{R_1! \dots R_u!} \tag{1.18}$$

The coefficients are $B[E; R_1, \dots, R_u]$ arbitrary constants, real or complex.

We will note $a_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \dots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1, K_1; \dots; N_v, K_v]$ and

$$b_u = \frac{(-E)_{F_1 L_1 + \dots + F_u L_u} B(E; L_1, \dots, L_u)}{L_1! \dots L_u!} \tag{1.19}$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [6 ,page 39 eq .30]

$$\frac{\prod_{j=1}^P \Gamma(A_j)}{\prod_{j=1}^Q \Gamma(B_j)} {}_P F_Q [(A_P); (B_Q); -(x_1 + \dots + x_r)]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \frac{\prod_{j=1}^P \Gamma(A_j + s_1 + \dots + s_r)}{\prod_{j=1}^Q \Gamma(B_j + s_1 + \dots + s_r)} \Gamma(-s_1) \dots \Gamma(-s_r) x_1^{s_1} \dots x_r^{s_r} ds_1 \dots ds_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \dots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \dots, r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \dots, r$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^l [1-\tau_j(t-a)^{h_j}]^{-\lambda_j} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j} dt = (b-a)^{\alpha+\beta-1} B(\alpha, \beta) \prod_{j=1}^k (af_j + g_j)^{\sigma_j}$$

$$F_{1:0, \dots, 0; 0, \dots, 0}^{1:1, \dots, 1; 1, \dots, 1} \left(\begin{matrix} (\alpha : h_1, \dots, h_l, 1, \dots, 1) : (\lambda_1 : 1), \dots, (\lambda_l : 1); (-\sigma_1 : 1), \dots, (-\sigma_k : 1) \\ \dots \\ (\alpha + \beta : h_1, \dots, h_l, 1, \dots, 1) : -, \dots, -; -, \dots, - \end{matrix} \right)$$

$$; \tau_1(b-a)^{h_1}, \dots, \tau_l(b-a)^{h_l}, -\frac{(b-a)f_1}{af_1 + g_1}, \dots, -\frac{(b-a)f_k}{af_k + g_k} \quad (2.2)$$

where $a, b \in \mathbb{R}(a < b), \alpha, \beta, f_i, g_i, \sigma_i, \tau_j, h_j \in \mathbb{C}, \lambda_j \in \mathbb{R}^+(i = 1, \dots, k; j = 1, \dots, l)$

$$\min(\operatorname{Re}(\alpha), \operatorname{Re}(\beta)) > 0, \max_{1 \leq j \leq l} \{|\tau_j(b-a)^{h_j}|\} < 1, \max_{1 \leq j \leq k} \left\{ \left| \frac{(b-a)f_j}{af_j + g_j} \right| \right\} < 1,$$

and $F_{1:0, \dots, 0; 0, \dots, 0}^{1:1, \dots, 1; 1, \dots, 1}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust [4, page 454] given by :

$$F_{1:0, \dots, 0; 0, \dots, 0}^{1:1, \dots, 1; 1, \dots, 1} \left(\begin{matrix} (\alpha : h_1, \dots, h_l, 1, \dots, 1) : (\lambda_1 : 1), \dots, (\lambda_l : 1); (-\sigma_1 : 1), \dots, (-\sigma_k : 1) \\ \dots \\ (\alpha + \beta : h_1, \dots, h_l, 1, \dots, 1) : -, \dots, -; -, \dots, - \end{matrix} \right)$$

$$; \tau_1(b-a)^{h_1}, \dots, \tau_l(b-a)^{h_l}, -\frac{(b-a)f_1}{af_1 + g_1}, \dots, -\frac{(b-a)f_k}{af_k + g_k} \Bigg) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \prod_{j=1}^l \Gamma(\lambda_j) \prod_{j=1}^k \Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}} \int_{L_1} \dots \int_{L_{l+k}} \frac{\Gamma\left(\alpha + \sum_{j=1}^l h_j s_j + \sum_{j=1}^k s_{l+j}\right)}{\Gamma\left(\alpha + \beta + \sum_{j=1}^l h_j s_j + \sum_{j=1}^k s_{l+j}\right)} \prod_{j=1}^l \Gamma(\lambda_j + s_j) \prod_{j=1}^k \Gamma(-\sigma_j + s_{l+j})$$

$$\prod_{j=1}^{l+k} \Gamma(-s_j) z_1^{s_1} \dots z_l^{s_l} z_{l+1}^{s_{l+1}} \dots z_{l+k}^{s_{l+k}} ds_1 \dots ds_{l+k} \quad (2.3)$$

Here the contour $L_j s$ are defined by $L_j = L_{w\zeta_j \infty}(\operatorname{Re}(\zeta_j) = v_j'')$ starting at the point $v_j'' - \omega\infty$ and terminating at the point $v_j'' + \omega\infty$ with $v_j'' \in \mathbb{R}(j = 1, \dots, l)$ and each of the remaining contour L_{l+1}, \dots, L_{l+k} run from $-\omega\infty$ to $\omega\infty$

(2.2) can be easily established by expanding $\prod_{j=1}^l [1-\tau_j(t-a)^{h_j}]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} z^r (|z| < 1) \quad (2.4)$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [6, page 454].

3. Eulerian integral

In this section , we note :

$$\theta_i = \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j^{(i)}}, \zeta_j^{(i)} > 0 (i = 1, \dots, r); \theta'_i = \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j'^{(i)}}, \zeta_j'^{(i)} > 0 (i = 1, \dots, s)$$

$$\theta''_i = \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j''^{(i)}}, \zeta_j''^{(i)} > 0 (i = 1, \dots, u)$$

$$\theta'''_i = \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta_j'''^{(i)}}, \zeta_j'''^{(i)} > 0 (i = 1, \dots, v) \tag{3.1}$$

$$X = m_1, n_1; \dots; m_r, n_r; m'_1, n'_1; \dots; m'_s, n'_s; 1, 0; \dots; 1, 0; 1, 0; \dots; 1, 0 \tag{3.2}$$

$$Y = p_1, q_1; \dots; p_r, q_r; p'_1, q'_1; \dots; p'_s, q'_s; 0, 1; \dots; 0, 1; 0, 1; \dots; 0, 1 \tag{3.3}$$

$$A = (a_j; A_j^{(1)}, \dots, A_j^{(r)}, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0)_{1,p} \tag{3.4}$$

$$B = (b_j; B_j^{(1)}, \dots, B_j^{(r)}, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0)_{1,q} \tag{3.5}$$

$$A' = (a'_j; 0, \dots, 0, A_j'^{(1)}, \dots, A_j'^{(s)}, 0, \dots, 0, 0, \dots, 0)_{1,p'} \tag{3.6}$$

$$B' = (b'_j; 0, \dots, 0, B_j'^{(1)}, \dots, B_j'^{(s)}, 0, \dots, 0, 0, \dots, 0)_{1,q'} \tag{3.7}$$

$$C = (c_j^{(1)}, C_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)}, C_j^{(r)})_{1,p_r}; (c_j'^{(1)}, C_j'^{(1)})_{1,p'_1}; \dots; (c_j'^{(r)}, C_j'^{(s)})_{1,p'_s}$$

$$(1, 0); \dots; (1, 0); (1, 0); \dots; (1, 0) \tag{3.8}$$

$$D = (d_j^{(1)}, D_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)}, D_j^{(r)})_{1,q_r}; (d_j'^{(1)}, D_j'^{(1)})_{1,q'_1}; \dots; (d_j'^{(s)}, D_j'^{(s)})_{1,q'_s};$$

$$(0, 1); \dots; (0, 1); (0, 1); \dots; (0, 1) \tag{3.9}$$

$$K_1 = (1 - \alpha - \sum_{i=1}^u R_i a_i - \sum_{i=1}^v K_i a'_i; \mu_1, \dots, \mu_r, \mu'_1, \dots, \mu'_s, h_1, \dots, h_l, 1, \dots, 1) \tag{3.10}$$

$$K_2 = (1 - \beta - \sum_{i=1}^u R_i b_i - \sum_{i=1}^v K_i b'_i; \rho_1, \dots, \rho_r, \rho'_1, \dots, \rho'_s, 0, \dots, 0, 0, \dots, 0) \tag{3.11}$$

$$K_j = [1 - \lambda_j - \sum_{i=1}^u R_i \zeta_j''^{(i)} - \sum_{i=1}^v K_i \zeta_j'''^{(i)}; \zeta_j^{(1)}, \dots, \zeta_j^{(r)}, \zeta_j'^{(1)}, \dots, \zeta_j'^{(s)},$$

$$0, \dots, 1, \dots, 0, 0, \dots, 0]_{1,l} \tag{3.12}$$

$$K'_j = [1 + \sigma_j - \sum_{i=1}^u R_i \lambda_j''^{(i)} - \sum_{i=1}^v K_i \lambda_j'''^{(i)}; \lambda_j^{(1)}, \dots, \lambda_j^{(r)}, \lambda_j'^{(1)}, \dots, \lambda_j'^{(s)},$$

$$0, \dots, 0, 0, \dots, 1, \dots, 0]_{1,k} \tag{3.13}$$

$$L_1 = (1 - \alpha - \beta - \sum_{i=1}^u R_i(a_i + b_i) - \sum_{i=1}^v (a'_i + b'_i)K_i; \mu_1 + \rho_1, \dots, \mu_r + \rho_r, \mu'_1 + \rho'_1, \dots, \mu'_r + \rho'_r, h_1, \dots, h_l, 1, \dots, 1) \tag{3.14}$$

$$L_j = [1 - \lambda_j - \sum_{i=1}^u R_i \zeta_j^{(i)} - \sum_{i=1}^v \zeta_j^{(i)} K_i; \zeta_j^{(1)}, \dots, \zeta_j^{(r)}, \zeta_j^{(1)}, \dots, \zeta_j^{(s)}, 0, \dots, 0, 0, \dots, 0]_{1,l} \tag{3.15}$$

$$L'_j = [1 + \sigma_j - \sum_{i=1}^u R_i \lambda_j^{(i)} - \sum_{i=1}^v \lambda_j^{(i)} K_i; \lambda_j^{(1)}, \dots, \lambda_j^{(r)}, \lambda_j^{(1)}, \dots, \lambda_j^{(s)}, 0, \dots, 0, 0, \dots, 0]_{1,k} \tag{3.16}$$

$$P_1 = (b - a)^{\alpha + \beta - 1} \left\{ \prod_{j=1}^h (af_j + g_j)^{\sigma_j} \right\} \tag{3.17}$$

$$B_{u,v} = (b - a)^{\sum_{i=1}^v K_i(a'_i + b'_i) + \sum_{i=1}^u (a_i + b_i)R_i} \left\{ \prod_{j=1}^h (af_j + g_j)^{-\sum_{i=1}^v K_i \lambda_i^{(i)} - \sum_{i=1}^u \lambda_i^{(i)} R_i} \right\} \tag{3.18}$$

$$\mathfrak{A} = A, A'; \mathfrak{B} = B, B' \tag{3.19}$$

We have the following integral :

$$\int_a^b (t - a)^{\alpha - 1} (b - t)^{\beta - 1} \prod_{j=1}^l [1 - \tau_j(t - a)^{h_j}]^{-\lambda_j} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j} S_L^{h_1, \dots, h_u} \left(\begin{matrix} z_1'' \theta_1''(t - a)^{a_1} (b - t)^{b_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_u'' \theta_u''(t - a)^{a_u} (b - t)^{b_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(u)}} \end{matrix} \right) S_{N_1, \dots, N_v}^{\mathfrak{M}_1, \dots, \mathfrak{M}_v} \left(\begin{matrix} z_1''' \theta_1'''(t - a)^{a'_1} (b - t)^{b'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)'}} \\ \vdots \\ z_v''' \theta_v'''(t - a)^{a'_v} (b - t)^{b'_v} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(v)'}} \end{matrix} \right) A \left(\begin{matrix} z_1 \theta_1(t - a)^{\mu_1} (b - t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_r \theta_r(t - a)^{\mu_r} (b - t)^{\rho_r} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(r)}} \end{matrix} \right)$$

$$\begin{aligned}
 & A \begin{pmatrix} z'_1 \theta'_1 (t-a)^{\mu'_1} (b-t)^{\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z'_s \theta'_s (t-a)^{\mu'_s} (b-t)^{\rho'_s} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(s)}} \end{pmatrix} dt \\
 &= P_1 \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v] h_1 R_1 + \cdots + h_u R_u \leq L} \sum_{R_1, \dots, R_u=0} a_v b_u B_{u,v} \prod_{k=1}^u z''^{R_k} \prod_{k=1}^v z'''^{K_k} \\
 & A_{p+p'+l+k+2, q+q'+l+k+1; Y}^{m+m', n+n'+l+k+2; X} \left(\begin{array}{c|c} \frac{z_1 (b-a)^{\mu_1 + \rho_1}}{\prod_{j=1}^k (a f_j + g_j)^{\lambda_j^{(1)}}} & \mathfrak{A}, K_1, K_2, K_j, K'_j : C \\ \vdots & \vdots \\ \frac{z_r (b-a)^{\mu_r + \rho_r}}{\prod_{j=1}^k (a f_j + g_j)^{\lambda_j^{(r)}}} & \vdots \\ \frac{z'_1 (b-a)^{\mu'_1 + \rho'_1}}{\prod_{j=1}^k (a f_j + g_j)^{\lambda_j^{(1)'}}} & \vdots \\ \vdots & \vdots \\ \frac{z'_s (b-a)^{\mu'_s + \rho'_s}}{\prod_{j=1}^k (a f_j + g_j)^{\lambda_j^{(s)'}}} & \vdots \\ \tau_1 (b-a)^{h_1} & \vdots \\ \vdots & \vdots \\ \tau_l (b-a)^{h_l} & \vdots \\ \frac{(b-a) f_1}{a f_1 + g_1} & \vdots \\ \vdots & \vdots \\ \frac{(b-a) f_k}{a f_k + g_k} & \mathfrak{B}, L_1, L_j, L'_j : D \end{array} \right) \quad (3.20)
 \end{aligned}$$

where $\mathfrak{A}, \mathfrak{B}, C, D, X, K_1, K_2, K_j, K'_j, L_1, L_j, L'_j, P_1, P_u, B_u$ and \mathfrak{B}_1 are defined above.

Provided that

- (A) $a, b \in \mathbb{R} (a < b); \mu_i, \mu'_u, \rho_i, \rho'_u, \lambda_j^{(i)}, \lambda_j^{(u)}, h_v \in \mathbb{R}^+, f_i, g_j, \tau_v, \sigma_j, \lambda_v \in \mathbb{C} (i = 1, \dots, r; j = 1, \dots, k;$
- $u = 1, \dots, s; v = 1, \dots, l), a_i, b_i, \lambda_j^{(i)'}, \zeta_j^{(i)' } \in \mathbb{R}^+, (i = 1, \dots, u; j = 1, \dots, k)$
- $a'_i, b'_i, \lambda_j^{(i)''}, \zeta_j^{(i)'' } \in \mathbb{R}^+, (i = 1, \dots, v; j = 1, \dots, k)$

(B) $m, n, p, m_i, n_i, p_i, c_i \in \mathbb{N}^*; i = 1, \dots, r; a_j, b_j, c_j^{(i)}, d_j^{(i)}, A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C}$

$m', n', p', m'_i, n'_i, p'_i, c'_i \in \mathbb{N}^*; i = 1, \dots, r; a'_j, b'_j, c'_j^{(i)}, d'_j^{(i)}, A'_j^{(i)}, B'_j^{(i)}, C'_j^{(i)}, D'_j^{(i)} \in \mathbb{C}$

(C) $\max_{1 \leq j \leq k} \left\{ \left| \frac{(b-a)f_i}{af_i + g_i} \right| \right\} < 1$

(D) $Re \left[\alpha + \sum_{j=1}^r \mu_j \min_{1 \leq k \leq m_i} \frac{d_k^{(j)}}{D_k^{(j)}} + \sum_{j=1}^s \mu'_i \min_{1 \leq k \leq m'_i} \frac{d'_k{}^{(j)}}{D'_k{}^{(j)}} \right] > 0$

$Re \left[\beta + \sum_{j=1}^r \rho_j \min_{1 \leq k \leq m_i} \frac{D_k^{(j)}}{D_k^{(j)}} + \sum_{j=1}^s \rho'_j \min_{1 \leq k \leq m'_i} \frac{D'_k{}^{(j)}}{D'_k{}^{(j)}} \right] > 0$

(E) $Re \left(\alpha + \sum_{i=1}^v K_i a'_i + \sum_{i=1}^u R_i a_i + \sum_{i=1}^r \mu_i s_i + \sum_{i=1}^s t_i \mu'_i \right) > 0$

$Re \left(\beta + \sum_{i=1}^v K_i b'_i + \sum_{i=1}^u R_i b_i + \sum_{i=1}^r v_i s_i + \sum_{i=1}^s t_i \rho'_i \right) > 0$

$Re \left(\lambda_j + \sum_{i=1}^v K_i \lambda_j^{u(i)} + \sum_{i=1}^u R_i \lambda_j^{v(i)} + \sum_{i=1}^r s_i \zeta_j^{(i)} + \sum_{i=1}^s t_i \zeta_j'^{(i)} \right) > 0 (j = 1, \dots, l);$

$Re \left(-\sigma_j + \sum_{i=1}^v K_i \lambda_j^{m(i)} + \sum_{i=1}^u R_i \lambda_j^{n(i)} + \sum_{i=1}^r s_i \lambda_j^{(i)} + \sum_{i=1}^s t_i \lambda_j'^{(i)} \right) > 0 (j = 1, \dots, k)$

(F) $|arg(\Omega_i)z_k| < \frac{1}{2}\eta_i\pi, \xi^* = 0, \eta_i > 0$

$\Omega_i = \prod_{j=1}^p \{A_j^{(i)}\} A_j^{(i)} \prod_{j=1}^q \{B_j^{(i)}\}^{-B_j^{(i)}} \prod_{j=1}^{q_i} \{D_j^{(i)}\} D_j^{(i)} \prod_{j=1}^{p_i} \{C_j^{(i)}\}^{-C_j^{(i)}}; i = 1, \dots, r$

$\xi_i^* = Im \left(\sum_{j=1}^p A_j^{(i)} - \sum_{j=1}^q B_j^{(i)} + \sum_{j=1}^{q_i} D_j^{(i)} - \sum_{j=1}^{p_i} C_j^{(i)} \right); i = 1, \dots, r$

$\eta_i = Re \left(\sum_{j=1}^n A_j^{(i)} - \sum_{j=n+1}^p A_j^{(i)} + \sum_{j=1}^m B_j^{(i)} - \sum_{j=m+1}^q B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)} \right)$

$-\mu_i - \rho_i - \sum_{l=1}^k \lambda_j^{(i)} - \sum_{l=1}^l \zeta_j^{(i)} > 0; i = 1, \dots, r$

$|arg(\Omega'_i)z'_k| < \frac{1}{2}\eta'_i\pi, \xi'^* = 0, \eta'_i > 0$

$\Omega'_i = \prod_{j=1}^{p'} \{A'_j{}^{(i)}\} A'_j{}^{(i)} \prod_{j=1}^{q'} \{B'_j{}^{(i)}\}^{-B'_j{}^{(i)}} \prod_{j=1}^{q'_i} \{D'_j{}^{(i)}\} D'_j{}^{(i)} \prod_{j=1}^{p'_i} \{C'_j{}^{(i)}\}^{-C'_j{}^{(i)}}; i = 1, \dots, s$

$\xi_i'^* = Im \left(\sum_{j=1}^{p'} A'_j{}^{(i)} - \sum_{j=1}^{q'} B'_j{}^{(i)} + \sum_{j=1}^{q'_i} D'_j{}^{(i)} - \sum_{j=1}^{p'_i} C'_j{}^{(i)} \right); i = 1, \dots, s$

$\eta'_i = Re \left(\sum_{j=1}^{n'} A'_j{}^{(i)} - \sum_{j=n'+1}^{p'} A'_j{}^{(i)} + \sum_{j=1}^{m'} B'_j{}^{(i)} - \sum_{j=m'+1}^{q'} B'_j{}^{(i)} + \sum_{j=1}^{m'_i} D'_j{}^{(i)} - \sum_{j=m'_i+1}^{q'_i} D'_j{}^{(i)} + \sum_{j=1}^{n'_i} C'_j{}^{(i)} - \sum_{j=n'_i+1}^{p'_i} C'_j{}^{(i)} \right)$

$$-\mu'_i - \rho'_i - \sum_{l=1}^k \lambda'_j{}^{(i)} - \sum_{l=1}^l \zeta'_j{}^{(i)} > 0; i = 1, \dots, s$$

$$(H) \left| \arg \left(z_i \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\zeta'_j{}^{(i)}} \prod_{j=1}^k (f_j t + g_j)^{-\lambda'_j{}^{(i)}} \right) \right| < \frac{1}{2} \eta_i \pi \quad (a \leq t \leq b; i = 1, \dots, r)$$

$$\left| \arg \left(z'_i \prod_{j=1}^l [1 - \tau'_j(t-a)^{h'_i}]^{-\zeta'_j{}^{(i)}} \prod_{j=1}^k (f_j t + g_j)^{-\lambda'_j{}^{(i)}} \right) \right| < \frac{1}{2} \eta'_i \pi \quad (a \leq t \leq b; i = 1, \dots, s)$$

Proof

To prove (3.20), first expressing a class of multivariable polynomials $S_{N_1, \dots, N_v}^{\mathfrak{M}_1, \dots, \mathfrak{M}_v} [\cdot]$ defined by Srivastava [3] in serie with the help of (1.17), a class of multivariable polynomials $S_L^{h_1, \dots, h_u} [\cdot]$ defined by Srivastava et al [5] in serie with the help of (1.18) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the I-functions of r-variables and s-variables defined by Kumari et al [2] in terms of Mellin-Barnes type contour integral with the help of (1.2) and (1.10) respectively and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process . Now collect the power of $[1 - \tau_j(t-a)^{h_i}]$ with $(i = 1, \dots, r; j = 1, \dots, l)$ and collect the power of $(f_j t + g_j)$ with $j = 1, \dots, k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r + s + k + l)$ dimensional Mellin-Barnes integral in multivariable I-function defined by Kumari et al [2], we obtain the equation (3.20).

Remarks

If a) $\rho_1 = \dots, \rho_r = \rho'_1 = \dots, \rho'_s = 0$; b) $\mu_1 = \dots, \mu_r = \mu'_1 = \dots, \mu'_s = 0$, we obtain the similar formulas that (3.20) with the corresponding simplifications.

4. Particular cases

a)If $A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R}, m = 0$ and $A'_j{}^{(i)}, B'_j{}^{(i)}, C'_j{}^{(i)}, D'_j{}^{(i)} \in \mathbb{R}$ and $m' = 0$, the multivariable A-functions reduces to multivariable H-functions defined by Srivastava et al [7], we obtain the following result.

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^l [1 - \tau_j(t-a)^{h_i}]^{-\lambda_j} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$$

$$S_L^{h_1, \dots, h_u} \left(\begin{matrix} z_1'' \theta_1'' (t-a)^{a_1} (b-t)^{b_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j''(1)} \\ \vdots \\ z_u'' \theta_u'' (t-a)^{a_u} (b-t)^{b_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j''(u)} \end{matrix} \right)$$

$$S_{N_1, \dots, N_v}^{\mathfrak{M}_1, \dots, \mathfrak{M}_v} \left(\begin{matrix} z_1''' \theta_1''' (t-a)^{a'_1} (b-t)^{b'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j'''(1)} \\ \vdots \\ z_v''' \theta_v''' (t-a)^{a'_v} (b-t)^{b'_v} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j'''(v)} \end{matrix} \right)$$

$$H \begin{pmatrix} z_1 \theta_1 (t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_r \theta_r (t-a)^{\mu_r} (b-t)^{\rho_r} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(r)}} \end{pmatrix}$$

$$H \begin{pmatrix} z'_1 \theta'_1 (t-a)^{\mu'_1} (b-t)^{\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j'^{(1)}} \\ \vdots \\ z'_s \theta'_s (t-a)^{\mu'_s} (b-t)^{\rho'_s} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j'^{(s)}} \end{pmatrix} dt$$

$$= P_1 \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \sum_{R_1, \dots, R_u=0}^{h_1 R_1 + \dots + h_u R_u \leq L} a_v b_u B_{u,v} \prod_{k=1}^u z^{u R_k} \prod_{k=1}^v z^{v K_k}$$

$$H_{p+p'+l+k+2, q+q'+l+k+1; Y}^{m+m', n+n'+l+k+2; X} \left(\begin{array}{c} \frac{z_1 (b-a)^{\mu_1 + \rho_1}}{\prod_{j=1}^k (a f_j + g_j)^{\lambda_j^{(1)}}} \\ \dots \\ \frac{z_r (b-a)^{\mu_r + \rho_r}}{\prod_{j=1}^k (a f_j + g_j)^{\lambda_j^{(r)}}} \\ \frac{z'_1 (b-a)^{\mu'_1 + \rho'_1}}{\prod_{j=1}^k (a f_j + g_j)^{\lambda_j'^{(1)}}} \\ \dots \\ \frac{z'_s (b-a)^{\mu'_s + \rho'_s}}{\prod_{j=1}^k (a f_j + g_j)^{\lambda_j'^{(s)}}} \\ \tau_1 (b-a)^{h_1} \\ \dots \\ \tau_l (b-a)^{h_l} \\ \frac{(b-a) f_1}{a f_1 + g_1} \\ \dots \\ \frac{(b-a) f_k}{a f_k + g_k} \end{array} \middle| \begin{array}{l} \mathfrak{A}, K_1, K_2, K_j, K'_j : C \\ \vdots \\ \vdots \\ \mathfrak{B}, L_1, L_j, L'_j : D \end{array} \right) \quad (4.1)$$

under the same notations and validity conditions that (3.21) with $A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R}, m = 0$ and $A_j^{\prime(i)}, B_j^{\prime(i)}, C_j^{\prime(i)}, D_j^{\prime(i)} \in \mathbb{R}$ and $m' = 0$

$$b) \text{ If } B(L; R_1, \dots, R_u) = \frac{\prod_{j=1}^{\bar{A}} (a_j)_{R_1 \theta'_j + \dots + R_u \theta_j^{(u)}} \prod_{j=1}^{B'} (b'_j)_{R_1 \phi'_j} \dots \prod_{j=1}^{B^{(u)}} (b_j^{(u)})_{R_u \phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}} (c_j)_{R_1 \psi'_j + \dots + R_u \psi_j^{(u)}} \prod_{j=1}^{D'} (d'_j)_{R_1 \delta'_j} \dots \prod_{j=1}^{D^{(u)}} (d_j^{(u)})_{R_u \delta_j^{(u)}}} \quad (4.2)$$

then the general class of multivariable polynomial $S_L^{h_1, \dots, h_u} [z_1, \dots, z_u]$ reduces to generalized Lauricella function defined by Srivastava et al [4]. We have

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^l [1 - \tau_j (t-a)^{h_j}]^{-\lambda_j} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j}$$

$$F_{\bar{C}; D'; \dots; D^{(u)}}^{1+\bar{A}; B'; \dots; B^{(u)}} \left(\begin{matrix} z_1' \theta_1'' (t-a)^{a_1} (b-t)^{b_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_u' \theta_u'' (t-a)^{a_u} (b-t)^{b_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(u)}} \end{matrix} \right)$$

$$\left(\begin{matrix} [(-L); R_1, \dots, R_u] [(a); \theta', \dots, \theta^{(u)}] : [(b'); \phi']; \dots; [(b^{(u)}); \phi^{(u)}] \\ [(c); \psi', \dots, \psi^{(u)}] : [(d'); \delta']; \dots; [(d^{(u)}); \delta^{(u)}] \end{matrix} \right)$$

$$S_{N_1, \dots, N_v}^{m_1, \dots, m_v} \left(\begin{matrix} z_1''' \theta_1''' (t-a)^{a'_1} (b-t)^{b'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)'}} \\ \vdots \\ z_v''' \theta_v''' (t-a)^{a'_v} (b-t)^{b'_v} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(v)'}} \end{matrix} \right)$$

$$A \left(\begin{matrix} z_1 \theta_1 (t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \vdots \\ z_r \theta_r (t-a)^{\mu_r} (b-t)^{\rho_r} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(r)}} \end{matrix} \right)$$

$$A \left(\begin{matrix} z_1' \theta_1' (t-a)^{\mu'_1} (b-t)^{\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)'}} \\ \vdots \\ z_s' \theta_s' (t-a)^{\mu'_s} (b-t)^{\rho'_s} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(s)'}} \end{matrix} \right) dt$$

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