On general Eulerian integral of certain products of multivariable A-function, the multivariable I-function and a class of polynomials

$F.Y. \ AYANT^1$

1 Teacher in High School , France

ABSTRACT
The object of this paper is to establish an general Eulerian integral involving the product of the multivariable A-function, the multivariable I-function defined by Nambisan et al [3], a general class of multivariable polynomials and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular case concerning the multivariable H-function defined by Srivastava et al [9] and the Srivastava-Daoust polynomial [7].

Keywords: Eulerian integral, multivariable I-function, Lauricella function of several variables, multivariable A-function, generalized hypergeometric function, multivariable H-function, Srivastava-Daoust polynomial

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1. Introduction

In this paper, we evaluate a general Eulerian integral concerning the product the multivariable I-functions defined by Nambisan et al [3],the multivariable A-function defined by Gautam [2], a generalized hypergeometric function and a class of multivariable polynomials. We will give a serie expansion of a multivariable A-function. The multivariable A-function is an extension of the multivariable H-function defined by Srivastava et al [9]. We will given a contracted form.
The serie representation of the multivariable A-function is given by Gautam [2] as

$$A[u_1,\cdots,u_v]=A^{0,\lambda:(\alpha',\beta');\cdots;(\alpha^{(v)},\beta^{(v)})}_{A,C:(M',N');\cdots;(M^{(v)},N^{(v)})}\left(\begin{array}{c}\mathrm{u}_1\\ \cdot\\ \cdot\\ \mathrm{u}_v\end{array}\middle|\begin{array}{c}[(\mathrm{g}_j);\gamma',\cdots,\gamma^{(v)}]_{1,A}:\\ \cdots\\ [(\mathrm{f}_j);\xi',\cdots,\xi^{(v)}]_{1,C}:\end{array}\right.$$

$$\left.\begin{array}{c}(\mathrm{q}^{(1)},\eta^{(1)})_{1,M^{(1)}};\cdots;(q^{(v)},\eta^{(v)})_{1,M^{(v)}}\\ \cdots\\ \cdots\\ (\mathrm{p}^{(1)},\epsilon^{(1)})_{1,N^{(1)}};\cdots;(p^{(v)},\epsilon^{(v)})_{1,N^{(v)}}\end{array}\right)=\sum_{G_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{v}\phi_i u_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon_{G_i}^{(i)}g_i!} \tag{1.1}$$

where

$$\phi_1=\frac{\prod_{j=1}^{\lambda}\Gamma\left(1-g_j+\sum_{i=1}^{v}\gamma_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\lambda'+1}^{A}\Gamma\left(g_j-\sum_{i=1}^{v}\gamma_j^{(i)}U_i\right)\prod_{j=1}^{C}\Gamma\left(1-f_j+\sum_{i=1}^{v}\xi_j^{(i)}\eta_{G_i,g_i}\right)} \tag{1.2}$$

$$\phi_i=\frac{\prod_{j=1,j\neq m_i}^{\alpha^{(i)}}\Gamma\left(p_j^{(i)}-\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=1}^{\beta^{(i)}}\Gamma\left(1-q_j^{(i)}+\eta_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\alpha^{(i)}+1}^{N^{(i)}}\Gamma\left(1-p_j^{(i)}+\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=\beta^{(i)}+1}^{M^{(i)}}\Gamma\left(q_j^{(i)}-\eta_j^{(i)}\eta_{G_i,g_i}\right)},i=1,\cdots,v \tag{1.3}$$

and $\eta_{G_i,g_i}=\dfrac{p_{G_i}^{(i)}+g_i}{\epsilon_{G_i}^{(i)}},i=1,\cdots,v$ (1.4)

which is valid under the following conditions : $\epsilon_{m_i}^{(i)}[p_j^{(i)}+p_i']\neq\epsilon_j^{(i)}[p_{m_i}+g_i]$ (1.5) and

$$u_i \neq 0, \sum_{j=1}^{A} \gamma_j^{(i)} - \sum_{j=1}^{C} \xi_j^{(i)} + \sum_{j=1}^{M^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{N^{(i)}} \epsilon_j^{(i)} < 0, i = 1, \cdots, v \quad (1.6)$$

Here $\lambda, A, C, \alpha_i, \beta_i, m_i, n_i \in \mathbb{N}^*; i = 1, \cdots, v; f_j, g_j, p_j^{(i)}, q_j^{(i)}, \gamma_j^{(i)}, \xi_j^{(i)}, \eta_j^{(i)}, \epsilon_j^{(i)} \in \mathbb{C}$

The multivariable I-function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1, \cdots, z_r) = I_{p,q:p'_1,q'_1;\cdots;p'_r,q'_r}^{0,n:m'_1,n'_1;\cdots;m'_r,n'_r} \left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}; A_j)_{1,p} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j; \beta_j^{(1)}, \cdots, \beta_j^{(r)}; B_j)_{1,q} : \end{array} \right.$$

$$\left. \begin{array}{c} (\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \cdots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d'}_j^{(1)}, \delta_j'^{(1)}; D_j^{(1)})_{1,q_1}; \cdots; (d_j'^{(r)}, \delta_j'^{(r)}; D_j^{(r)})_{1,q_r} \end{array} \right) \quad (1.7)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \phi(s_1, \cdots, s_r) \prod_{i=1}^{r} \theta_i(s_i) z_i^{s_i} \mathrm{d}s_1 \cdots \mathrm{d}s_r \quad (1.8)$$

where $\phi(s_1, \cdots, s_r), \theta_i(s_i), i = 1, \cdots, r$ are given by :

$$\phi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{n} \Gamma^{A_j} \left(1 - aj + \sum_{i=1}^{r} \alpha_j^{(i)} s_j\right)}{\prod_{j=n+1}^{p} \Gamma^{A_j} \left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} s_j\right) \prod_{j=1}^{q} \Gamma^{B_j} \left(1 - bj + \sum_{i=1}^{r} \beta_j^{(i)} s_j\right)} \quad (1.9)$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n'_i} \Gamma^{C_j^{(i)}} \left(1 - c_j^{(i)} + \gamma_j^{(i)} s_i\right) \prod_{j=1}^{m'_i} \Gamma^{D_j^{(i)}} \left(d_j'^{(i)} - \delta_j'^{(i)} s_i\right)}{\prod_{j=n'_i+1}^{p'_i} \Gamma^{C_j^{(i)}} \left(c_j^{(i)} - \gamma_j^{(i)} s_i\right) \prod_{j=m'_i+1}^{q'_i} \Gamma^{D_j^{(i)}} \left(1 - d_j^{(i)} + \delta_j^{(i)} s_i\right)} \quad (1.10)$$

For more details, see Nambisan et al [3].

Following the result of Braaksma [1] the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^{p} A_j \alpha_j^{(i)} - \sum_{j=1}^{q} B_j \beta_j^{(i)} + \sum_{j=1}^{p'_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q'_i} D_j^{(i)} \delta_j'^{(i)} \leqslant 0, i = 1, \cdots, r \quad (1.11)$$

The integral (2.1) converges absolutely if

$|arg(z_k)| < \frac{1}{2} \Delta_k \pi, k = 1, \cdots, r$ where

$$\Delta_k = -\sum_{j=n+1}^{p} A_j \alpha_j^{(k)} - \sum_{j=1}^{q} B_j \beta_j^{(k)} + \sum_{j=1}^{m'_k} D_j^{(k)} \delta_j'^{(k)} - \sum_{j=m'_k+1}^{q_k} D_j^{(k)} \delta_j'^{(k)} + \sum_{j=1}^{n'_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n'_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)} > 0 \quad (1.12)$$

Srivastava and Garg [7] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]=\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)\frac{z_1^{R_1}\cdots z_u^{R_u}}{R_1!\cdots R_u!} \tag{1.13}$$

The coefficients are $B[E;R_1,\ldots,R_v]$ arbitrary constants, real or complex.

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [8 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j),j=1,\cdots,r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j),j=1,\cdots,r$

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t\;=(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right) \tag{2.2}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i,\tau_j,h_j\in\mathbb{C},\lambda_j\in\mathbb{R}^+(i=1,\cdots,k;j=1,\cdots,l)$

$$min(Re(\alpha),Re(\beta))>0,\ \max_{1\leqslant j\leqslant l}\left\{\left|\tau_j(b-a)^{h_j}\right|\right\}<1,\ \max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1,$$

and $F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[6,page 454] given by :

$$F_{1:0,\cdots,0;0,\cdots,0}^{1:1,\cdots,1;1\cdots,1}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots\\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\Bigg)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\prod_{j=1}^{l}\Gamma(\lambda_j)\prod_{j=1}^{k}\Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}}\int_{L_1}\cdots\int_{L_{l+k}}\frac{\Gamma\left(\alpha+\sum_{j=1}^{l}h_jw_j+\sum_{j=1}^{k}w_{l+j}\right)}{\Gamma\left(\alpha+\beta+\sum_{j=1}^{l}h_jw_j+\sum_{j=1}^{k}w_{l+j}\right)}\prod_{j=1}^{l}\Gamma(\lambda_j+w_j)\prod_{j=1}^{k}\Gamma(-\sigma_j+w_{l+j})$$

$$\prod_{j=1}^{l+k}\Gamma(-w_j)z_1^{w_1}\cdots z_l^{w_l}z_{l+1}^{w_{l+1}}\cdots,w_{l+k}^{w_{l+k}}\,\mathrm{d}w_1\cdots\mathrm{d}w_{l+k} \tag{2.3}$$

Here the contour $L_j's$ are defined by $L_j=L_{w\zeta_j\infty}(Re(\zeta_j)=v_j'')$ starting at the point $v_j''-\omega\infty$ and terminating at the point $v_j''+\omega\infty$ with $v_j''\in\mathbb{R}(j=1,\cdots,l)$ and each of the remaining contour $L_{l+1},\cdots,L_{l+k}$ run from $-\omega\infty$ to $\omega\infty$

(2.2) can be easily established by expanding $\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [4, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [6, page 454].

3. Eulerian integral

In this section , we evaluate a general Eulerian integral with the product of the multivariable Aleph-function, the multivariable I-function defined by Nambisan et al [3], a class of multivariable polynomials and generalized hypergeometric function. We note

$$B_u=\frac{(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!} \tag{3.1}$$

and $$B_{u,v}=(b-a)^{\sum_{i=1}^{v}(a_i'+b_i')\eta_{G_i,g_i}+\sum_{i=1}^{u}(a_i+b_i)R_i}\left\{\prod_{j=1}^{h}(af_j+g_j)^{-\sum_{i=1}^{v}\lambda_i'''\eta_{G_i,g_i}-\sum_{i=1}^{u}\lambda_i''R_i}\right\}G_v \tag{3.2}$$

where $$G_v=\phi_1\frac{\prod_{i=1}^{v}\phi_iu_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon_{G_i}^{(i)}g_i!} \tag{3.3}$$

ϕ_1,ϕ_i for $i=1,\cdots,v$ are defined respectively by (1.2) and (1.3)

$$\theta_i=\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}},\zeta_j^{(i)}>0(i=1,\cdots,r)\,;\,\theta_i'=\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'^{(i)}},\zeta_j'^{(i)}>0(i=1,\cdots,s)$$

$$\theta_i''=\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j''^{(i)}},\zeta_j''^{(i)}>0(i=1,\cdots,u)$$

$$\theta_i'''=\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j'''^{(i)}},\zeta_j'''^{(i)}>0(i=1,\cdots,v) \tag{3.4}$$

$$X=m_1',n_1';\cdots;m_r',n_r';1,0;\cdots;1,0;1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{3.5}$$

$$Y = p'_1, q'_1; \cdots ; p'_r, q'_r; 0,1; \cdots ; 0,1; 0,1; \cdots ; 0,1; 0,1; \cdots ; 0,1 \tag{3.6}$$

$$A = (a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0; A_j)_{1,p} \tag{3.7}$$

$$B = (b_j; \beta_j^{(1)}, \cdots, \beta_j^{(r)}, 0 \cdots, 0, 0 \cdots, 0, 0 \cdots, 0; B_j)_{1,q} \tag{3.8}$$

$$\mathrm{C} = (\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \cdots ; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r}; (1,0;1) \cdots ; (1,0;1)$$

$$(1,0;1); \cdots ; (1,0;1); (1,0;1); \cdots ; (1,0;1) \tag{3.9}$$

$$D = (\mathrm{d}'^{(1)}_j, \delta'^{(1)}_j; D_j^{(1)})_{1,q_1}; \cdots ; (d'^{(r)}_j, \delta'^{(r)}_j; D_j^{(r)})_{1,q_r}; (0,1;1); \cdots ; (0,1;1);$$

$$(0,1;1); \cdots ; (0,1;1); (0,1;1); \cdots ; (0,1;1) \tag{3.10}$$

$$K_1 = (1 - \alpha - \sum_{i=1}^{u} R_i a_i - \sum_{i=1}^{v} \eta_{G_i,g_i} a'_i; \mu_1, \cdots, \mu_r, \mu'_1, \cdots, \mu'_s, h_1, \cdots, h_l, 1, \cdots, 1; 1) \tag{3.11}$$

$$K_2 = (1 - \beta - \sum_{i=1}^{u} R_i b_i - \sum_{i=1}^{v} \eta_{G_i,g_i} b'_i; \rho_1, \cdots, \rho_r, \rho'_1, \cdots, \rho'_s, 0, \cdots, 0, 0 \cdots, 0; 1) \tag{3.12}$$

$$K_P = [1 - A_j; 0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0, 0, \cdots, 0; 1]_{1,P} \tag{3.13}$$

$$K_j = [1 - \lambda_j - \sum_{i=1}^{u} R_i \zeta_j''^{(i)} - \sum_{i=1}^{v} \eta_{G_i,g_i} \zeta_j'''^{(i)}; \zeta_j^{(1)}, \cdots, \zeta_j^{(r)}, \zeta_j'^{(1)} \cdots, \zeta_j'^{(s)},$$

$$0, \cdots, \underset{\mathrm{j}}{1}, \cdots, 0, 0 \cdots, 0; 1]_{1,l} \tag{3.14}$$

$$K'_j = [1 + \sigma_j - \sum_{i=1}^{u} R_i \lambda_j''^{(i)} - \sum_{i=1}^{v} \eta_{G_i,g_i} \lambda_j'''^{(i)}; \lambda_j^{(1)}, \cdots, \lambda_j^{(r)}, \lambda_j'^{(1)} \cdots, \lambda_j'^{(s)},$$

$$0, \cdots, 0, 0 \cdots, \underset{\mathrm{j}}{1}, \cdots, 0; 1]_{1,k} \tag{3.15}$$

$$L_1 = (1 - \alpha - \beta - \sum_{i=1}^{u} R_i (a_i + b_i) - \sum_{i=1}^{v} (a'_i + b'_i) \eta_{G_i,g_i}; \mu_1 + \rho_1, \cdots, \mu_r + \rho_r, \mu'_1 + \rho'_1, \cdots, \mu'_r + \rho'_r,$$

$$h_1, \cdots, h_l, 1, \cdots, 1; 1) \tag{3.16}$$

$$L_Q = [1 - B_j; 0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0, 0 \cdots, 0; 1]_{1,Q} \tag{3.17}$$

$$L_j = [1 - \lambda_j - \sum_{i=1}^{u} R_i \zeta_j''^{(i)} - \sum_{i=1}^{s} \zeta_j'''^{(i)} \eta_{G_i,g_i}; \zeta_j^{(1)}, \cdots, \zeta_j^{(r)}, \zeta_j'^{(1)} \cdots, \zeta_j'^{(s)}, 0, \cdots, 0, 0 \cdots, 0; 1]_{1,l} \tag{3.18}$$

$$L'_j = [1+\sigma_j - \sum_{i=1}^{u} R_i \lambda_j''^{(i)} - \sum_{i=1}^{v} \lambda_j'''^{(i)} \eta_{G_i,g_i}; \lambda_j^{(1)}, \cdots, \lambda_j^{(r)}, \lambda_j'^{(1)} \cdots, \lambda_j'^{(s)}, 0, \cdots, 0, 0, \cdots, 0; 1]_{1,k} \quad (3.19)$$

We have the general Eulerian integral

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{l} \left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j} \prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u} \left(\begin{array}{c} \mathrm{z}_1''\theta_1''(t-a)^{a_1}(b-t)^{b_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_u''\theta_u''(t-a)^{a_u}(b-t)^{b_u} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j''^{(u)}} \end{array} \right)$$

$$A \left(\begin{array}{c} \mathrm{z}_1'''\theta_1'''(t-a)^{a_1'}(b-t)^{b_1'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_v'''\theta_v'''(t-a)^{a_v'}(b-t)^{b_v'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'''^{(v)}} \end{array} \right)$$

$$I \left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(r)}} \end{array} \right)$$

$${}_PF_Q \left[(A_P);(B_Q); -\sum_{i=1}^{s} z_i'\theta_i'(t-a)^{\mu_i'}(b-t)^{\rho_i'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(i)}} \right] \mathrm{d}t =$$

$$(b-a)^{\alpha+\beta-1} \frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)} \prod_{j=1}^{} (af_j+g_j)^{\sigma_j} \sum_{g_1,\cdots,g_v=0}^{\infty} \sum_{G_1=0}^{\alpha^{(1)}} \cdots \sum_{G_v=0}^{\alpha^{(v)}} \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} \prod_{i=1}^{v} z_i'''^{\eta_{h_i,k_i}} \prod_{k=1}^{u} z''^{K_k} B_u B_{u,v}$$

$$I^{0,n+P+l+k+2;X}_{p+P+l+k+2,q+Q+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\mathrm{K}_1,K_2,K_3,K_j,K'_j : C \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(s)}_j}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \mathrm{B}\ ,\mathrm{L}_1,L_2,L_j,L'_j : D \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (3.20)$$

This result is an extansion the formula given by Saxena et al [4]. Provided that

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\mu'_u,\rho_i,\rho'_u,\lambda_j^{(i)},\lambda_j^{\prime(u)},h_v\in\mathbb{R}^+,f_i,g_j,\tau_v,\sigma_j,\lambda_v\in\mathbb{C}\ (i=1,\cdots,r;j=1,\cdots;k;$

$u=1,\cdots,s;v=1,\cdots,l),a_i,b_i,\lambda_j^{\prime\prime(i)},\zeta_j^{\prime\prime(i)}\in\mathbb{R}^+,(i=1,\cdots,u;j=1,\cdots,k)$

$a'_i,b'_i,\lambda_j^{\prime\prime\prime(i)},\zeta_j^{\prime\prime\prime(i)}\in\mathbb{R}^+,(i=1,\cdots,v;j=1,\cdots,k)$

(B) $m'_j,n'_j,p'_j,q'_j(j=1,\cdots,r),n,p,q\in\mathbb{N}^*$ **;**$\delta_j^{\prime(i)}\in\mathbb{R}_+(j=1,\cdots,q_i;i=1,\cdots,r)$

$\alpha_j^{(i)}\in\mathbb{R}_+(j=1,\cdots,p;i=1,\cdots,r),\beta_j^{(i)}\in\mathbb{R}_+(j=1,\cdots,q;i=1,\cdots,r),\gamma_j^{(i)}\in\mathbb{R}_+(j=1,\cdots,p_i;i=1,\cdots,r)$

$a_j(j=1,\cdots,p),b_j(j=1,\cdots,q),c_j^{(i)}(j=1,\cdots,p_i,i=1,\cdots,r),d_j^{\prime(i)}(j=1,\cdots,q_i,i=1,\cdots,r)\in\mathbb{C}$

The exposants $A_j(j=1,\cdots,p),B_j(j=1,\cdots,q),C_j^{(i)}(j=1,\cdots,p_i;i=1,\cdots,r),D_j^{(i)}(j=1,\cdots,q_i;i=1,\cdots,r)$ of various gamma function involved in (1.3) and (1.4) may take non integer values.

(C) $\max\limits_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$

(D) $Re\left[\alpha+\sum_{i=1}^{v}a'_i\min\limits_{1\leqslant j\leqslant\alpha^{(i)}}\frac{p_j^{(i)}}{\epsilon_j^{(i)}}+\sum_{i=1}^{r}\mu_i\min\limits_{1\leqslant j\leqslant m'_i}\frac{d_j^{\prime(i)}}{\delta_j^{\prime(i)}}\right]>0$ and

$$Re[\beta + \sum_{i=1}^{v} b'_i \min_{1 \leqslant j \leqslant \alpha^{(i)}} \frac{p_j^{(i)}}{\epsilon_j^{(i)}} + \sum_{i=1}^{r} \rho_i \min_{1 \leqslant j \leqslant m'_i} \frac{d_j'^{(i)}}{\delta_j'^{(i)}}] > 0$$

(E) $Re\left(\alpha + \sum_{i=1}^{v} \eta_{G_i,g_i} a'_i + \sum_{i=1}^{u} R_i a_i + \sum_{i=1}^{r} \mu_i s_i + \sum_{i=1}^{l} h_i w_i\right) > 0 \,; Re\left(\beta + \sum_{i=1}^{v} \eta_{G_i,g_i} b'_i + \sum_{i=1}^{u} R_i b_i + \sum_{i=1}^{r} \rho_i s_i\right) > 0$

$$Re\left(\lambda_j + \sum_{i=1}^{v} \eta_{G_i,g_i} \lambda_j'''^{(i)} + \sum_{i=1}^{u} R_i \lambda_j''^{(i)} + \sum_{i=1}^{r} s_i \zeta_j^{(i)}\right) > 0 (j = 1, \cdots, l)$$

$$Re\left(-\sigma_j + \sum_{i=1}^{v} \eta_{G_i,g_i} \lambda'''^{(i)} + \sum_{i=1}^{u} R_i \lambda_j''^{(i)} + \sum_{i=1}^{r} s_i \lambda_j^{(i)}\right) > 0 (j = 1, \cdots, k)$$

(F) $U_i = \sum_{j=1}^{p} A_j \alpha_j^{(i)} - \sum_{j=1}^{q} B_j \beta_j^{(i)} + \sum_{j=1}^{p'_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q'_i} D_j^{(i)} \delta_j^{(i)} \leqslant 0, i = 1, \cdots, r$

(G) $\Delta_k = -\sum_{j=n+1}^{p} A_j \alpha_j^{(k)} - \sum_{j=1}^{q} B_j \beta_j^{(k)} + \sum_{j=1}^{m'_k} D_j^{(k)} \delta_j'^{(k)} - \sum_{j=m_k+1}^{q'_k} D_j^{(k)} \delta_j'^{(k)} + \sum_{j=1}^{n'_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p'_k} C_j^{(k)} \gamma_j^{(k)}$

$$-\mu_i - \rho_i - \sum_{l=1}^{k} \lambda_l^{(i)} > 0 \quad (i = 1, \cdots, r)$$

(H) $\left| arg\left(z_i \prod_{j=1}^{l} \left[1 - \tau_j (t-a)^{h_i}\right]^{-\zeta_j^{(i)}} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(i)}}\right)\right| < \frac{1}{2} \Delta_i \pi \;\; (a \leqslant t \leqslant b; i = 1, \cdots, r)$

(I) $P \leqslant Q + 1$. The equality holds, when , in addition,

either $P > Q$ and $\left| \left(z'_i \sum_{i=1}^{l} \left[1 - \tau_j (t-a)^{h_i}\right]^{-\zeta_j'^{(i)}} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(i)}}\right) \right|^{\frac{1}{Q-P}} < 1 \; (a \leqslant t \leqslant b)$

or $P \leqslant Q$ and $\max_{1 \leqslant i \leqslant k} \left[\left| \left(z'_i \sum_{j=1}^{l} \left[1 - \tau_j (t-a)^{h_i}\right]^{-\zeta_j'^{(i)}} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(i)}}\right) \right| \right] < 1 \;\; (a \leqslant t \leqslant b)$

(J) The multiple series occuring on the right-hand side of (3.20) is absolutely and uniformly convergent.

Proof

To prove (3.20), first, we express in serie the multivariable A-function with the help of (1.1), a class of multivariable polynomials defined by Srivastava et al [7] $S_L^{h_1,\cdots,h_u}[.]$ in serie with the help of (1.13) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the I-functions of r-variables defined by Nambisan et al [3] and in terms of Mellin-Barnes type contour integral with the help of (1.8), the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process..Now collect the power of $\left[1 - \tau_j (t-a)^{h_i}\right]$ with $(i = 1, \cdots, r; j = 1, \cdots, l)$ and collect the power of $(f_j t + g_j)$ with $j = 1, \cdots, k$. Use the equations (2.2) and (2.3) and express the result in Mellin-Barnes contour integral. Interpreting the $(r + s + k + l)$ dimensional Mellin-Barnes integral to multivariable I-function, we obtain the equation (3.20).

4. Particular cases

a) If $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$, the multivariable I-function defined by Nambisan et al [3] reduces to multivariable H-function defined by Srivastava et al [9]. We have the following result.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c} z_1''\theta_1''(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j''^{(1)}} \\ . \\ . \\ . \\ z_u''\theta_u''(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j''^{(u)}} \end{array}\right)$$

$$A\left(\begin{array}{c} z_1'''\theta_1'''(t-a)^{a_1'}(b-t)^{b_1'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'''^{(1)}} \\ . \\ . \\ . \\ z_v'''\theta_v'''(t-a)^{a_v'}(b-t)^{b_v'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'''^{(v)}} \end{array}\right) H\left(\begin{array}{c} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$=(b-a)^{\alpha+\beta-1}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}(af_j+g_j)^{\sigma_j}\sum_{g_1,\cdots,g_v=0}^{\infty}\sum_{G_1=0}^{\alpha^{(1)}}\cdots\sum_{G_v=0}^{\alpha^{(v)}}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u}z''^{K_k}B_uB_{u,v}$$

$$H^{0,n+P+l+k+2;X}_{p+P+l+k+2,q+Q+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_3,K_j,K_j':C \\ \cdots & . \\ \cdots & . \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & . \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(s)}}} & . \\ \tau_1(b-a)^{h_1} & . \\ \cdots & . \\ \cdots & . \\ \tau_l(b-a)^{h_l} & . \\ \frac{(b-a)f_1}{af_1+g_1} & . \\ \cdots & \mathrm{B}\ ,\ \mathrm{L}_1,L_2,L_j,L_j':D \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad (4.1)$$

under the same notations and conditions that (3.20) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$:

b) If $B(L;R_1,\cdots,R_u) = \dfrac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(r)}}(b_j^{(r)})_{R_r\phi_j^{(r)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{R_1\psi'_j+\cdots+R_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{R_r\delta_j^{(r)}}}$ (4.2)

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]$ reduces to generalized Lauricella function defined by Srivastava et al [6]. We have

$$F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c|c} \mathrm{z}_1 & [(\mathrm{a});\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{(u)});\phi^{(u)}] \\ \cdot & \cdot \\ \cdot & [(\mathrm{c});\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)});\delta^{(u)}] \\ \mathrm{z}_u & \end{array}\right) \tag{4.3}$$

and we have the two following formulas

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c} \mathrm{z}''_1\theta''_1(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}''_u\theta''_u(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}} \end{array}\right)$$

$$A\left(\begin{array}{c} \mathrm{z}'''_1\theta'''_1(t-a)^{a'_1}(b-t)^{b'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}'''_v\theta'''_v(t-a)^{a'_v}(b-t)^{b'_v}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(v)}} \end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z}_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{s}z'_i\theta'_i(t-a)^{\mu'_i}(b-t)^{\rho'_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right]\mathrm{d}t=$$

$$(b-a)^{\alpha+\beta-1}\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}(af_j+g_j)^{\sigma_j}\sum_{g_1,\cdots,g_v=0}^{\infty}\sum_{G_1=0}^{\alpha^{(1)}}\cdots\sum_{G_v=0}^{\alpha^{(v)}}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u}z''^{K_k}B_u'B_{u,v}$$

$$I_{p+P+l+k+2,q+Q+l+k+1;Y}^{0,n+P+l+k+2;X}\left(\begin{array}{c|c}\frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{K}_1,K_2,K_3,K_j,K_j':C\\ \cdots & \cdot\\ \cdots & \cdot\\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot\\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot\\ \cdots & \cdot\\ \cdots & \cdot\\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot\\ \tau_1(b-a)^{h_1} & \cdot\\ \cdots & \cdot\\ \cdots & \cdot\\ \tau_l(b-a)^{h_l} & \cdot\\ \frac{(b-a)f_1}{af_1+g_1} & \cdot\\ \cdots & \mathrm{B}\ ,\ \mathrm{L}_1,L_2,L_j,L_j':D\\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad \textbf{(4.4)}$$

under the same notations and conditions that (3.20)

where $B_u'=\dfrac{(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!}$, $B[E;R_1,\ldots,R_v]$ is defined by (4.2)

Remark:

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions and a class of multivariable polynomials defined by Srivastava et al [6].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of the multivariable A-function, the multivariable I-function defined by Nambisan et al [2], a class of multivariable polynomials and generalized hypergeometric function with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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Personal adress : 411 Avenue Joseph Raynaud
Le parc Fleuri , Bat B
83140 , Six-Fours les plages
Tel : 06-83-12-49-68
Department : VAR
Country : FRANCE