On general Eulerian integral of certain products of I-functions and multivariable A-function

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ABSTRACT
The object of this paper is to establish an general Eulerian integral involving the product of two multivariable I-functions defined by Prasad [2], the multivariable A-function and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular case concerning the multivariable H-function defined by Srivastava et al [7].

Keywords: Eulerian integral, multivariable I-function, Lauricella function of several variables, multivariable H-function, generalized hypergeometric function

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1.Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable I-functions defined by Prasad [2]. The A-function of several variables is an extension of multivariable H-function defined by Srivastava et al [4].

The serie representation of the multivariable A-function is given by Gautam [1] as

$$A[u_1,\cdots,u_v] = A^{0,\lambda:(\alpha',\beta');\cdots;(\alpha^{(v)},\beta^{(v)})}_{A,C:(M',N');\cdots;(M^{(v)},N^{(v)})}\left(\begin{array}{c} \mathrm{u}_1 \\ \cdot \\ \cdot \\ \mathrm{u}_v \end{array}\right| \begin{array}{c} [(\mathrm{g}_j);\gamma',\cdots,\gamma^{(v)}]_{1,A}: \\ \cdots \\ [(\mathrm{f}_j);\xi',\cdots,\xi^{(v)}]_{1,C}: \end{array}$$

$$\left.\begin{array}{c} (\mathrm{q}^{(1)},\eta^{(1)})_{1,M^{(1)}};\cdots;(q^{(v)},\eta^{(v)})_{1,M^{(v)}} \\ \cdots \\ \cdots \\ (\mathrm{p}^{(1)},\epsilon^{(1)})_{1,N^{(1)}};\cdots;(p^{(v)},\epsilon^{(v)})_{1,N^{(v)}} \end{array}\right) = \sum_{G_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{v}\phi_i u_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon_{G_i}^{(i)}g_i!} \tag{1.1}$$

where

$$\phi_1 = \frac{\prod_{j=1}^{\lambda}\Gamma\left(1-g_j+\sum_{i=1}^{v}\gamma_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\lambda'+1}^{A}\Gamma\left(g_j-\sum_{i=1}^{v}\gamma_j^{(i)}U_i\right)\prod_{j=1}^{C}\Gamma\left(1-f_j+\sum_{i=1}^{v}\xi_j^{(i)}\eta_{G_i,g_i}\right)} \tag{1.2}$$

$$\phi_i = \frac{\prod_{j=1,j\neq m_i}^{\alpha^{(i)}}\Gamma\left(p_j^{(i)}-\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=1}^{\beta^{(i)}}\Gamma\left(1-q_j^{(i)}+\eta_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\alpha^{(i)}+1}^{N^{(i)}}\Gamma\left(1-p_j^{(i)}+\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=\beta^{(i)}+1}^{M^{(i)}}\Gamma\left(q_j^{(i)}-\eta_j^{(i)}\eta_{G_i,g_i}\right)}, i=1,\cdots,v \tag{1.3}$$

and $\eta_{G_i,g_i} = \dfrac{p_{G_i}^{(i)}+g_i}{\epsilon_{G_i}^{(i)}}, i=1,\cdots,v$ (1.4)

which is valid under the following conditions : $\epsilon_{m_i}^{(i)}[p_j^{(i)}+p_i'] \neq \epsilon_j^{(i)}[p_{m_i}+g_i]$ (1.5) and

$$u_i \neq 0, \sum_{j=1}^{A} \gamma_j^{(i)} - \sum_{j=1}^{C} \xi_j^{(i)} + \sum_{j=1}^{M^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{N^{(i)}} \epsilon_j^{(i)} < 0, i = 1, \cdots, v \tag{1.6}$$

Here $\lambda, A, C, \alpha_i, \beta_i, m_i, n_i \in \mathbb{N}^*; i = 1, \cdots, v; f_j, g_j, p_j^{(i)}, q_j^{(i)}, \gamma_j^{(i)}, \xi_j^{(i)}, \eta_j^{(i)}, \epsilon_j^{(i)} \in \mathbb{C}$

The multivariable I-function of r-variables is defined by Prasad [2] in term of multiple Mellin-Barnes type integral :

$$I(z'_1, \cdots, z'_s) = I^{0,n_2;0,n_3;\cdots;0,n_s:m^{(1)},n^{(1)};\cdots;m^{(s)},n^{(s)}}_{p_2,q_2,p_3,q_3;\cdots;p_s,q_s:p^{(1)},q^{(1)};\cdots;p^{(s)},q^{(s)}} \left(\begin{array}{c} z'_1 \\ \cdot \\ \cdot \\ \cdot \\ z'_s \end{array} \middle| \begin{array}{c} (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2}; \cdots; \\ \\ (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1,q_2}; \cdots; \end{array} \right.$$

$$\left. \begin{array}{c} (a_{sj}; \alpha_{sj}^{(1)}, \cdots, \alpha_{sj}^{(s)})_{1,p_s} : (a_j^{(1)}, \alpha_j^{(1)})_{1,p^{(1)}}; \cdots; (a_j^{(s)}, \alpha_j^{(s)})_{1,p^{(s)}} \\ \\ (b_{rj}; \beta_{sj}^{(1)}, \cdots, \beta_{sj}^{(s)})_{1,q_s} : (b_j^{(1)}, \beta_j^{(1)})_{1,q^{(1)}}; \cdots; (b_j^{(s)}, \beta_j^{(s)})_{1,q^{(s)}} \end{array} \right) \tag{1.7}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L'_1} \cdots \int_{L'_s} \phi(t_1, \cdots, t_s) \prod_{i=1}^{s} \zeta_i(t_i) z_i'^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.8}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [2]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|arg z'_i| < \frac{1}{2}\Omega_i \pi \,, \quad \text{where}$$

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} \right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)} \right) \tag{1.9}$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z'_1,\cdots,z'_s)=0(\,|z'_1|^{\alpha_1},\cdots,|z'_s|^{\alpha_r}\,),max(\,|z'_1|,\cdots,|z'_s|\,)\rightarrow 0$$

$$I(z'_1,\cdots,z'_s)\;=0(\,|z'_1|^{\beta_1},\cdots,|z'_s|^{\beta_r}\,),min(\,|z'_1|,\cdots,|z'_s|\,)\rightarrow\infty$$

where $k=1,\cdots,r:\alpha'_k=min[Re(b_j^{(k)}/\beta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta'_k=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})],j=1,\cdots,n_k$$

We will use these following notations in this section :

$$I(z''_1,\cdots,z''_u)=I^{0,n'_2;0,n'_3;\cdots;0,n'_u:m'^{(1)},n'^{(1)};\cdots;m'^{(u)},n'^{(u)}}_{p'_2,q'_2,p'_3,q'_3;\cdots;p'_u,q'_u:p'^{(1)},q'^{(1)};\cdots;p'^{(u)},q'^{(u)}}\left(\begin{array}{c|c} z''_1 & (a'_{2j};\alpha'^{(1)}_{2j},\alpha'^{(2)}_{2j})_{1,p'_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ z''_u & (b'_{2j};\beta'^{(1)}_{2j},\beta'^{(2)}_{2j})_{1,q'_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (a'_{uj};\alpha'^{(1)}_{uj},\cdots,\alpha'_{uj}{}^{(u)})_{1,p'_u}:(a'^{(1)}_j,\alpha'^{(1)}_j)_{1,p'^{(1)}};\cdots;(a'_j{}^{(u)},\alpha'_j{}^{(u)})_{1,p'^{(u)}} \\ \\ (b'_{uj};\beta'^{(1)}_{uj},\cdots,\beta'_{uj}{}^{(u)})_{1,q'_u}:(b'^{(1)}_j,\beta'^{(1)}_j)_{1,q'^{(1)}};\cdots;(b'_j{}^{(u)},\beta'_j{}^{(u)})_{1,q'^{(u)}} \end{array}\right) \tag{1.10}$$

$$=\frac{1}{(2\pi\omega)^u}\int_{L''_1}\cdots\int_{L''_u}\psi(x_1,\cdots,x_u)\prod_{i=1}^{u}\xi_i(x_i)z_i''^{x_i}\mathrm{d}x_1\cdots\mathrm{d}x_u \tag{1.11}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [2]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

where $|argz''_i|<\frac{1}{2}\Omega''_i\pi$,

$$\Omega'_i=\sum_{k=1}^{n'^{(i)}}\alpha'_k{}^{(i)}-\sum_{k=n'^{(i)}+1}^{p'^{(i)}}\alpha'_k{}^{(i)}+\sum_{k=1}^{m'^{(i)}}\beta'_k{}^{(i)}-\sum_{k=m^{(i)}+1}^{q'^{(i)}}\beta'_k{}^{(i)}+\left(\sum_{k=1}^{n'_2}\alpha'_{2k}{}^{(i)}-\sum_{k=n_2+1}^{p'_2}\alpha'_{2k}{}^{(i)}\right)$$

$$+\cdots+\left(\sum_{k=1}^{n'_u}\alpha'_{uk}{}^{(i)}-\sum_{k=n'_u+1}^{p'_u}\alpha'_{uk}{}^{(i)}\right)-\left(\sum_{k=1}^{q'_2}\beta'_{2k}{}^{(i)}+\sum_{k=1}^{q'_3}\beta'_{3k}{}^{(i)}+\cdots+\sum_{k=1}^{q'_u}\beta'_{uk}{}^{(i)}\right) \tag{1.12}$$

where $i=1,\cdots,u$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z''_1,\cdots,z''_u)=0(\,|z''_1|^{\alpha'_1},\cdots,|z''_u|^{\alpha'_s}\,),max(\,|z''_1|,\cdots,|z''_u|\,)\rightarrow 0$$

$$I(z''_1,\cdots,z''_u)\;=0(\,|z''_1|^{\beta'_1}\;,\cdots,|z''_u|^{\beta'_s}\,),min(\,|z''_1|,\cdots,|z''_u|\,)\rightarrow \infty$$

where $k=1,\cdots,z:\alpha''_k=min[Re(b_j'^{(k)}/\beta_j'^{(k)})],j=1,\cdots,m'_k$ and

$\beta''_k=max[Re((a_j'^{(k)}-1)/\alpha_j'^{(k)})],j=1,\cdots,n'_k$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [5 ,page 39 eq .30]

$$\frac{\prod_{j=1}^P\Gamma(A_j)}{\prod_{j=1}^Q\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^P\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^Q\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j),j=1,\cdots,r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j),j=1,\cdots,r$

The Lauricella function $F_D^{(k)}$ is defined as

$$F_D^{(k)}\left[a,b_1,\cdots,b_k;c;x_1,\cdots,x_k\right]=\frac{\Gamma(c)}{\Gamma(a)\prod_{j=1}^k\Gamma(b_j)}\frac{1}{(2\pi\omega)^k}\int_{L_1}\cdots\int_{L_k}\frac{\Gamma\left(a+\sum_{j=1}^k\zeta_j\right)\Gamma(b_1+\zeta_1),\cdots,\Gamma(b_k+\zeta_k)}{\Gamma\left(c+\sum_{j=1}^k\zeta_j\right)}$$

$$\prod_{j=1}^k\Gamma(-\zeta_j)(-x_j)^{\zeta_i}\,\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_k \tag{2.2}$$

where $max\left[|arg(-x_1)|,\cdots,|arg(-x_k)|\right]<\pi,c\neq 0,-1,-2,\cdots$.

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^k(f_jt+g_j)^{\sigma_j}\mathrm{d}t=(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^k(af_j+g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha,-\sigma_1,\cdots,-\sigma_k;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right] \tag{2.3}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i\in\mathbb{C},(i=1,\cdots,k)$; $min(Re(\alpha),Re(\beta))>0$ and

$$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava et al ([5], page60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [3, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [5, page 454].

3. Eulerian integral

In this section , we note :

$$U=p_2,q_2;p_3,q_3;\cdots;p_{s-1},q_{s-1};p'_2,q'_2;p'_3,q'_3;\cdots;p'_{u-1},q'_{u-1};0,0;\cdots;0,0;0,0;\cdots;0,0 \tag{3.1}$$

$$V=0,n_2;0,n_3;\cdots;0,n_{s-1};0,n'_2;0,n'_3;\cdots;0,n'_{u-1};0,0;\cdots;0,0;0,0;\cdots;0,0 \tag{3.2}$$

$$X=m^{(1)},n^{(1)};\cdots;m^{(s)},n^{(s)};m'^{(1)},n'^{(1)};\cdots;m'^{(u)},n'^{(u)};1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{3.3}$$

$$Y=p^{(1)},q^{(1)};\cdots;p^{(s)},q^{(s)}\ ;p'^{(1)},q'^{(1)};\cdots;p'^{(u)},q'^{(u)};0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.4}$$

$$A=(a_{2k};\alpha_{2k}^{(1)},\alpha_{2k}^{(2)})_{1,p_2};\cdots;(a_{(s-1)k};\alpha_{(s-1)k}^{(1)},\alpha_{(s-1)k}^{(2)},\cdots,\alpha_{(s-1)k}^{(s-1)})_{1,p_{s-1}}:(a'_{2k};\alpha'^{(1)}_{2k},\alpha'^{(2)}_{2k})_{1,p'_2}$$

$$;\cdots;(a'_{(u-1)k};\alpha'^{(1)}_{(u-1)k},\alpha'^{(2)}_{(u-1)k},\cdots,\alpha'^{(u-1)}_{(u-1)k})_{1,p'_{u-1}} \tag{3.5}$$

$$B=(b_{2k};\beta_{2k}^{(1)},\beta_{2k}^{(2)})_{1,q_2};\cdots;(b_{(s-1)k};\beta_{(s-1)k}^{(1)},\beta_{(s-1)k}^{(2)},\cdots,\beta_{(s-1)k}^{(s-1)})_{1,q_{s-1}};(b'_{2k};\beta'^{(1)}_{2k},\beta'^{(2)}_{2k})_{1,q'_2};$$

$$;\cdots;(b'_{(u-1)k};\beta'^{(1)}_{(u-1)k},\beta'^{(2)}_{(u-1)k},\cdots,\beta'^{(u-1)}_{(u-1)k})_{1,q'_{u-1}} \tag{3.6}$$

$$\mathfrak{A}=(a_{sk};\alpha_{sk}^{(1)},\alpha_{sk}^{(2)},\cdots,\alpha_{sk}^{(s)},0,\cdots,0,0\cdots,0,0,\cdots,0)_{1,p_s} \tag{3.7}$$

$$\mathfrak{A}'=(a'_{uk};0,\cdots,0,\alpha'^{(1)}_{uk},\alpha'^{(2)}_{uk},\cdots,\alpha'^{(u)}_{uk},0,\cdots,0,0,\cdots,0)_{1,p'_u} \tag{3.8}$$

$$\mathfrak{B}=(b_{sk};\beta_{sk}^{(1)},\beta_{sk}^{(2)},\cdots,\beta_{sk}^{(s)},0,\cdots,0,0,\cdots,0,0,\cdots,0)_{1,q_s} \tag{3.9}$$

$$\mathfrak{B}'=(b'_{uk};0,\cdots,0,\beta'^{(1)}_{uk},\beta'^{(2)}_{uk},\cdots,\beta'^{(u)}_{uk},0,\cdots,0,0,\cdots,0)_{1,q'_u} \tag{3.10}$$

$$A' = (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \cdots ; (a_k^{(s)}, \alpha_k^{(s)})_{1,p^{(s)}}; (a_k'^{(1)}, \alpha_k'^{(1)})_{1,p^{(1)}}; \cdots ; (a_k'^{(u)}, \alpha_k'^{(u)})_{1,p'^{(u)}};$$

$$(1,0); \cdots ; (1,0); (1,0); \cdots ; (1,0) \quad (3.11)$$

$$B' = (b_k^{(1)}, \beta_k^{(1)})_{1,q^{(1)}}; \cdots ; (b_k^{(s)}, \beta_k^{(s)})_{1,q^{(s)}}; (b_k'^{(1)}, \beta_k'^{(1)})_{1,q'^{(1)}}; \cdots ; (b_k'^{(u)}, \beta_k'^{(u)})_{1,q'^{(u)}};$$

$$(0,1); \cdots ; (0,1); (0,1); \cdots ; (0,1) \quad (3.12)$$

$$K_1 = (1 - \alpha - \sum_{i=1}^{v} \eta_{G_i,g_i}(\mu_i + \mu_i'); \mu_1, \cdots , \mu_s, \mu_1', \cdots , \mu_u', 1, \cdots , 1, \upsilon_1, \cdots , \upsilon_l) \quad (3.13)$$

$$K_2 = (1 - \beta - \sum_{i=1}^{v} \eta_{G_i,g_i}(\rho_i + \rho_i'); \rho_1, \cdots , \rho_s, \rho_1', \cdots , \rho_u', 0, \cdots , 0, \tau_1, \cdots , \tau_l) \quad (3.14)$$

$$K_P = [1 - A_j; 0, \cdots , 0, 0, \cdots , 0, 0, \cdots , 0, 1, \cdots , 1]_{1,P} \quad (3.15)$$

$$K_j = [1 + \sigma_j - \sum_{i=1}^{v} \eta_{G_i,g_i}(\lambda_i^{(j)} + \lambda_i'^{(j)}); \lambda_j^{(1)}, \cdots , \lambda_j^{(s)}, \lambda_j'^{(1)} \cdots , \lambda_j'^{(u)}, 0 \cdots , 1, \cdots , 0, \zeta_j', \cdots , \zeta_j^{(l)}]_{1,k} \quad (3.16)$$

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$$L_1 = (1 - \alpha - \beta - \sum_{i=1}^{v} \eta_{G_i,g_i}(\mu_i + \mu_i' + \rho_i + \rho_i'); \mu_1 + \rho_1, \cdots , \mu_s + \rho_s, \mu_1' + \rho_1', \cdots , \mu_u' + \rho_u',$$

$$1, \cdots , 1, \upsilon_1 + \tau_1, \cdots , \upsilon_l + \tau_l) \quad (3.17)$$

$$L_Q = [1 - B_j; 0, \cdots , 0, 0, \cdots , 0, 0, \cdots , 0, 1 \cdots , 1]_{1,Q} \quad (3.18)$$

$$L_j = [1 + \sigma_j - \sum_{i=1}^{v} \eta_{G_i,g_i}(\lambda_i^{(j)} + \lambda_i'^{(j)}); \lambda_j^{(1)}, \cdots , \lambda_j^{(s)}, \lambda_j'^{(1)} \cdots , \lambda_j'^{(u)}, 0 \cdots , 0, \zeta_j', \cdots , \zeta_j^{(l)}]_{1,k} \quad (3.19)$$

$$P_1 = (b-a)^{\alpha+\beta-1} \left\{ \prod_{j=1}^{h} (af_j + g_j)^{\sigma_j} \right\} \quad (3.20)$$

$$B_{u,v} = (b-a)^{\sum_{i=1}^{v}(\mu_i+\mu_i'+\rho_i+\rho_i')\eta_{G_i,g_i}} \left\{ \prod_{j=1}^{h} (af_j + g_j)^{-\sum_{i=1}^{v}(\lambda_i+\lambda_i')\eta_{G_i,g_i}} \right\} G_v \quad (3.21)$$

where $G_v = \phi_1 \frac{\prod_{i=1}^{v} \phi_i u_i^{\eta_{G_i,g_i}} (-)^{\sum_{i=1}^{v} g_i}}{\prod_{i=1}^{v} \epsilon_{G_i}^{(i)} g_i!}$

ϕ_1, ϕ_i for $i = 1, \cdots , v$ are defined respectively by (1.2) and (1.3)

We have the following result

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^k (f_jt+g_j)^{\sigma_j}$$

$$A\begin{pmatrix} \mathrm{z'''}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{z'''}_v(t-a)^{\mu_v+\mu'_v}(b-t)^{\rho_v+\rho'_v}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(v)}-\lambda_j'^{(v)}} \end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ . \\ . \\ . \\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(s)}} \end{pmatrix}$$

$$I\begin{pmatrix} \mathrm{z'}_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ \mathrm{z'}_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(u)}} \end{pmatrix}$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^l z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^k(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t = (b-a)^{\alpha+\beta-1}$$

$$= P_1\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\sum_{g_1,\cdots,g_v=0}^{\infty}\sum_{M_1=0}^{\alpha^{(1)}}\cdots\sum_{M_v=0}^{\alpha^{(v)}}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^v z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^u z''^{R_k}B_uB_{u,v}$$

$$I_{U:p_s+p'_u+l+k+2,q_s+q'_u+l+k+1;Y}^{V;0,n_s+n'_u+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdot\;\cdot\;\cdot & \\ \cdot\;\cdot\;\cdot & \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(s)}}} & \mathrm{A}\;;\mathrm{K}_1,K_2,K_P,K_j,\mathfrak{A},\mathfrak{A}';A' \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdot\;\cdot\;\cdot & \cdot \\ \cdot\;\cdot\;\cdot & \cdot \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(u)}_j}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdot\;\cdot\;\cdot & \cdot \\ \cdot\;\cdot\;\cdot & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & \mathrm{B}\;;\mathrm{L}_1,L_j,L_Q,\mathfrak{B},\mathfrak{B}';B' \\ \cdot\;\cdot\;\cdot & \\ \cdot\;\cdot\;\cdot & \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \end{array}\right) \tag{3.22}$$

We obtain the I-function of $s+u+k+l$ variables.

Provided that

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\rho_i,\mu'_j,\rho'_j\lambda_v^{(i)};\lambda'^{(j)}_v\in\mathbb{R}^+,f_i,g_j,\tau_v,\sigma_j\in\mathbb{C}\;\;(i=1,\cdots,s;j=1,\cdots;u;v=1,\cdots,k)$

(B) $a'_{ij},b'_{ik},\in\mathbb{C}\,(i=1,\cdots,s;j=1,\cdots,p'_i;k=1,\cdots,q'_i);a'^{(i)}_j,b'^{(k)}_j\in\mathbb{C}$

$(i=1,\cdots,s;j=1,\cdots,p'^{(i)};k=1,\cdots,q'^{(i)})$

$a''_{ij},b''_{ik},\in\mathbb{C}\,(i=1,\cdots,u';j=1,\cdots,p''_i;k=1,\cdots,q''_i);a''^{(i)}_j,b''^{(k)}_j,\in\mathbb{C}$

$(i=1,\cdots,u;j=1,\cdots,p''^{i)};k=1,\cdots,q''^{(i)})$

$\alpha'^{(k)}_{ij},\beta'^{(k)}_{ij}\in\mathbb{R}^+\;\;(i=1,\cdots,s,j=1,\cdots,p'_i,k=1,\cdots,s)\,;\alpha'^{(i)}_j,\beta'^{(i)}_i\in\mathbb{R}^+\,(i=1,\cdots,s;j=1,\cdots,p'_i)$

$\alpha''^{(k)}_{ij},\beta''^{(k)}_{ij}\in\mathbb{R}^+\;\;(i=1,\cdots,u,j=1,\cdots,p''_i,k=1,\cdots,u)\,;\alpha''^{(i)}_j,\beta''^{(i)}_i\in\mathbb{R}^+\,(i=1,\cdots,u;j=1,\cdots,p''_i)$

(C) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$

(D) $Re\big[\alpha+\sum_{i=1}^{s}\mu_i \min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}+\sum_{i=1}^{u}\mu'_i \min_{1\leqslant j\leqslant m'^{(i)}}\frac{b'^{(i)}_j}{\beta'^{(i)}_j}\big]>0$

$Re\big[\beta+\sum_{i=1}^{s}\rho_i \min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}+\sum_{i=1}^{u}\rho'_i \min_{1\leqslant j\leqslant m'^{(i)}}\frac{b'^{(i)}_j}{\beta'^{(i)}_j}\big]>0$

(E) $\Omega_i=\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right)+\cdots+$

$\left(\sum_{k=1}^{n_s}\alpha_{sk}^{(i)}-\sum_{k=n_s+1}^{p_s}\alpha_{sk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_s}\beta_{sk}^{(i)}\right)-\mu_i-\rho_i$

$-\sum_{j=1}^{k}\lambda_j^{(i)}>0 \quad (i=1,\cdots,s)$

$\Omega'_i=\sum_{k=1}^{n'^{(i)}}\alpha'^{(i)}_k-\sum_{k=n'^{(i)}+1}^{p'^{(i)}}\alpha'^{(i)}_k+\sum_{k=1}^{m'^{(i)}}\beta'^{(i)}_k-\sum_{k=m^{(i)}+1}^{q'^{(i)}}\beta'^{(i)}_k+\left(\sum_{k=1}^{n'_2}\alpha'^{(i)}_{2k}-\sum_{k=n_2+1}^{p'_2}\alpha'^{(i)}_{2k}\right)$

$+\cdots+\left(\sum_{k=1}^{n'_u}\alpha'^{(i)}_{uk}-\sum_{k=n'_u+1}^{p'_u}\alpha'^{(i)}_{uk}\right)-\left(\sum_{k=1}^{q'_2}\beta'^{(i)}_{2k}+\sum_{k=1}^{q'_3}\beta'^{(i)}_{3k}+\cdots+\sum_{k=1}^{q'_u}\beta'^{(i)}_{uk}\right)-\mu'_i-\rho'_i$

$-\sum_{j=1}^{k}\lambda'^{(i)}_j>0 \quad (i=1,\cdots,u)$

(F) $\left|arg\left(z_i\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right|<\frac{1}{2}\Omega_i\pi \quad (a\leqslant t\leqslant b; i=1,\cdots,s)$

$\left|arg\left(z'_i\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(i)}_j}\right)\right|<\frac{1}{2}\Omega'_i\pi \quad (a\leqslant t\leqslant b; i=1,\cdots,u)$

(G) $P\leqslant Q+1$. The equality holds, when , in addition, either $P > Q$ and $\left| z_i'' \left(\prod_{j=1}^{k} (f_j t + g_j)^{-\zeta_j^{(i)}} \right) \right|^{\frac{1}{Q-P}} < 1 \qquad (a \leqslant t \leqslant b)$

or $P \leqslant Q$ and $\max_{1 \leqslant i \leqslant k} \left[\left| \left(z_i'' \prod_{j=1}^{k} (f_j t + g_j)^{-\zeta_j^{(i)}} \right) \right| \right] < 1 \qquad (a \leqslant t \leqslant b)$

(H) The multiple series occuring on the right-hand side of (3.22) is absolutely and uniformly convergent.

Proof

First expressing the multivariable A-function in serie with the help of (1.1) nd we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the I-function of s-variables and u-variables defined by Prasad [2] by the Mellin-Barnes contour integral with the help of the equation (1.8) and (1.11) respectively, the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1). Now collect the power of $(f_j t + g_j)$ with $j = 1, \cdots, k$ and use the equations (2.1) and (2.2) and we obtain $k-$Mellin-Barnes contour integral and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process . Interpreting $(r + s + k + l)-$Mellin-barnes contour integral in multivariable I-function defined by Prasad [2], we obtain the desired result.

4. Multivariable H-function

If $A = B = U = V = 0$, the multivariable I-function reduces to the multivariable H-function and we obtain

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j} A \begin{pmatrix} \text{z''}_1 (t-a)^{\mu_1+\mu_1'} (b-t)^{\rho_1+\rho_1'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)} - \lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z''}_v (t-a)^{\mu_v+\mu_v'} (b-t)^{\rho_v+\rho_v'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(v)} - \lambda_j'^{(v)}} \end{pmatrix}$$

$$H \begin{pmatrix} \text{z}_1 (t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z}_s (t-a)^{\mu_s} (b-t)^{\rho_s} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(s)}} \end{pmatrix}$$

$$H \begin{pmatrix} \text{z'}_1 (t-a)^{\mu_1'} (b-t)^{\rho_1'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ \text{z'}_u (t-a)^{\mu_u'} (b-t)^{\rho_u'} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j'^{(u)}} \end{pmatrix}$$

$${}_PF_Q \left[(A_P); (B_Q); -\sum_{i=1}^{l} z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i} \prod_{j=1}^{k} (f_j t + g_j)^{-\zeta_j^{(i)}} \right] \mathrm{d}t = (b-a)^{\alpha+\beta-1}$$

$$= P_1 \frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)} \sum_{g_1,\cdots,g_v=0}^{\infty} \sum_{M_1=0}^{\alpha^{(1)}} \cdots \sum_{M_v=0}^{\alpha^{(v)}} \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} \prod_{i=1}^{v} z_i'''^{\eta_{h_i,k_i}} \prod_{k=1}^{u} z''^{R_k} B_u B_{u,v}$$

$$H_{p_s+p'_u+l+k+2,q_s+q'_u+l+k+1;Y}^{0,n_s+n'_u+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{K}_1, K_2, K_P, K_j, \mathfrak{A}, \mathfrak{A}'; A' \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(u)}_j}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdot\cdot\cdot & \cdot \\ \cdot\cdot\cdot & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & \mathrm{L}_1, L_j, L_Q, \mathfrak{B}, \mathfrak{B}'; B' \\ \cdot\cdot\cdot & \\ \cdot\cdot\cdot & \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \end{array}\right) \tag{4.1}$$

under the same conditions that (3.22) with $A = B = U = V = 0$

Remark

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable I-functions defined by Prasad [2].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable I-functions, defined by Prasad [2], a expansion of multivariable A-function and a generalized hypergeometric function with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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