Number Theoretic Functions: Augmentation & Analytical Results

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Abstract — *In number theory, there exist many number theoretic functions, which includes Divisor function $\tau(n)$, Sigma function $\sigma(n)$, Euler phi function $\phi(n)$ and Mobius function $\mu(n)$. All these functions play very important role in the field of number theory. In this paper we have given some results for number theoretic functions.*

Keywords — Number theory, Number theoretic functions.

I. INTRODUCTION

A function f is called an arithmetic function or a number-theoretic function [1, 2] if it assigns to each positive integer n a unique real or complex number f(n). Typically, an arithmetic function is a real-valued function whose domain is the set of positive integers.

A real function f defined on the positive integers is said to be multiplicative if

$$f(m)\, f(n) = f(mn),\ \forall\ m, n \in Z,$$

where gcd(m, n) = 1. If

$$f(m)\, f(n) = f(mn),\ \forall\ m, n \in Z,$$

then f is completely multiplicative. Every completely multiplicative function is multiplicative.

A. *The Divisor Function d(n) and $\sigma_k(n)$ Function*

The function d(n) [3] is the number of divisors of n, including 1 and n, while $\sigma_k(n)$ is the sum of the k^{th} powers of the divisors of n. Thus

$$\sigma_k(n) = \sum_{d|n} d^k, \qquad d(n) = \sum_{d|n} 1$$

and $d(n) = \sigma_0(n)$. We Write $\sigma(n)$ for $\sigma_1(n)$, the sum of the divisors of n. The divisor function is usually denoted by d(n) or $\tau(n)$.

B. *Euler's Phi-Function $\phi(n)$*

The function $\phi(n)$ [4] was defined for n > 1, as the number of positive integers less than and prime to n. Also

$$\phi(n) = n \prod_{p|n} (1-1/p)$$

C. *Mobius Function $\mu(n)$*

The Mobius function [3] $\mu(n)$ is defined as follows :

1) $\mu(1) = 1$;
2) $\mu(n) = 0$ if n has a square factor;
3) $\mu(p_1.p_2...p_k) = (-1)^k$ if all the primes $p_1, p_2,..., p_k$, are different.

Thus $\mu(2) = -1$, $\mu(4) = 0$, $\mu(6) = 1$.

$\mu(n)$ is multiplicative. i.e, for any two positive numbers a and b

$$\mu(ab) = \mu(a)\,\mu(b)$$

Also,

$$\sum_{d|n} \mu(d) = 1 \text{ (for n=1)}, \sum_{d|n} \mu(d) = 0 \text{ (for n > 1)}$$

If n > 1, and k is the number of different prime factors of n, then

$$\sum_{d|n} |\mu(d)| = 2^k$$

II. RESULTS FOR DIVISOR FUNCTION

A. If $\tau_k(p)$ is the number of divisors of p (prime) which are greater than equal to k, then

$$\tau_k(p) = \begin{cases} 2, \text{ if } k = 1 \\ 1, \text{ if } 1 < k \le p \\ 0, \text{ if } k > p \end{cases}$$

eg: $\tau_3(7) = 1$, $\tau_1(13) = 2$, $\tau_{19}(7) = 0$

B. If $\tau_k(p^\alpha)$ is the number of all the divisors of p^α which are greater than equal to k, then

$$\tau_k(p^\alpha) = \lambda, \text{ if } k \le p^{\alpha+1-\lambda}$$

for maximum integer value of λ.

eg: $\tau_4(2^8) = 7$, since $4 \le 2^{8+1-7}$ for $\lambda = 7$.

C. If $\tau_p^{p^k}(p^\alpha)$ is the number of divisors of p^α lies in the interval $[p, p^k]$, then

$$\tau_p^{p^k}(p^\alpha) = \begin{cases} k, \text{ if } k \le \alpha \\ \alpha, \text{ if } k > \alpha \end{cases}$$

eg: $\tau_{11}^{11^4}(11^7) = 4$

III. RESULTS FOR SIGMA FUNCTION

A. If $\sigma[(\alpha,\beta); n]$ be the sum of prime divisors of n, which lies in the interval $[\alpha,\beta]$, where $n = p_1 . p_2 . p_3 \ldots\ldots p_k$, then

$$\sigma[(\alpha,\beta); n] = \sigma[(\alpha,\beta); p_1] + \sigma[(\alpha,\beta); p_2] + \ldots\ldots\ldots + \sigma[(\alpha,\beta); p_k]$$

where, $\sigma[(\alpha,\beta); p] = \begin{cases} p, \text{ if } p \in [\alpha,\beta] \text{ and } \alpha \le \beta \\ 0, \text{ otherwise} \end{cases}$

Therefore, $\sigma[(\alpha,\beta); n] = \sum_{\substack{i=1 \\ p_i | n}}^{k} \sigma[(\alpha,\beta); p_i]$

eg: $\sigma[(2,6); 70] = \sigma[(2,6); 2.5.7] = 2+5 = 7$

B. $\sigma_k(p^\alpha)$ is the sum of the k^{th} powers of positive divisors of p^α.

i.e., $\sigma_k(p^\alpha) = 1 + p^k + p^{2k} + \ldots\ldots + p^{\alpha k}$

$$= \sum_{n=0}^{\alpha} p^{nk} = (p^{k(\alpha+1)} - 1) / (p^k - 1)$$

eg: $\sigma_3(2^5) = 1 + 2^3 + 2^{2.3} + 2^{3.3} + 2^{4.3} + 2^{5.3}$
$= 1 + 2^3 + 2^6 + 2^9 + 2^{12} + 2^{15}$
$= 37449$

C. $\sigma_{k_1}^{k_2}(n)$ be the sum of the $(k_2\text{-}k_1)^{th}$ power of positive divisors of n.

i.e, $$\sigma_{k_1}^{k_2}(n) = \sum_{d\,|\,n} d^{k_2 - k_1} \qquad (\text{for } k_2 > k_1)$$

$$\sigma_{k_1}^{k_2}(p) = 1 + p^{k_2 - k_1}$$

$$\sigma_{k_1}^{k_2}(p^\alpha) = \sum_{d\,|\,p^\alpha} d^{k_2 - k_1}$$

$$= 1 + p^{k_2 - k_1} + p^{2(k_2 - k_1)} + \ldots\ldots + p^{\alpha(k_2 - k_1)}$$

$$= \sum_{r=0}^{\alpha} p^{r(k_2 - k_1)}$$

IV. RESULTS FOR EULER PHI FUNCTION

$\phi_\alpha^\beta(n)$ be the number of integers lies in $[\alpha, \beta]$, which are co-prime to n.

eg: $\phi_3^8(15) = 3$

Particular Cases (for $\alpha = 1$)

Case I: When $\beta = k$ then

$$\phi_1^k(n) = \phi^k(n)$$

be the number of integers less than equal to k, which are co-prime to n.

eg: $\phi^3(6) = 1$

Case II: When $\beta = 1$ then

$$\phi^1(n) = 1$$

eg: $\phi^1(24) = 1$

Case III: When $\beta = 2$ then

$$\phi^2(n) = \begin{cases} 1, \text{ if n is even.} \\ 2, \text{ if n is odd.} \end{cases}$$

eg: $\phi^2(72) = 1$, $\phi^2(81) = 2$

Case IV: When $\beta = n$ then

$$\phi^n(n) = \phi(n)$$

eg: $\phi^7(7) = 6 = \phi(7)$

V. RESULTS FOR MOBIUS FUNCTION

The function $[\mu(\alpha,\beta);n]$ is known as the Mobius (α,β) function for all positive numbers n, having value $(-1)^k$ and 0, depends upon prime factors of n lies in the interval $[\alpha,\beta]$. The function $[\mu(\alpha,\beta);n]$ is defined as follows:

$$[\mu(\alpha,\beta);n] = \begin{cases} (-1)^k, & \text{if all } p_i \text{ are distinct and lies in the interval } [\alpha,\beta]. \\ 0, & \text{if } p_i^2 | n \text{ and } p_i \in [\alpha,\beta]. \end{cases}$$

where, $\alpha, \beta \in N$ and k is the number of distinct prime factors of n lies in $[\alpha,\beta]$.

and $\qquad [\mu(1,\beta);1] = 1$

eg: $[\mu(3,12);195] = [\mu(3,12);3.5.13] = (-1)^2 = 1$

VI. CONCLUSION

New results for number theoretic functions are very useful in the field of number theory.

VII. FURTHER STUDY

Further we will generate the algorithms for these number theoretic functions.

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