

Number Theoretic Functions: Augmentation & Analytical Results

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Abstract — In number theory, there exist many number theoretic functions, which includes Divisor function $\tau(n)$, Sigma function $\sigma(n)$, Euler phi function $\phi(n)$ and Mobius function $\mu(n)$. All these functions play very important role in the field of number theory. In this paper we have given some results for number theoretic functions.

Keywords — Number theory, Number theoretic functions.

I. INTRODUCTION

A function f is called an arithmetic function or a number-theoretic function [1, 2] if it assigns to each positive integer n a unique real or complex number $f(n)$. Typically, an arithmetic function is a real-valued function whose domain is the set of positive integers.

A real function f defined on the positive integers is said to be multiplicative if

$$f(m)f(n) = f(mn), \forall m, n \in \mathbb{Z},$$

where $\gcd(m, n) = 1$. If

$$f(m)f(n) = f(mn), \forall m, n \in \mathbb{Z},$$

then f is completely multiplicative. Every completely multiplicative function is multiplicative.

A. The Divisor Function $d(n)$ and $\sigma_k(n)$ Function

The function $d(n)$ [3] is the number of divisors of n , including 1 and n , while $\sigma_k(n)$ is the sum of the k^{th} powers of the divisors of n . Thus

$$\sigma_k(n) = \sum_{d|n} d^k, \quad d(n) = \sum_{d|n} 1$$

and $d(n) = \sigma_0(n)$. We Write $\sigma(n)$ for $\sigma_1(n)$, the sum of the divisors of n . The divisor function is usually denoted by $d(n)$ or $\tau(n)$.

B. Euler's Phi-Function $\phi(n)$

The function $\phi(n)$ [4] was defined for $n > 1$, as the number of positive integers less than and prime to n . Also

$$\phi(n) = n \prod_{p|n} (1 - 1/p)$$

C. Mobius Function $\mu(n)$

The Mobius function [3] $\mu(n)$ is defined as follows :

- 1) $\mu(1) = 1$;
- 2) $\mu(n) = 0$ if n has a square factor;
- 3) $\mu(p_1 p_2 \dots p_k) = (-1)^k$ if all the primes p_1, p_2, \dots, p_k , are different.

Thus $\mu(2) = -1, \mu(4) = 0, \mu(6) = 1$.

$\mu(n)$ is multiplicative. i.e, for any two positive numbers a and b

$$\mu(ab) = \mu(a) \mu(b)$$

Also,

$$\sum_{d|n} \mu(d) = 1 \text{ (for } n=1), \sum_{d|n} \mu(d) = 0 \text{ (for } n > 1)$$

If $n > 1$, and k is the number of different prime factors of n , then

$$\sum_{d|n} |\mu(d)| = 2^k$$

II. RESULTS FOR DIVISOR FUNCTION

A. If $\tau_k(p)$ is the number of divisors of p (prime) which are greater than equal to k , then

$$\tau_k(p) = \begin{cases} 2, & \text{if } k=1 \\ 1, & \text{if } 1 < k \leq p \\ 0, & \text{if } k > p \end{cases}$$

eg: $\tau_3(7) = 1, \tau_1(13) = 2, \tau_{19}(7) = 0$

B. If $\tau_k(p^\alpha)$ is the number of all the divisors of p^α which are greater than equal to k , then

$$\tau_k(p^\alpha) = \lambda, \text{ if } k \leq p^{\alpha+1-\lambda}$$

for maximum integer value of λ .

eg: $\tau_4(2^8) = 7$, since $4 \leq 2^{8+1-7}$ for $\lambda = 7$.

C. If $\tau_p^k(p^\alpha)$ is the number of divisors of p^α lies in

the interval $[p, p^k]$, then

$$\tau_p^k(p^\alpha) = \begin{cases} k, & \text{if } k \leq \alpha \\ \alpha, & \text{if } k > \alpha \end{cases}$$

eg: $\tau_{11}^{11}(11^7) = 4$

III. RESULTS FOR SIGMA FUNCTION

A. If $\sigma[(\alpha, \beta); n]$ be the sum of prime divisors of n , which lies in the interval $[\alpha, \beta]$, where $n = p_1 p_2 \dots p_k$, then

$$\sigma[(\alpha, \beta); n] = \sigma[(\alpha, \beta); p_1] + \sigma[(\alpha, \beta); p_2] + \dots + \sigma[(\alpha, \beta); p_k]$$

where, $\sigma[(\alpha, \beta); p] = \begin{cases} p, & \text{if } p \in [\alpha, \beta] \text{ and } \alpha \leq \beta \\ 0, & \text{otherwise} \end{cases}$

Therefore, $\sigma[(\alpha, \beta); n] = \sum_{i=1}^k \sigma[(\alpha, \beta); p_i]$

eg: $\sigma[(2,6); 70] = \sigma[(2,6); 2.5.7] = 2+5 = 7$

B. $\sigma_k(p^\alpha)$ is the sum of the k^{th} powers of positive divisors of p^α .

i.e., $\sigma_k(p^\alpha) = 1 + p^k + p^{2k} + \dots + p^{\alpha k}$
 $= \sum_{n=0}^{\alpha} p^{nk} = (p^{k(\alpha+1)} - 1) / (p^k - 1)$

eg: $\sigma_3(2^5) = 1 + 2^3 + 2^{2 \cdot 3} + 2^{3 \cdot 3} + 2^{4 \cdot 3} + 2^{5 \cdot 3}$
 $= 1 + 2^3 + 2^6 + 2^9 + 2^{12} + 2^{15}$
 $= 37449$

C. $\sigma_{k_2}^{k_1}(n)$ be the sum of the $(k_2-k_1)^{\text{th}}$ power of positive divisors of n .

i.e., $\sigma_{k_2}^{k_1}(n) = \sum_{d|n} d^{k_2 - k_1}$ (for $k_2 > k_1$)
 $\sigma_{k_2}^{k_1}(p) = 1 + p^{k_2 - k_1}$
 $\sigma_{k_2}^{k_1}(p^\alpha) = \sum_{d|p^\alpha} d^{k_2 - k_1}$
 $= 1 + p^{k_2 - k_1} + p^{2(k_2 - k_1)} + \dots + p^{\alpha(k_2 - k_1)}$
 $= \sum_{r=0}^{\alpha} p^{r(k_2 - k_1)}$

IV. RESULTS FOR EULER PHI FUNCTION

$\phi_{\alpha}^{\beta}(n)$ be the number of integers lies in $[\alpha, \beta]$,

which are co-prime to n .

eg: $\phi_3^8(15) = 3$

Particular Cases (for $\alpha = 1$)

Case I: When $\beta = k$ then

$$\phi_1^k(n) = \phi^k(n)$$

be the number of integers less than equal to k , which are co-prime to n .

eg: $\phi_3^6(6) = 1$

Case II: When $\beta = 1$ then

$$\phi_1^1(n) = 1$$

eg: $\phi_1^1(24) = 1$

Case III: When $\beta = 2$ then

$$\phi_2^2(n) = \begin{cases} 1, & \text{if } n \text{ is even.} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

eg: $\phi_2^2(72) = 1, \phi_2^2(81) = 2$

Case IV: When $\beta = n$ then

$$\phi^n(n) = \phi(n)$$

eg: $\phi^7(7) = 6 = \phi(7)$

V. RESULTS FOR MOBIUS FUNCTION

The function $[\mu(\alpha, \beta); n]$ is known as the Mobius (α, β) function for all positive numbers n , having value $(-1)^k$ and 0, depends upon prime factors of n lies in the interval $[\alpha, \beta]$. The function $[\mu(\alpha, \beta); n]$ is defined as follows:

$$[\mu(\alpha, \beta); n] = \begin{cases} (-1)^k, & \text{if all } p_i \text{ are distinct and lies in the interval } [\alpha, \beta]. \\ 0, & \text{if } p_i^2 | n \text{ and } p_i \in [\alpha, \beta]. \end{cases}$$

where, $\alpha, \beta \in \mathbb{N}$ and k is the number of distinct prime factors of n lies in $[\alpha, \beta]$.

and $[\mu(1, \beta); 1] = 1$

eg: $[\mu(3, 12); 195] = [\mu(3, 12); 3.5.13] = (-1)^2 = 1$

VI. CONCLUSION

New results for number theoretic functions are very useful in the field of number theory.

VII. FURTHER STUDY

Further we will generate the algorithms for these number theoretic functions.

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