Second Order Fuzzy Uniform Structures

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Abstract — *In this paper the definition of fuzzy uniformity introduced by Hutton, B. [4] is extended to the second order case. Given a first order fuzzy uniform structure (Hutton [4]) $\mathcal{U}$ on a set X, a second order fuzzy uniform structure $\hat{\mathcal{U}}$ on X is constructed. Every second order fuzzy uniformity $\hat{\mathcal{U}}$ induces a second order fuzzy topology $\hat{\delta}(\hat{\mathcal{U}})$. It is proved that the associations $\mathcal{U} \rightarrow \hat{\mathcal{U}}$ and $\hat{\mathcal{U}} \rightarrow \hat{\delta}(\hat{\mathcal{U}})$ are functorial.*

Keywords — *Fuzzy uniformity, second order H-fuzzy uniformity, second order H-uniformly continuous.*

I. INTRODUCTION

A **fuzzy set** on a set X is a map defined on X with values in I, where I is the closed unit interval [0, 1]. Equivalently fuzzy sets which are named as first order fuzzy sets in this paper deal with crisply defined membership functions or degrees of membership. It is doubtful whether, for instance, human beings have or can have a crisp image of membership functions in their minds. Zadeh [8] therefore suggested the notion of a fuzzy set whose membership function itself is a fuzzy set. This leads to the following definition of a second order fuzzy set or a fuzzy set of type 2. A **second order fuzzy set** on a nonempty set X is a map from X to I^I.

First order fuzzy sets are denoted by f, g, h, ... and second order fuzzy sets are denoted by $\hat{f}, \hat{g}, \hat{h}$, ...

In this paper the terms 'fuzzy set' and 'first order fuzzy set' are used synonymously.

Whenever a fuzzy set is considered without mentioning the order, it always refers to a first order fuzzy set.

Similar terminology applies to all concepts related to first order fuzzy sets.

Fundamental definitions and properties of second order fuzzy sets and second order fuzzy topological spaces are introduced in [5].

In this paper the definition of fuzzy uniformity introduced by Hutton, B. [4] is extended to the second order case. Given a first order fuzzy uniform structure (Hutton [4]) $\mathcal{U}$ on a set X, a second order fuzzy uniform structure $\hat{\mathcal{U}}$ on X is constructed. Every second order fuzzy uniformity $\hat{\mathcal{U}}$ induces a second order fuzzy topology $\hat{\delta}(\hat{\mathcal{U}})$. It is proved that the associations $\mathcal{U} \rightarrow \hat{\mathcal{U}}$ and $\hat{\mathcal{U}} \rightarrow \hat{\delta}(\hat{\mathcal{U}})$ are functorial.

II. FUNDAMENTAL DEFINITIONS AND NOTATIONS

Definition : 2.1 [5]

A second order Chang fuzzy topology $\hat{\delta}$ on a nonempty set X is a collection of second order fuzzy sets on X satisfying the following conditions :

(i) $\hat{0}, \hat{1} \in \hat{\delta}$ where, for any x ∈ X.

$\hat{0}(x)$ = the zero function **0** for I $\hat{1}(x)$ = the constant function **I** on I.

(ii) $\hat{f}_\lambda \in \hat{\delta}$ for each $\lambda \in \Lambda$ implies $(\bigvee_{\lambda \in \Lambda} \hat{f}_\lambda) \in \hat{\delta}$

(iii) $\hat{f}_i \in \hat{\delta}$ for i = 1, 2, . . . , implies $(\bigwedge_{\lambda \in \Lambda}^{m} \hat{f}_i) \in \hat{\delta}$

The pair (X, $\hat{\delta}$) is called a **second order fuzzy topological space.**

Definition : 2.2 [7]

$(X, \hat{\delta}_1)$, $(Y, \hat{\delta}_2)$ be two second order fuzzy topological spaces. Then a function $\theta : X \rightarrow Y$ is said to be **2-f continuous** if the following condition is satisfied.

$\theta^{-1}(\hat{f}) \in \hat{\delta}_1$ if $\hat{f} \in \hat{\delta}_2$.

Definition : 2.3 [4]

A **H-fuzzy uniformity** $\mathcal{U}$ on a set X is a collection of maps $\mu : I^X \rightarrow I^X$ satisfying the following conditions :

(HU1) $\mathcal{U} \neq \varphi$

(HU2) $f \leq \mu(f)$ for all $f \in I^X$ and $\mu(\mathbf{0}) = \mathbf{0}$.

(HU3) $\mu(\bigvee_{\lambda \in \Lambda} f_\lambda) = \bigvee_{\lambda \in \Lambda} \mu(f_\lambda)$

(HU4) $\mu \in \mathcal{U}, \mu \leq \mu_1 \Rightarrow \mu_1 \in \mathcal{U}$

(HU5) $\mu_1, \mu_2 \in \mathcal{U} \Rightarrow \mu_1 \wedge \mu_2 \in \mathcal{U}$.

(HU6) $\mu \in \mathcal{U} \Rightarrow$ there exists $\nu \in \mathcal{U}$ such that $\nu . \nu \leq \mu$ where $\nu . \nu$ denotes the composition of mappings.

(HU7) $\mu \in \mathcal{U} \Rightarrow \mu^{-1} \in \mathcal{U}$ where $\mu^{-1}(g) = \Lambda \{h \in I^X / \mu(h^c) \leq g^c\}$

The pair (X, $\mathcal{U}$) is called a **H-fuzzy uniform space**.

Definition : 2.4 [4]

Let (X, $\mathcal{U}$) be a H-fuzzy uniform space. Define int : $I^X \rightarrow I^X$ as int f = V {g ∈ I^X / μ(g) ≤ f for some

$\mu \in \mathcal{U}\}$. The function 'int' is an interior operator on I^X. The collection $\delta(\mathcal{U}) = \{f \in I^X / \text{int } f = f\}$ is a fuzzy topology on X and it is called the **fuzzy topology induced by $\mathcal{U}$**.

Definition : 2.5 [4]

Let (X, $\mathcal{U}_1$), (Y, $\mathcal{U}_2$) be two H-fuzzy uniform spaces. A map $\theta : X \to Y$ is said to be **H-uniformly continuous** if the following condition is satisfied :

For every $\mu_2 \in \mathcal{U}_2$, there exists a $\mu_1 \in \mathcal{U}_1$ such that $\mu_1 \leq \theta^{-1}(\mu_2)$ (i.e.) for every $f \in I^X$, $\mu_1(f) \leq \theta^{-1}(\mu_2(\theta(f)))$.

III. SECOND ORDER FUZZY UNIFORM STRUCTURES

Definition : 3.1

A second order H-fuzzy uniformity $\hat{\mathcal{U}}$ on a set X is a collection of maps $\hat{\mu} = (I^I)^X \to (I^I)^X$ satisfying the following axioms :

(SHU1) $\hat{\mathcal{U}} \neq \varphi$

(SHU2) $\hat{\mu}(\hat{f}) \geq \hat{f}$ for every $\hat{f} \in (I^I)^X$ and $\hat{\mu}(\hat{0}) = \hat{0}$

(SHU3) $\hat{\mu}\left(\bigvee_{\lambda \in \Lambda} \hat{f}_\lambda\right) = \bigvee_{\lambda \in \Lambda} \hat{\mu}(\hat{f}_\lambda)$

(SHU4) $\hat{\mu} \in \hat{\mathcal{U}}$, $\hat{\mu} \leq \hat{\mu}_1 \Rightarrow \hat{\mu}_1 \in \hat{\mathcal{U}}$

(SHU5) $\hat{\mu}_1, \hat{\mu}_2 \in \hat{\mathcal{U}} \Rightarrow \hat{\mu}_1 \wedge \hat{\mu}_2 \in \hat{\mathcal{U}}$.

(SHU6) $\hat{\mu} \in \hat{\mathcal{U}} \Rightarrow$ there exists $\hat{\nu} \in \hat{\mathcal{U}}$ such that $\hat{\nu} . \hat{\nu} \leq \hat{\mu}$ where $\hat{\nu} . \hat{\nu}$ denotes the composition of mappings.

(SHU7) $\hat{\mu} \in \hat{\mathcal{U}} \Rightarrow (\hat{\mu})^{-1} \in \hat{\mathcal{U}}$ where

$$(\hat{\mu})^{-1}(\hat{g}) = \wedge \{\hat{h} / \hat{\mu}((\hat{h})^C) \leq (\hat{g})^C\}$$

The pair (X, $\hat{\mathcal{U}}$) is said to be a **second order H-fuzzy uniform space**.

Theorem : 3.2

Let $\mathcal{U}$ be a H-fuzzy uniformity on X. For $\mu \in \mathcal{U}$, define $\hat{\mu} : (I^I)^X \to (I^I)^X$ as follows :

$(\hat{\mu}(\hat{f}))(x)(\alpha) = (\mu(f_\alpha))(x)$, for every $x \in X$ and for every $\alpha \in I$.

where $f_\alpha(x) = \hat{f}(x)(\alpha)$.

Let $\hat{B} = (\hat{\mu} / \mu \in \mathcal{U}\}$.

Let $\hat{\mathcal{U}} = \{\hat{\nu} : (I^I)^X \to (I^I)^X / \hat{\nu} \geq \hat{\mu}$ for some $\hat{\mu} \in \hat{B}\}$

Then $\hat{\mathcal{U}}$ is a second order H-fuzzy uniformity on X.

Proof :

Axiom (SHU4) is obvious. $\therefore$ To prove the theorem it is enough to prove $\hat{B}$ satisfies all the remaining axioms.

(SHU1) since $\mathcal{U} \neq \varphi$, $\hat{\mathcal{U}} \neq \varphi$

(SHU2) For $x \in X$, $\alpha \in I$, $\hat{\mu} \in \hat{B}$

$(\hat{\mu}(\hat{f}))(x)(\alpha)$

$= (\mu(f_\alpha))(x)$

$\geq f_\alpha(x)$

$= \hat{f}(x)(\alpha)$

$\therefore \hat{\mu}(\hat{f}) \geq \hat{f}$

since $\mu(\mathbf{0}) = \mathbf{0}$, $\hat{\mu}(\hat{0}) = \hat{0}$

(SHU3) If $\hat{g} = \bigvee_{\lambda \in \Lambda} \hat{f}_\lambda$, then

$$g_\alpha = \bigvee_{\lambda \in \Lambda} (f_\lambda)_\alpha$$

$\therefore$ For $x \in X$, $\alpha \in I$

$(\hat{\mu}(\bigvee_{\lambda \in \Lambda} \hat{f}_\lambda))(x)(\alpha)$

$= (\mu(\bigvee_{\lambda \in \Lambda} (f_\lambda)_\alpha))(x)$

$= (\bigvee_{\lambda \in \Lambda} \mu((f_\lambda)_\alpha))(x)$

$= \bigvee_{\lambda \in \Lambda} (\mu((f_\lambda)_\alpha))(x)$

$= \bigvee_{\lambda \in \Lambda} (\hat{\mu}(\hat{f}_\lambda))(x)(\alpha)$

$= (\bigvee_{\lambda \in \Lambda} \hat{\mu}(\hat{f}_\lambda))(x)(\alpha)$

$\therefore \hat{\mu}(\bigvee_{\lambda \in \Lambda} \hat{f}_\lambda) = \bigvee_{\lambda \in \Lambda} \hat{\mu}(\hat{f}_\lambda)$

(SHU5) $\hat{\mu}_1, \hat{\mu}_2 \in \hat{B}$

$\Rightarrow \mu_1, \mu_2 \in \mathcal{U}$

$\Rightarrow \mu_1 \wedge \mu_2 \in \mathcal{U}$

$\Rightarrow \widehat{\mu_1 \wedge \mu_2} \in \hat{B}$

For $x \in X$ and $\alpha \in I$,

$(\widehat{\mu_1 \wedge \mu_2}(\hat{f}_\lambda))(x)(\alpha)$

$= ((\mu_1 \wedge \mu_2)(f_\alpha))(x)$

$= (\mu_1(f_\alpha))(x) \wedge (\mu_2(f_\alpha))(x)$

$= (\hat{\mu}_1(\hat{f}))(x)(\alpha) \wedge (\hat{\mu}_2(\hat{f}))(x)(\alpha)$

$= ((\hat{\mu}_1 \wedge \hat{\mu}_2)(\hat{f}))(x)(\alpha)$

$\therefore \widehat{\mu_1 \wedge \mu_2} = \hat{\mu}_1 \wedge \hat{\mu}_2$

$\therefore \hat{\mu}_1 \wedge \hat{\mu}_2 \in \hat{B}$

(SHU6)

Let $\mu \in \mathcal{U}$

$\therefore$ There exists $\nu \in \mathcal{U}$ such that $\nu . \nu \leq \mu$.

$\nu \in \mathcal{U} \Rightarrow \hat{\nu} \in \hat{B}$

For $x \in X$, $\alpha \in I$, $\hat{f} \in (I^I)^X$

$(\widehat{\nu . \nu}(\hat{f}))(x)(\alpha)$

$= ((\nu . \nu)(f_\alpha))(x)$

$= (\nu(\nu(f_\alpha)))(x)$

$= (\nu(g_\alpha))(x)$ where $g_\alpha = \nu(f_\alpha)$

Let $\hat{g} : X \to I^I$ be such that

$\hat{g}(x)(\alpha) = g_\alpha(x)$.

$\therefore \; \hat{g} = \hat{\nu}(\hat{f})$

$\therefore \; (\widehat{\nu . \nu}(\hat{f}))(x)(\alpha) = (\hat{\nu}(\hat{g}))(x)(\alpha)$

$= (\hat{\nu}(\hat{\nu}(\hat{f})))(x)(\alpha)$

$= ((\hat{\nu} . \hat{\nu})(\hat{f}))(x)(\alpha)$

$\therefore \; \widehat{\nu . \nu} = \hat{\nu} . \hat{\nu}$

Consider $((\hat{\nu} . \hat{\nu})(\hat{f}))(x)(\alpha)$

$= (\widehat{\nu . \nu}(\hat{f}))(x)(\alpha)$

$= ((\nu . \nu)(f_\alpha))(x)$

$\leq (\mu(f_\alpha))(x)$

$= (\hat{\mu}(\hat{f}))(x)(\alpha)$

$\therefore \; \hat{\nu} . \hat{\nu} \leq \hat{\mu}$

(SHU7) Let $\hat{\mu} \in \hat{B}$

$\therefore \; \mu \in \mathcal{U}$

$\therefore \; \mu^{-1} \in \mathcal{U}$ where

$\mu^{-1}(g) = \Lambda \{h / \mu(h^C) \leq g^C\}$

$\therefore \; \widehat{(\mu^{-1})} \in \hat{B}$

To prove $(\hat{\mu})^{-1} \in \hat{B}$, in view of (SHU4), it is enough to prove

$\widehat{(\mu^{-1})} \leq (\hat{\mu})^{-1}$

To prove $\widehat{(\mu^{-1})} \leq (\hat{\mu})^{-1}$

(i.e.) to prove $\widehat{(\mu^{-1})}(\hat{g}) \leq (\hat{\mu})^{-1}(\hat{g})$ for all $\hat{g} \in (I^I)^X$

(i.e.) to prove $\widehat{(\mu^{-1})}(\hat{g}) \leq \hat{h}$ for all $\hat{h} \in (I^I)^X$

for which $\hat{\mu}((\hat{h})^C) \leq (\hat{g})^C$

$(\because (\hat{\mu})^{-1}(\hat{g}) = \Lambda(\hat{h} / \hat{\mu}((\hat{h})^C) \leq (\hat{g})^C\})$

(i.e.) to prove $(\widehat{(\mu^{-1})}(\hat{g}))(x)(\alpha) \leq \hat{h}(x)(\alpha)$, $\forall x \in X$, $\forall \alpha \in I$ and for all $\hat{h} \in (I^I)^X$ for which $\hat{\mu}((\hat{h})^C) \leq (\hat{g})^C)$

(i.e.) to prove $(\mu^{-1}(g_\alpha))(x) \leq h_\alpha(x)$, $\forall x \in X$ and for all $\hat{h} \in (I^I)^X$ for which $\hat{\mu}((\hat{h})^C) \leq (\hat{g})^C$

(i.e.) to prove $\mu^{-1}(g_\alpha) \leq h_\alpha$.

for all $\hat{h} \in (I^I)^X$ for which $\hat{\mu}((\hat{h})^C) \leq (\hat{g})^C$ (1)

Take $\hat{h} \in (I^I)^X$ satisfying $\hat{\mu}((\hat{h})^C) \leq (\hat{g})^C$

$\therefore (\hat{\mu}((\hat{h})^C))(x)(\alpha) \leq (\hat{g})^C(x)(\alpha)$ for every $\alpha \in I$, for every $x \in X$.

$\therefore \mu(h_\alpha^C)(x) \leq g_\alpha^C(x)$ for every $\alpha \in I$ and for every $x \in X$.

$\therefore \mu(h_\alpha^C) \leq g_\alpha^C$

$\therefore \mu^{-1}(g_\alpha) \leq h_\alpha$.

Hence (1) is true.

$\therefore \; \widehat{(\mu^{-1})} \leq (\hat{\mu})^{-1}$

$\therefore \; (\hat{\mu})^{-1} \in \hat{B}$

$\therefore \; \hat{\mathcal{U}}$ is a second order H-fuzzy uniformity on X.

Definition : 3.3

Let $(X, \hat{\mathcal{U}})$ be a second order H-fuzzy uniform space. Define int : $(I^I)^X \to (I^I)^X$ as follows :

int $\hat{f} = V\{\hat{g} \in (I^I)^X / \hat{\mu}(\hat{g}) \leq \hat{f}$, for some $\hat{\mu} \in \hat{\mathcal{U}}\}$

Theorem : 3.4

int : $(I^I)^X \to (I^I)^X$ is an interior operator.

Proof

(1) int $\hat{1} = \hat{1}$ is obvious.

(2) int $\hat{f} \leq \hat{f}$ is also trivially true.

(3) int (int $\hat{f}$) $\leq$ int $\hat{f}$ ($\because$ by (2))

Consider $\hat{g} \in (I^I)^X$ such that $\hat{\mu}(\hat{g}) \leq \hat{f}$ for some $\hat{\mu} \in \hat{\mathcal{U}}$. Since $\hat{\mu} \in \hat{\mathcal{U}}$, there exists $\hat{\nu} \in \hat{\mathcal{U}}$ such that $\hat{\nu} . \hat{\nu} \leq \hat{\mu}$

$\therefore \; \hat{\nu} . \hat{\nu}(\hat{g}) \leq \hat{\mu}(\hat{g}) \leq \hat{f}$

$\therefore \; \hat{\nu}(\hat{g}) \leq$ int $\hat{f}$

$\therefore \; \hat{g} \leq$ int (int $\hat{f}$)

$\therefore$ whenever $\hat{\mu}(\hat{g}) \leq \hat{f}$ for some $\hat{\mu} \in \hat{\mathcal{U}}$

then $\hat{g} \leq$ int (int $\hat{f}$)

$\therefore$ int $\hat{f} \leq$ int (int $\hat{f}$)

$\therefore$ int (int $\hat{f}$) = int $\hat{f}$

(4) Obviously int $(\hat{f} \Lambda \hat{g}) \leq$ (int $\hat{f}$) Λ (int $\hat{g}$)

Suppose int $(\hat{f} \Lambda \hat{g}) <$ (int $\hat{f}$) Λ (int $\hat{g}$), then there exist $x \in X$, $\alpha \in I$ such that

$t_1 = (\text{int}(\hat{f} \Lambda \hat{g}))(x)(\alpha) < t < ((\text{int}\hat{f}) \Lambda (\text{int}\hat{g}))(x)(\alpha) = t_2$

$t < t_2 \Rightarrow$ there exist $\hat{h}_1, \hat{h}_2 \in (I^I)^X$ such that $\hat{\mu}_1(\hat{h}_1) \leq \hat{f}$, $\hat{\mu}_2(\hat{h}_2) \leq \hat{g}$ for some $\hat{\mu}_1, \hat{\mu}_2 \in \hat{\mathcal{U}}$ and $\hat{h}_1(x)(\alpha) > t$, $\hat{h}_2(x)(\alpha) > t$

Let $\hat{\mu}_3 = \hat{\mu}_1 \Lambda \hat{\mu}_2$. $\therefore \hat{\mu}_3 \in \hat{\mathcal{U}}$

and $\hat{\mu}_3(\hat{h}_1 \Lambda \hat{h}_2) \leq \hat{f} \Lambda \hat{g}$

Also $(\hat{h}_1 \Lambda \hat{h}_2)(x)(\alpha) > t$

$\therefore$ There is a $\hat{\mu}_3 \in \hat{\mathcal{U}}$ such that $\hat{\mu}_3(\hat{h}_1 \Lambda \hat{h}_2) \leq \hat{f} \Lambda \hat{g}$ and $(\hat{h}_1 \Lambda \hat{h}_2)(x)(\alpha) > t$ (1)

$t_1 = (\text{int}(\hat{f} \Lambda \hat{g}))(x)(\alpha) < t \Rightarrow \forall \hat{h} \in (I^I)^X$ $\hat{h}(x)(\alpha) < t$ whenever $\hat{\mu}(\hat{h}) \leq \hat{f} \Lambda \hat{g}$ for some $\hat{\mu} \in \hat{\mathcal{U}}$ (2)

$\therefore$ (1) and (2) are contradictory.

$\therefore$ int $(\hat{f} \Lambda \hat{g})$ = (int $\hat{f}$) Λ (int $\hat{g}$)

$\therefore$ The function $\hat{f} \to$ int $\hat{f}$ is an interior operator.

Definition : 3.5

Let (X, $\hat{\mathcal{U}}$) be a second order H-fuzzy uniform space. The **second order fuzzy topology induced by** $\hat{\mathcal{U}}$ is denoted by $\hat{\delta}(\hat{\mathcal{U}})$ and it consists of all second order fuzzy sets $\hat{f} \in (I^I)^X$ for which $\hat{f} = \text{int } \hat{f}$.

Definition : 3.6

Let (X, $\hat{\mathcal{U}}_1$) and (Y, $\hat{\mathcal{U}}_2$) be two second order H-fuzzy uniform spaces. A function $\theta : X \to Y$ is said to be **second order H-uniformly continuous** if the following condition is satisfied : For every $\hat{\mu}_2 \in \hat{\mathcal{U}}_2$, there exists $\hat{\mu}_1 \in \hat{\mathcal{U}}_1$ such that $\hat{\mu}_1 \leq \theta^{-1}(\hat{\mu}_2)$. That is for every $\hat{f} \in (I^I)^X$,

$$\hat{\mu}_1(\hat{f}) \leq \theta^{-1}(\hat{\mu}_2(\theta(\hat{f}))).$$

Theorem : 3.7

Let (X, $\hat{\mathcal{U}}_1$) and (Y, $\hat{\mathcal{U}}_2$) be two second order H-fuzzy uniform spaces. Let $\theta : (X, \hat{\mathcal{U}}_1) \to (X, \hat{\mathcal{U}}_2)$ be a second order H-uniformly continuous onto function. Then

$\theta : (X, \hat{\delta}(\hat{\mathcal{U}}_1)) \to (Y, \hat{\delta}(\hat{\mathcal{U}}_2))$ is 2-f continuous.

Proof

Consider $\hat{g} \in \hat{\delta}(\hat{\mathcal{U}}_2)$

$\therefore \quad \hat{g} = \text{int } \hat{g}$

$= \vee \{ \hat{h} \in (I^I)^Y / \hat{\mu}_2(\hat{h}) \leq \hat{g}$ for some $\hat{\mu}_2 \in \hat{\mathcal{U}}_2 \}$

Let $\hat{h} \in (I^I)^Y$ such that $\hat{\mu}_2(\hat{h}) \leq \hat{g}$ for some $\hat{\mu}_2 \in \hat{\mathcal{U}}_2$.

$\therefore$ There exists $\hat{\mu}_1 \in \hat{\mathcal{U}}_1$ such that

$\hat{\mu}_1(\theta^{-1}(\hat{h})) \leq \theta^{-1}(\hat{\mu}_2(\theta(\theta^{-1}(\hat{h}))))$

($\because$ θ is second order H-uniformly continuous)

$\therefore \quad \hat{\mu}_1(\theta^{-1}(\hat{h})) \leq \theta^{-1}(\hat{\mu}_2(\hat{h}))$

$\leq \theta^{-1}(\hat{g})$

$\therefore \quad \theta^{-1}(\hat{h}) \quad \leq \text{int}(\theta^{-1}(\hat{g}))$

$\therefore \quad \vee \{\theta^{-1}(\hat{h}) / \hat{\mu}_2(\hat{h}) \leq \hat{g}$ for some $\hat{\mu}_2 \in \hat{\mathcal{U}}_2\}$

$\leq \quad \text{int}(\theta^{-1}(\hat{g}))$

$\theta^{-1}(\text{int } \hat{g}) \leq \text{int } \theta^{-1}(\hat{g})$

$$(\because \theta^{-1}(\bigvee_{\lambda \in \Lambda} \hat{f}_\lambda) = \bigvee_{\lambda \in \Lambda} \theta^{-1}(\hat{f}_\lambda))$$

$\therefore \quad \theta^{-1}(\hat{g}) \leq \text{int } \theta^{-1}(\hat{g})$ ($\because \hat{g} = \text{int } \hat{g}$)

$\therefore \quad \theta^{-1}(\hat{g}) = \text{int } \theta^{-1}(\hat{g})$

$\therefore \quad \theta^{-1}(\hat{g}) \in \hat{\delta}(\hat{\mathcal{U}}_1)$

$\therefore \quad \theta$ is 2-f continuous.

In Theorem 3.2, it is proved that every first order H-fuzzy uniformity $\mathcal{U}$ induces a second order H-fuzzy uniformity $\hat{\mathcal{U}}$. The following theorem shows that the map $\mathcal{U} \to \hat{\mathcal{U}}$ is functorial.

Theorem : 3.8

Let θ : (X, $\mathcal{U}_1$) $\to$ (Y, $\mathcal{U}_2$) be a H-uniformly continuous onto map. Then θ : (X, $\hat{\mathcal{U}}_1$) $\to$ (Y, $\hat{\mathcal{U}}_2$) is a second order H-uniformly continuous map.

Proof

For $\mu_2 \in \mathcal{U}_2$, there exists a $\mu_1 \in \mathcal{U}_1$ such that $\mu_1 \leq \theta^{-1}(\mu_2)$. For $x \in X$, $\alpha \in I$ and $\hat{f} \in (I^I)^X$, consider

$(\hat{\mu}_1(\hat{f}))(x)(\alpha)$

$= \quad \mu_1(f_\alpha)(x)$

$\leq \quad (\theta^{-1}(\mu_2(\theta(f_\alpha))))(x)$

$= \quad (\mu_2(\theta(f_\alpha)))\,\theta(x) \qquad (1)$

For $y \in Y$, $\alpha \in I$, $(\theta(\hat{f}))(y)(\alpha) = (\bigvee_{\theta(x)=y} \hat{f}(x))(\alpha)$

$= \bigvee_{\theta(x)=y} \hat{f}(x)(\alpha)$

$= \bigvee_{\theta(x)=y} f_\alpha(x)$

$= (\theta(f_\alpha))(y) \qquad (2)$

Therefore from (1) and (2),

$(\hat{\mu}_1(\hat{f}))(x)(\alpha) \leq (\hat{\mu}_2(\theta(\hat{f})))(\theta(x))(\alpha)$

$\leq (\theta^{-1}(\hat{\mu}_2(\theta(\hat{f}))))(x)(\alpha)$

$\therefore \quad \hat{\mu}_1(\hat{f}) \leq \theta^{-1}(\hat{\mu}_2(\theta(\hat{f})))$, for every $\hat{f} \in (I^I)^X$.

$\therefore \quad \theta$: (X, $\hat{\mathcal{U}}_1$) $\to$ (Y, $\hat{\mathcal{U}}_2$) is a second order H-uniformly continuous map.

IV. CONCLUSION

In this paper a second order fuzzy uniformity is constructed using the definition of fuzzy uniformity introduced by Hutton,B. It is proved that every second order fuzzy uniformity induces a second order fuzzy topology.

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