# One Period Binomial Model: The risk-neutral probability measure assumption and the state price deflator approach

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**Abstract**-*The aim of this paper is to find the value of the option that provides a payoff at some future date based on the value of a non-dividend paying share at the future date using a Binomial or lattice model to calculate the value of the derivative at time"$t = 0$". We will show that how to find the value of the derivative with the help of the risk-neutral probability ($Q$) measure and the state price deflator approach.In this report, we compare the value of a call option and put option under the risk neutral valuation (the $Q$ -measure) and the real world valuation (The $P$-measure). Under the risky neutral probability measure ($Q$ measure) we will see that the expectedreturn on the risky stock is the same as that on risk free investment cash and also it investigates what will happen to the state price deflator $A_t$ if$p > q$.*

*Key words: Call and Put option, risk-neutral probability, state price deflator ($A_t$) approach, and Binominal or lattice model, synthetic probability ($q$)*

## I. INTRODUCTION

The idea of an option is not new. Options are traded in Ancient Romans against outdoing cargoes from their seaport. The options are the main dynamic segment of the security market, since the origin of the Chicago Board option exchange (CBOF) in April, 1997. It is the largest option exchange in the world with more than one million contracts per day [1]. Option is a type of a derivative. It is defined as "It gives the holder the rights but not obligation to buy in case of call option or sell in case put option of an underlying asset at a fixed price (exercise or strike price) on the maturity/expiry date". Options are used for hedging and speculation [4]. Option model was exists in the market namely Binomial model developed by Cox-Ross-Rubinstein in 1979[2,3].BM is a simple and easy to understand. The Binominal model (BM) is the necessary techniques/methods that can be used to estimate the option pricing problems. BM is a simple statistical method. A very popular method that is used to calculate the price of options at $t = 0$ is to construct a binominal tree/lattice. In this paper we start to develop a single model that can be used to value of the call and put options using risk neutral valuation ($Q$ measure) and the state price deflator. The binominal graph represents the different paths, each path having some probability. If we talk about synthetic probability ($q$) then the graph moves up with probability $q$ and down $1 - q$ and if we talk about real world probability ($p$) then the graph moves up with probability $p$ and down $1 - p$.

It is mentioned that real world probability (p) is greater than the synthetic probability ($q$). Synthetic probability ($q$) is simple number defined as:

$$q = \frac{e^r - d}{u - d}$$

So, $q$ depends upon $r, d, u$ but not $p$.

Essentially the deflator are pricing kernel/ stochastic discount factor/ state price density/deflator/ state price deflator that are used in the "$P$" measure to maintain the arbitrage true, market consistency property.

In this paper we will look first at risk neutral of binominal model using no-arbitrage opportunity that is $d < e^r < u$ after that we will use the real world probability ($p$) that is state price deflator method to find the value of the option.

The state price deflator approach and risk neutral measure ($Q$ measure) plays an important role in any general equilibrium or arbitrage free model of asset price.In this paper we are going to find the value at an option at time $t = 0$, that is now of a put and call option that provides a payoff at time $t = 1$ (for call option payoff, $c_t = \max(0, S_t - k)$ and for put option payoff, $p_t = \max(0, k - S_t)$ based on the value of the stock at time $t = 1$.To find the value of the call and put option using"Q" measure and state price deflator that pays out an amount that depends directly on the value of the stock price at time $t = 1$ (say $S_1$).

In "$Q$" measure we are using only synthetic probability ($q$) but in state price deflator we are using both probabilities synthetic probability ($q$) and the real world probability ($p$). We will show that there is an any difference between the two methods for calculating the value of the an option using some examples and under "$Q$" measure, the expected return on the risky stock is same as that on a risk free investment in cash and under "$P$" measure the expected return on the stock will not normally be equal to the return on the risk free cash.

Here we will try to show why we are using no arbitrage opportunity in $Q$ measure. It is mention that that $p > q$ but sometimes it may not be right when arbitrage opportunity exist in the market. We will show some example that shows us that sometimes q>1 that is the new thing in this paper. If we talk about no arbitrage opportunity that is $d < e^r < u$ then q's value is between 0 and 1 for modelling the price of the option under no arbitrage opportunity. So we will try to describe the same assumption for the no arbitrage opportunity at r=5% in a different way.So it is only possible only when no arbitrage opportunity exist in the market. It is mentioned that $d < e^r < u$ is the no arbitrage opportunity. At last we will show some graphs that show how state price deflator $A_t$ increases/decreases when stock price goes up/down.

## II. DEFINITIONS

### A. *Derivative*

A derivative is a contract/security which gives us promise to make payment on some future date. It is financial tool whose value is depends on some underlying assets [4]. The underlying assets are: shares, bonds, index, interest rate, currency, commodity (gold, wheat etc).

### B. *Options*

It is contract between individuals or firms in which one party is ready to buy and another party is ready to sell. Option is a contract that gives the right to buy or sell any underlying asset at a fixed price/exercise/strike price that predetermined price of an underlying asset [3] on the future date.

There are two types of options:
**(i)** *Call option*: It gives the holder the right but not obligation to buy an underlying asset at a fixed price/exercise/strike price on the future date [7, 8].
*(ii) Put option*: It gives the holder the right but not obligation to sell an underlying asset at a fixed price/exercise/strike price on the future date [7, 8].

### C. *Option Style*

There are two styles of the options which are defined below:

*(i) American options:*American option is a contract that gives the holder right to buy or sell any underlying asset at a fixed price/exercise/strike price on before expiry date.
*(ii) European options:* A European Option is a contract that gives the right to buy or sell any underlying asset at a fixed price/exercise/strike price on the expiry/exercise date.
The only difference between an American and a European option is that with an American option the holder can exercise the contract early (before the expiry date) and in case of European option the holder can exercise only at expiry date. In this paper we are working only on European options [4].

### D. *Notations*

Below are some notations that are using in this paper:

t - The current time;
$S_t$ - The underlying share price at timet;
K - The strike/exercise price;
T - The option exercise date;
r- The interest rate;

### E. *Payoff*

The money realised by the holder of a derivative (an option) at the end of its life[9]. The profit made by the holder at the maturity date is known as payoff.

*(i) Payoff of European call option*: Let us consider a call option with $S_t$ is the price of the underlying at time t and fixed strike price $K$.

At expiry time T we have two different cases:

(a) At expiry time T, if the price of underlying asset $S_T$ is greater than strike price . The call option is then exercised I.e. the holder buys the underlying asset at price $K$ and immediately sell it in the market at price $S_T$, the holder realize the profit.

$$\text{I.e.} Payoff = S_T - k$$

(b) At expiry time T, if the price of underlying asset $S_T$ is less than strike price $K$. The call option then not exercised. The option expires worthless with $payoff = 0$.

Combine the above two cases, at time T. The value of a call option is given by a function

$$C_u = \max (S_T - k, 0)$$

Figure2.1 shows payoff of a European call option without any premium.

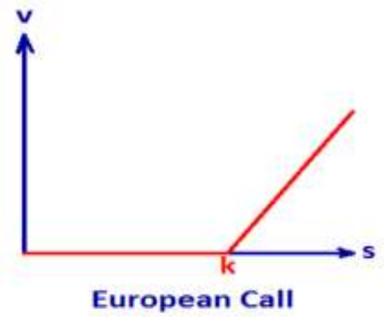



Figure 1: payoff of a European call option

Note: In call option the holder expect that the price of the underlying asset will increase in future [4].

*(ii) Payoff of European put option*: Let us consider a put option with $S_t$ is the price of the underlying at time t and fixed strike price $K$.

At expiry time T we have two different cases:

(a) At the expiry date T, if the price of the underlying asset $S_T$ is less than strike price $K$. Then the put option is exercised, i.e. the holder buys the underlying asset from the market price $S_T$ and sell it to the writer at price $K$. The holder realizes the profit of $K - S_T$.

(b) At expiry time T, if the price of underlying asset $S_T$ is greater than strike price $K$. The put option then not exercised. The option expires worthless with payoff=0.
Combine the above two cases, at time T. The value of a call option is given by a function

$$c_d = \max\,(k - S_T, 0)$$

Figure 2 shows payoff of a European put option without premium.

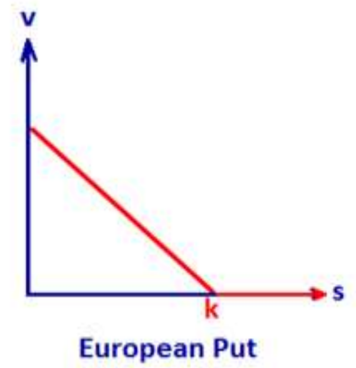



**Figure 2: Payoff of a European put option**

Note: The holder expect that the price of the underlying asset will decrease [4].

Holder of an option is the buyer while the writer is known as seller of the option. The writer grants the holder a right to buy or sell a particular underlying asset in exchange for certain money but in this paper we are working without premium for the obligation taken by him in the option contract.

### *F. Arbitrage*

Arbitrage means risk free trading profit, it is something described as a free breakfast.

Arbitrage opportunity exist if

(i) If we make immediate profit with probability of loss is zero.

(ii) With a zero cost initial investment if we can get money in future.

### *G. No arbitrage*

Noarbitrage means when arbitrage opportunity does not exist in the market. In this paper we are showing what it means.

## III. BINOMIAL MODEL

The binomial option pricing model is used to evaluate the price of an option. It was first proposed by Cox, Ross and Rubinstein in 1979. The BM is also known as Cox-Ross-Rubinstein (CRR) model because Cox, Ross and Rubinstein developed BM in year 1979 [4]. Options play a necessary role in financial market as a widely applied in financial derivatives [5] BM is a model to find the value if the derivative of an option at time $t = 0$ (at beginning) that provides a payoff at a future date based on the value of a non-dividend paying shares at a future date. The BM is based on the assumption that there is no arbitrage opportunity exist in the market. It is a popular model for pricing of options. The assumption is that the stock price follows a random walk [4]. In every step it has a certain probability of moving up or down.

### *A. One step Binomial model*

In one step binomial model, we assume the price of an asset today is So and that over a small interval $\Delta t$, it may move only to two values $S_o$*u or $S_o$*d where “u” represents the rise in stock price and “d” represents the fall in stock price. Probability “q” is assigned to the rise in the stock price and 1-q is assigned to the fall in the stock price. In simple words we can say that for a single time period one step binomial model is used [6].

The notations used above are:
$S_O$ = current stock price
u = the factor by which stock price rises
d = the factor by which stock price decreases
q = probability of rise in stock price
1-q = probability of fall in stock price

It is a graphical representation that depicts the future value of a stock at each final node in different time intervals. The value at each node depends upon the probability that the price of the stock will either increase or decrease as shown in figure 3.

Figure 3: One step Binominal model.

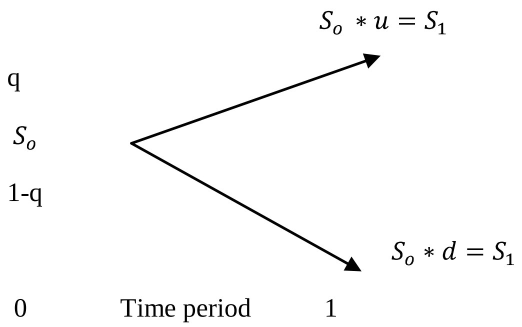



**Figure 3: one period Binominal model**

### B. *Value of the derivative at time $t = 0$*

We consider a one period binomial model shown as figure 1.1 we have two possibilities at time $t = 1$ i.e.

$$S_1 = \begin{cases} S_0 * u & \text{if the share price goes up} \\ S_0 * d & \text{if the share price goes down} \end{cases}$$

Here $u > 1$and $d < 1$. In order to avoid arbitrage we must have $d < e^r < u$

Where,

- $r$ is the risk free rate of interest

Let at $t = 0$, we hold a portfolio that consists of $\alpha$ unit of stock and $\beta$ unit of cash. So the value of the portfolio at time $t = 0$ is:

$$V_0 = \alpha * S_0 + \beta \dots\dots\dots\dots\dots(1)$$

The value of the portfolio at time $t = 1$ will be

$$V_1 = \begin{cases} \alpha * S_0 * u + \beta e^r & \text{if the share price goes up} \\ \alpha * S_0 * d \ + \beta e^r & \text{if the share price goes down} \end{cases} \dots(2)$$

Let the derivatives pays $c_u$ if the prices of the underlying stock go up and $c_d$ if the prices of the underlying stock go down.

Let us choose $\alpha$ *and* $\beta$ so that

$$V_1 = \begin{cases} c_u & \text{if the share price goes up} \\ c_d & \text{if the share price goes down} \end{cases} \dots\dots\dots(3)$$

So, from equation (2) and (3) we get

$$\alpha * S_0 * u + \beta e^r = c_u \dots\dots\dots\dots(4)$$

$$\alpha * S_0 * d \ + \beta e^r = c_d \dots\dots\dots\dots(5)$$

Subtract equation (4) and (5) we get

$$\alpha * S_0 * u - \alpha * S_0 * d \ = c_u - c_d$$

$$\alpha * S_0(u - d) = c_u - c_d$$

$$\boxed{\alpha = \frac{c_u - c_d}{S_0(u - d)}}$$

Put value of $\alpha$ in equation (4) we get

$$\boxed{\beta = e^{-r}\left(\frac{c_d * u - c_u * d}{u - d}\right)}$$

Substitute the value of $\alpha$ *and* $\beta$ in equation (1) we get

$$V_0 = e^{-r}\left\{c_u\left(\frac{e^r - d}{u - d}\right) + c_d\left(\frac{u - e^r}{u - d}\right)\right\}$$

$$V_0 = e^{-r}\left\{c_u\left(\frac{e^r - d}{u - d}\right) + c_d\left(1 - \frac{e^r - u}{u - d}\right)\right\}$$

$$\boxed{V_0 = e^{-r}(c_u * q + c_d(1 - q))}$$

Where, Synthetic probability (q) is simple number defined as:

$$\boxed{q = \frac{e^r - d}{u - d}}$$

If we denote the payoff of the options at time $t = 1$ by a random variable by $c_t$, we can write:

$$V_0 = e^{-r}E_Q(\, c_t)$$

Where Q is the probability which gives probability $q$ when stock price goes up and $1 - q$ when it goes down.

## IV. THE RISK NEUTRAL PROBABILITY MEASURE

Let us consider the real world probabilities of up and down move of the stock price. Suppose $p$ and $1 - p$ is the real world probability of up and down move respectively. This is defined as probability measure $P$.

Hence the expected stock price at time $t = 1$ under the real world probability measure P will be:

$$E_P(S_1) = S_O(p * u + (1 - p) * d)$$

Now, the expected stock value at time $t = 1$ under the synthetic probability $(q)$ measure (risk neutral probability measure) $Q$ is:

$$E_Q(S_1) = S_O(q * u + (1 - q) * d) \dots \dots \dots (6)$$

Where, q (Synthetic probability) is simple number defined as:

$$\boxed{q = \frac{e^r - d}{u - d}}$$

Put value of q in equation (6) we get

$$E_Q(S_1) =_{S_O} \left\{u * \left(\frac{e^r - d}{u - d}\right) + d * \left(1 - \frac{e^r - d}{u - d}\right)\right\}$$

$$= S_O \left\{\frac{e^r(u - d)}{u - d}\right\}$$

$$= S_O * e^r$$

Under $Q$ we see that the expected return on a risky stock is same as that on a risky free investment in cash. The $S_O * e^r$ simply shows us that it is accumulated value of an initial stock $S_O$ with risk free rate of return $r$ at time $t = 1$ and that is why $Q$ is sometimes called risk neutral probability measure.

If we talk about the real world probability $(p)$ the expected return on a stock price is not normally equal to the return on the risk free cash. Because under normal circumstances investor demand high return for accepting the risk in the stock price. So we would normally find $p > q$.

### A. *Prove that $p > q$*

Under normal circumstances investor demand high return for accepting the risk in the stock price. So, expected return calculated with respect to real world probability $(p)$ is greater than the risk neutral probability $(q)$ measure.

So,

$$E_p(S_1) > E_Q(S_1)$$

$$S_0(p * u + (1 - q) * d) > S_0(q * u + (1 - q) * d)$$

$$p * u - p * d > q * u - q * d$$

$$(p - q)(u - d) > 0$$

We know that $u > 1$ and $d < 1$ that means $u - d > 0$

Therefore $p > q$

## V. ASSUMPTION OF Q MEASURE

*Synthetic probability (q)*

We know that $0 < q < 1$but we will show that with the help of examples that sometimes $q > 1$. q Is only less than one if there is no arbitrage in the market.

That is the assumption for this model that consists that there is no arbitrage in the market after that we will show a general assumption where this model is not valid. This assumption is only for no arbitrage opportunity it is same as $d < e^r < u$ but we are describing in some new form. In below table we are using $r = 5\%$

*Table 1: The* synthetic *probability*

| S.No | interest rate goes up | $e^r$ | $d$ | $u$ | $e^r - d$ | $u - d$ | $q$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.001 | 1.051271 | 0.20248 | 1.001 | 0.848791 | 0.79852 | 1.062955 |
| 2 | 0.002 | 1.051271 | 0.285899 | 1.002 | 0.765372 | 0.716101 | 1.068805 |
| 3 | 0.003 | 1.051271 | 0.907215 | 1.003 | 0.144056 | 0.095785 | 1.503951 |
| 4 | 0.004 | 1.051271 | 0.262277 | 1.004 | 0.788994 | 0.741723 | 1.063731 |
| 5 | 0.005 | 1.051271 | 0.713755 | 1.005 | 0.337516 | 0.291245 | 1.158873 |
| 6 | 0.006 | 1.051271 | 0.780887 | 1.006 | 0.270384 | 0.225113 | 1.201104 |
| 7 | 0.007 | 1.051271 | 0.654379 | 1.007 | 0.396892 | 0.352621 | 1.125549 |
| 8 | 0.008 | 1.051271 | 0.327706 | 1.008 | 0.723565 | 0.680294 | 1.063606 |
| 9 | 0.009 | 1.051271 | 0.283002 | 1.009 | 0.768269 | 0.725998 | 1.058225 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10 | 0.01 | 1.051271 | 0.997309 | 1.01 | 0.053962 | 0.012691 | 4.252083 |
| 11 | 0.011 | 1.051271 | 0.76273 | 1.011 | 0.288542 | 0.24827 | 1.162207 |
| 12 | 0.012 | 1.051271 | 0.340827 | 1.012 | 0.710444 | 0.671173 | 1.058511 |
| 13 | 0.013 | 1.051271 | 0.347082 | 1.013 | 0.704189 | 0.665918 | 1.057471 |
| 14 | 0.014 | 1.051271 | 0.17645 | 1.014 | 0.874821 | 0.83755 | 1.0445 |
| 15 | 0.015 | 1.051271 | 0.80684 | 1.015 | 0.244431 | 0.20816 | 1.174246 |
| 16 | 0.016 | 1.051271 | 0.422485 | 1.016 | 0.628786 | 0.593515 | 1.059427 |
| 17 | 0.017 | 1.051271 | 0.796834 | 1.017 | 0.254437 | 0.220166 | 1.15566 |
| 18 | 0.018 | 1.051271 | 0.078233 | 1.018 | 0.973038 | 0.939767 | 1.035404 |
| 19 | 0.019 | 1.051271 | 0.928498 | 1.019 | 0.122773 | 0.090502 | 1.356578 |
| 20 | 0.02 | 1.051271 | 0.914483 | 1.02 | 0.136788 | 0.105517 | 1.296361 |
| 21 | 0.021 | 1.051271 | 0.574516 | 1.021 | 0.476755 | 0.446484 | 1.067799 |
| 22 | 0.022 | 1.051271 | 0.887236 | 1.022 | 0.164036 | 0.134764 | 1.217202 |
| 23 | 0.023 | 1.051271 | 0.67379 | 1.023 | 0.377481 | 0.34921 | 1.080957 |
| 24 | 0.024 | 1.051271 | 0.829 | 1.024 | 0.222271 | 0.195 | 1.139852 |
| 25 | 0.025 | 1.051271 | 0.809071 | 1.025 | 0.2422 | 0.215929 | 1.121666 |
| 26 | 0.026 | 1.051271 | 0.621841 | 1.026 | 0.42943 | 0.404159 | 1.062528 |
| 27 | 0.027 | 1.051271 | 0.473689 | 1.027 | 0.577582 | 0.553311 | 1.043865 |
| 28 | 0.028 | 1.051271 | 0.706995 | 1.028 | 0.344276 | 0.321005 | 1.072495 |
| 29 | 0.029 | 1.051271 | 0.6104 | 1.029 | 0.440872 | 0.4186 | 1.053204 |
| 30 | 0.03 | 1.051271 | 0.843682 | 1.03 | 0.207589 | 0.186318 | 1.114165 |
| 31 | 0.031 | 1.051271 | 0.480892 | 1.031 | 0.570379 | 0.550108 | 1.036849 |
| 32 | 0.032 | 1.051271 | 0.683669 | 1.032 | 0.367602 | 0.348331 | 1.055324 |
| 33 | 0.033 | 1.051271 | 0.068482 | 1.033 | 0.982789 | 0.964518 | 1.018943 |
| 34 | 0.034 | 1.051271 | 0.359776 | 1.034 | 0.691495 | 0.674224 | 1.025616 |
| 35 | 0.035 | 1.051271 | 0.099047 | 1.035 | 0.952224 | 0.935953 | 1.017385 |
| 36 | 0.036 | 1.051271 | 0.439995 | 1.036 | 0.611277 | 0.596005 | 1.025622 |
| 37 | 0.037 | 1.051271 | 0.954665 | 1.037 | 0.096606 | 0.082335 | 1.17333 |
| 38 | 0.038 | 1.051271 | 0.727796 | 1.038 | 0.323475 | 0.310204 | 1.042782 |
| 39 | 0.039 | 1.051271 | 0.385016 | 1.039 | 0.666255 | 0.653984 | 1.018764 |
| 40 | 0.04 | 1.051271 | 0.259191 | 1.04 | 0.792081 | 0.780809 | 1.014435 |
| 41 | 0.041 | 1.051271 | 0.304818 | 1.041 | 0.746453 | 0.736182 | 1.013952 |
| 42 | 0.042 | 1.051271 | 0.367446 | 1.042 | 0.683825 | 0.674554 | 1.013744 |
| 43 | 0.043 | 1.051271 | 0.297552 | 1.043 | 0.753719 | 0.745448 | 1.011095 |
| 44 | 0.044 | 1.051271 | 0.299111 | 1.044 | 0.75216 | 0.744889 | 1.009761 |
| 45 | 0.045 | 1.051271 | 0.157739 | 1.045 | 0.893532 | 0.887261 | 1.007068 |
| 46 | 0.046 | 1.051271 | 0.92595 | 1.046 | 0.125321 | 0.12005 | 1.043908 |
| 47 | 0.047 | 1.051271 | 0.006664 | 1.047 | 1.044607 | 1.040336 | 1.004105 |
| 48 | 0.048 | 1.051271 | 0.713739 | 1.048 | 0.337533 | 0.334261 | 1.009786 |
| 49 | 0.049 | 1.051271 | 0.671612 | 1.049 | 0.379659 | 0.377388 | 1.006018 |
| 50 | 0.05 | 1.051271 | 0.420566 | 1.05 | 0.630705 | 0.629434 | 1.002019 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 51 | 0.051 | 1.051271 | 0.808827 | 1.051 | 0.242445 | 0.242173 | 1.001119 |
| 52 | 0.052 | 1.051271 | 0.767593 | 1.052 | 0.283678 | 0.284407 | 0.997437 |
| 53 | 0.053 | 1.051271 | 0.119542 | 1.053 | 0.931729 | 0.933458 | 0.998148 |
| 54 | 0.054 | 1.051271 | 0.784909 | 1.054 | 0.266363 | 0.269091 | 0.989859 |

If we look at the table 1 that shows that every value of $q$ is greater than one at $r = 5\%$ except the last two entries. When we use the $Q$ measure for finding the value of the option using no arbitrage opportunity that is $d < e^r < u$ the main aim of this assumption is that:Synthetic probability: $q < 1$

Under the no arbitrage opportunity $(d < e^r < u)$ to find the value of the derivatives the synthetic probability $(q)$ must be less than 1. If there is the arbitrage in the market then synthetic probability $(q)$ must be greater than one. Above table1 shows us that $q > 1$ that is arbitrage opportunity. We can say that if there is arbitrage in the market then we can't use Q measure to find the value of the derivatives because the synthetic probability is greater than one that is not accepted (because probability won't be greater than one). we can say that when $q > 1$ then there is an arbitrage in the market.

**Example for Risk neutral probability for finding value of the option**

Let us consider a one step binominal model of stock price 100 at time $t = 0$.

Suppose that: Over a single period the stock price goes up with 120 and down 90.

Here we will find the value of the European call and put option v0, that has strike price 110.

The real world probability is 0.6 and we are assuming that $r = 5\%$

**Solution:**

We will draw a one step binominal model shown as below$S_1 = S_0 * u = 120$

$S_0 = 100$

$S_1 = S_0 * d = 90$

For calculating the option price first we have to see that $e^r$ is lying between $u$ and $d$ that is $d < e^r < u$ if it is lying between then there is no arbitrage in the market

We like wise consider a binominal tree in respect of payoff for call option $(c_1 = c_u)$ and put option $(c_1 = c_d)$ at time $t = 1$.

First we will draw for call option and the payoff is $c_t = \max(0, S_t - k)$

$$c_1 = c_u = 120 - 110 = 10$$

$V_0/c_0$

$$c_1 = c_d = 0$$

In order to find the value of call option $(V_0/c_0)$ we have to find $u, d, q, 1-q$

Therefore$u = \frac{120}{100} = 1.2$, $d = \frac{90}{100} = 0.9$,

$$q = \frac{e^r - d}{u - d} = 0.50423$$

Here it is satisfying the inequality $d < e^r < u$

The value of the call option is:

$$V_0 = e^{-r}(c_u * q + c_d * (1-q))$$

$$= e^{-0.05}(10 * 0.50423 + 0 * (1 - 0.50423))$$

$$= 4.796$$

**Note**

- We didn't use the real world probability p to find the value of the derivatives. $V_0$ Is independent on $p$.
- $p > q$
- If we use the real world probability $(p = 0.6)$ to find the value of the derivative (for call option)

$$V_0^{/} = e^{-r}(c_u * p + c_d * (1-p))$$
$$= e^{-0.05}(10 * 0.6 + 0 * (1 - 0.6))$$
$$= 5.70$$

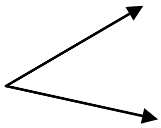
Note that $V_0^{/} > V_0$. This is because of $p > q$

**Put option**

We will draw for put option and the payoff is
$p_t = \max(0, k - S_t)$

$$p_1 = p = 0$$

$V_0/c_0$



$$p_1 = p_d = 110 - 90 = 20$$

Now we will find the value of put option $(V_0/c_0)$

The value of the put option is:

$$V_0 = e^{-r}(p_u * q + p * (1 - q))$$

$$= e^{-0.05}(0 * 0.50423 + 20 * (1 - 0.50423))$$

$$= 10.86$$

## VI. THE STATE PRICE DEFLATOR APPROACH

### A. *One period model*

We have the one period binominal model as

$$V_1 = \begin{cases} c_u\,; & q \\ c_d\,; & 1-q \end{cases}$$

$$V_0 = E_Q[c_1] = e^{-r}E_Q[V_1]$$

$$= e^{-r}[c_u * q + c_d * (1 - q)]$$

Now we can re express their values in terms of real world probability

$$V_0 = e^{-r}\left[c_u * p * \frac{q}{p} + c_d * (1 - q) * \frac{(1-p)}{(1-p)}\right]$$

$$V_0 = p * \left(\frac{e^{-r} * q}{p}\right) * c_u + (1 - p) * \frac{e^{-r} * (1-q)}{1-p} * c_d$$

$$= E_P\,[A_V * V_1]$$

Where $A_V$ is a random variable with

$$A_V = \begin{cases} e^{-r} * \dfrac{q}{p}\,; & p \\ e^{-r} * \dfrac{(1-q)}{(1-p)}; & 1-p \end{cases}$$

However, in this case we are using real world probabilities and a different discount factor. The discount factor $A_V$ depends on whether the stock price goes up or down. This means that it is a random variable and so we call it stochastic discount factor.$A_V$ Is called state price deflator and it is also known as:

- Deflator
- State price density
- Pricing kernel

**Example (same as above)**

Here we will find the value of the option using state price deflator

We know that

$$V_0 = p * \left(\frac{e^{-r} * q}{p}\right) * c_u + (1 - p) * \frac{e^{-r} * (1-q)}{1-p} * c_d$$

Every parameter are given in previous example we just substitute the values here

**For call option**

$$V_0 = 0.6 * \left(\frac{e^{-0.05} * 0.50423}{0.6}\right) * 10 + (1 - 0.6) * \frac{e^{-0.05} * (1 - 0.50423)}{1 - 0.6} * 0$$

$$= 0.50423$$

**For put option**

$$V_0 = 0.6 * \left(\frac{e^{-0.05} * 0.50423}{0.6}\right) * 0 + (1 - 0.6) * \frac{e^{-0.05} * (1 - 0.50423)}{1 - 0.6} * 20$$

$$= 10.86$$

*Note: Both risk neutral approach method and the state price deflator method is giving the same value for both the call and put option as shown in previous examples.*

**The state price deflator takes higher value if the stock value goes down**

Here we are using some examples shown in a below table 2 and assume that r=5 and p=0.7.

Table 2: The state price deflator

| $S_0$ | $S_0 * u$ | $S_0 * d$ | $q$ | $p$ | $e^{-r}$ | $A_V; p$ | $A_V; 1-p$ |
|---|---|---|---|---|---|---|---|
| **100** | 120 | 90 | 0.504237 | 0.7 | 0.951229 | 0.685207 | 1.571948 |
| **120** | 145 | 105 | 0.528813 | 0.7 | 0.951229 | 0.718604 | 1.494022 |
| **135** | 150 | 125 | 0.676864 | 0.7 | 0.951229 | 0.91979 | 1.024588 |
| **160** | 190 | 140 | 0.564068 | 0.7 | 0.951229 | 0.766511 | 1.382239 |

The above tables shows us that $A_V; 1-p$ have higher value than$A_V; p$

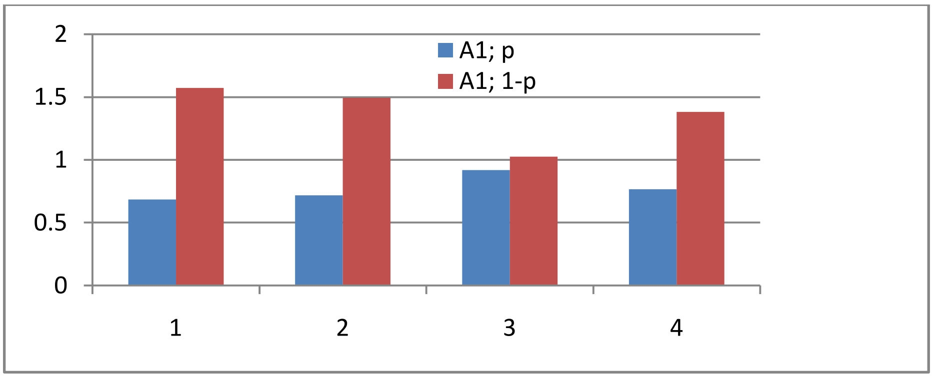



**Figure 4: price deflator**

The Figure 4 shows us that the when the stock price goes down then the state price deflator $A_V; 1-p$ gives higher value than $A_V; p$

***B.*** *Mathematically*

If the stock price goes up then the stock price deflator $A_V$ with probability p is (as shown above)

$$e^{-r} * \frac{q}{p}$$

If the stock price goes down then the stock price deflator $A_V$ with probability 1-p is (as shown above)

$$e^{-r} * \frac{(1-q)}{(1-p)}$$

So, we know that $p > q$

Then $A_V$ with probability $1-p$ will take higher value if the stock price goes down and $A_V$ with probability $p$ will take lower value if the stock price goes up. As shown in above figure.

**CONCLUSION**

*Summary of Results*

*In this paper:*

1. We obtained the same value for call and put options using the following methods:

- The Risk-Neutral approach
- The Real World with Deflators

2. In "$Q$" measure we are using only synthetic probability ($q$) but in state price deflator we are using both probabilities synthetic probability ($q$) and the real world probability ($p$) but both methods give us same result.

3. The difference between the two methods is:

- The value of the an option using "$Q$" measure, expected return on the stock is same as that on a risk free investment in cash and under "$P$" measure the expected return on the stock will not normally be equal to the return on the risk free cash.

4. Under normal circumstances investor demand high return for accepting the risk in the stock price. So, expected return calculated with respect to real world probability ($p$) is greater than the risk neutral probability ($q$) measure

5. Under$Q$ measure the assumption for no arbitrage is the synthetic probability should be less than

$1(q < 1)$. So we can't use the Q measure when $q > 1$

6. When the stock price goes down then the state price deflator $A_V; 1 - p$ gives higher value than $A_V; p$

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