Interpolation By Inverse Cubic Spline Method

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Abstract — A formula for inverse cubic spline is derived by applying a method called reversion of series and its error is also analyzed.

Keywords — Natural cubic spline, Rolle's Theorem, Series Reversion.

I. Introduction

The process of constructing a function say g(x) which satisfies the given set of data points $(x, f(x))$ is called interpolation. If the interpolated function is a polynomial it is known as a polynomial interpolation. A cubic spline consists of third degree polynomial bits joined together. [2], [3]

Consider a function $y = f(x)$ satisfied by a set of data points (a_i, y_i) , $i = 1,2,\ldots,n$. Let $I_i = [a_{i-1}, a_i]$, $i = 2,3,\ldots,n$ be a subinterval of $I = [a,b]$. A cubic spline S_i (x) is a polynomial of degree 3 and having continuous derivatives up to order 2.

We have ,
$$S_i(x) = \begin{cases} S_1(x), a_1 < x < a_2 \\ S_2(x), a_2 < x < a_3 \\ , \\ , \\ , \\ S_n(x), a_{n-1} < x < a_n \end{cases}$$

The conditions for the natural cubic spline are

(i) $S_1(x), S_2(x), ---, S_n(x)$ are all almost cubic

(ii) $S_i(x_i) = y_i$, $i = 1,2,..,n$

(iii) $S_i(x), S_i'(x)$ *and* $S_i''(x)$ are continuous in $I = [a,b]$

(iv) $S_i''(a) = S_i''(b) = 0$

Since S(x) is a cubic spline , [1],[5] $S''(x)$ is linear and continuous.

Let $S_{i-1}''(x) = M_{i-1}\dfrac{a_i - x}{h_i} + M_i\dfrac{x - a_{i-1}}{h_i}$

Integrating twice with respect to x

$$S_{i-1}(x) = M_{i-1}\frac{(a_i - x)^3}{6h_i} + M_i\frac{(x - a_{i-1})^3}{6h_i} + C_i(a_i - x) + d_i(x - a_{i-1})$$

Applying the conditions $S_{i-1}(x_{i-1}) = y_{i-1}$ and $S_i(x_i) = y_i$

$$S_{i-1}(x) = M_{i-1}\frac{(a_i - x)^3}{6h_i} + M_i\frac{(x - a_{i-1})^3}{6h_i} + \left(y_{i-1} - \frac{M_{i-1}h_i^2}{6}\right)\left(\frac{a_i - x}{h_i}\right) + \left(y_i - \frac{M_i h_i^2}{6}\right)\left(\frac{x - a_{i-1}}{h_i}\right), x \in [a_{i-1}, a_i]$$

------------------ (1)

Where $h_i = a_i - a_{i-1}$, $i = 2,3,---,n$ which is the cubic spline equation.

The M_i are obtained by the equations:

$$\left.\begin{aligned} &\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i,\ i = 1,2\ldots,n \\ &2M_1 + \lambda_1 M_2 = d_1 \\ &\mu_n M_{n-1} + 2M_n = d_n \end{aligned}\right\} \quad (2)$$

Where $\sigma_i = \dfrac{y_i - y_{i-1}}{h_i}$, i = 2,3,…..,n --------- (3)

$$\left.\begin{aligned} \lambda_i &= \frac{h_{i+1}}{h_i + h_{i+1}} \\ \mu_i &= 1 - \lambda_i \\ d_i &= \frac{6(\sigma_{i+1} - \sigma_i)}{h_i + h_{i+1}} \end{aligned}\right\} \quad \text{---------- (4)}$$

i = 2,3,….,(n-1)

II. SERIES REVERSION METHOD

Method of series reversion [6] can be applied to the cubic polynomial

$y = a_0 + a_1x + a_2x^2 + a_3x^3$ ------- (5)

Then $Y = a_1x + a_2x^2 + a_3x^3$ ------- (6) where

$Y = y - a_0$

Inverting the above

$x = A_1Y + A_2Y^2 + A_3Y^3$ ----------- (7)

Substituting (6) in (7), and equating we get,

$$A_1 = \frac{1}{a_1}$$

$$A_2 = \frac{-a_2}{a_1^3}$$

$$A_3 = \frac{2a_2^2 - a_3a_1}{a_1^5}$$

Hence the inverted cubic in Y is

$$x = \frac{1}{a_1}Y + \frac{-a_2}{a_1^3}Y^2 + \frac{2a_2^2 - a_3a_1}{a_1^5}Y^3$$

$$x = \frac{1}{a_1}(y - a_0) + \frac{-a_2}{a_1^3}(y - a_0)^2 + \frac{2a_2^2 - a_3a_1}{a_1^5}(y - a_0)^3$$

III. INVERSE CUBIC SPLINE

The natural cubic spline interpolation formula is

$$S_{i-1}(x) = M_{i-1}\frac{(a_i - x)^3}{6h_i} + M_i\frac{(x - a_{i-1})^3}{6h_i} + \left(y_{i-1} - \frac{M_{i-1}h_i^2}{6}\right)\left(\frac{a_i - x}{h_i}\right) + \left(y_i - \frac{M_ih_i^2}{6}\right)\left(\frac{x - a_{i-1}}{h_i}\right), x \in [a_{i-1}, a_i]$$

i = 2, 3, …., (n-1)

$$S_{i-1}(x) = M_{i-1}\frac{X_i^3}{6h_i} + M_i\frac{(h_i - X_i)^3}{6h_i} + \left(y_{i-1} - \frac{M_{i-1}h_i^2}{6}\right)\left(\frac{X_i}{h_i}\right) + \left(y_i - \frac{M_ih_i^2}{6}\right)\left(\frac{h_i - X_i}{h_i}\right)$$

Where $a_i - x = X_i$

$x - a_{i-1} = h_i - X_i$

$$S_{i-1}(x) = \frac{1}{6h_i}\left[\begin{array}{l}(M_{i-1} - M_i)X_i^3 + \\ 3M_ih_iX_i^2 - \\ 2M_ih_i^2X_i - M_{i-1}h_i^2X_i + \\ (6y_{i-1} - 6y_i)X_i + 6y_ih_i\end{array}\right]$$

$$S_{i-1}(x) = \frac{\alpha_{i-1}}{6h_i}X_i^3 + \frac{M_i}{2}X_i^2 + \left[\frac{\beta_{i-1}}{h_i} - (2M_i + M_{i-1})\frac{h_i}{6}\right]X_i + y_i$$

Where

$$\left.\begin{array}{l} M_{i-1} - M_i = \alpha_{i-1} \\ y_{i-1} - y_i = \beta_{i-1}\end{array}\right., \text{ i = 2,3,…..,n} \quad \text{----(8)}$$

Substitute

$$\left.\begin{array}{l} p_i = \left[\frac{\beta_{i-1}}{h_i} - (2M_i + M_{i-1})\frac{h_i}{6}\right] \\ q_i = \frac{M_i}{2} \\ r_i = \frac{\alpha_{i-1}}{6h_i}\end{array}\right\} \text{ i = 2,3,----,n} \quad \text{----------- (9)}$$

Hence the Natural cubic spline is

$$S_{i-1}(x) = y_i + p_iX_i + q_iX_i^2 + r_iX_i^3,$$

i = 2,3,---,n

By applying reversion of series method to the above equation inverse cubic spline is

$$S_{i-1}^{-1}(y) = a_i - \left[\begin{array}{l}\frac{1}{p_i}(y - y_i) - \frac{q_i}{p_i^3}(y - y_i)^2 \\ - \frac{2q_i^2 - p_ir_i}{p_i^5}(y - y_i)^3\end{array}\right]$$

Where $y \in [y_{i-1}, y_i]$, i = 2,3,…..,n ---------(10)

IV. ERROR ANALYSIS OF INVERSE CUBIC SPLINE

Theorem :

Let $S_n^{-1}(y) \in \pi_n$ (π_n - set of all polynomials not exceeding n), which interpolates at $f^{-1}(y)$ at $(n+1)$ distinct points say $y_0, y_1, ..., y_n \in [c,d]$. Then $\forall y \in [c,d]$ there exist a point $e_n \in (c,d)$ Such that $f^{-1}(y) - S_n^{-1}(y) =$

$$\frac{1}{(n+1)!}\left(\frac{d^{n+1}}{dy^{n+1}} f^{-1}(e_n)\right) \mathop{\pi}_{j=0}^{n} (y - y_j) \quad \text{--------------- (11)}$$

Proof

Given $S_n^{-1}(y) \in \pi_n$ is the approximating polynomial of $f^{-1}(y)$ at (n+1) points say $y_0, y_1, ..., y_n$. If y is one of the interpolation point then LHS and RHS of (11) are both zero. Therefore (11) is satisfied trivially.

Suppose $y \neq y_j, 0 \leq j \leq n$ then $\mathop{\pi}_{j=0}^{n}(y - y_j) \neq 0$

So let $q(y) = \mathop{\pi}_{j=0}^{n}(y - y_j)$

Construct a function $F(y) = f^{-1}(y) - S_n^{-1}(y) - \lambda q(y)$ where λ is chosen so that $F(y) = 0$.

Therefore $\lambda = \dfrac{f^{-1}(y) - S_n^{-1}(y)}{q(y)}$

Since $y \neq y_j$ gives $q(y) \neq 0$, λ is well defined. Since F(y) is a function which is (n + 1) times differentiable in $[c,d]$ and F(y) vanishes at (n + 1) Points $y_0, y_1, ..., y_n$ in (c , d) by Rolle's theorem[4] there exist a number e_n in (c , d) such that $\dfrac{d^{n+1}}{dy^{n+1}} F(e_n) = 0$

Hence

$$0 = \frac{d^{n+1}}{dy^{n+1}} F(e_n) = \frac{d^{n+1}}{dy^{n+1}}(f^{-1}(e_n)) - \frac{d^{n+1}}{dy^{n+1}}(S_n^{-1}(e_n)) - \lambda \frac{d^{n+1}}{dy^{n+1}}(q(e_n))$$

$$= \frac{d^{n+1}}{dy^{n+1}}(f^{-1}(e_n)) - \frac{f^{-1}(y) - S_n^{-1}(y)}{q(y)}(n+1)!$$

Hence $f^{-1}(y) - S_n^{-1}(y) =$

$$\frac{1}{(n+1)!}\left(\frac{d^{n+1}}{dy^{n+1}} f^{-1}(e_n)\right) \mathop{\pi}_{j=0}^{n} (y - y_j)$$

V. ILLUSTRATION

Consider the data points [7] (a_i, y_i)=(0.25,0.5), (0.3,0.5477) , (0.39,0.6245) , (0.45,0.6708), (0.53, 0.7280)

$h_2 = 0.05, h_3 = 0.09, h_4 = 0.06, h_5 = 0.08$

$\sigma_2 = 0.9540$ $\quad \lambda_2 = \dfrac{9}{14}$

$\sigma_3 = 0.8533$ $\quad \lambda_3 = \dfrac{2}{5}$

$\sigma_4 = 0.7717$ $\quad \lambda_4 = \dfrac{4}{7}$

$\sigma_5 = 0.7150$

$\mu_2 = \dfrac{5}{14}$ $\quad d_2 = -4.3157$

$\mu_3 = \dfrac{3}{5}$ $\quad d_3 = -3.2640$

$\mu_4 = \dfrac{3}{7}$ $\quad d_4 = 2.4300$

For natural spline S(x), $M_1 = M_5 = 0$

$M_2 = -1.8806$ $\quad M_3 = -0.8226$

$M_4 = -1.0261$

$p_2 = -0.922657$ $\quad q_2 = -0.9403$

$p_3 = -0.8004463$ $\quad q_3 = -0.4113$

$p_4 = -0.74292$ $\quad q_4 = -0.51305$

$p_5 = -0.7013$ $\quad q_5 = 0$

$r_2 = 6.268667$

$r_3 = -1.959259259$

$r_4 = 0.565278$

$r_5 = -2.13771$

Hence the inverse cubic spline is

$$S_1^{-1}(y) = -11.28y^3 + 19.81y^2 - 10.472y + 1.943 \quad , \; y \in [0.5, 0.5477]$$

$$S_2^{-1}(y) = 3.75y^3 - 6.225y^2 + 4.635y - 0.993 , \quad y \in [0.5477, 0.6245]$$

$$S_3^{-1}(y) = -4.18y^3 + 9.6612y^2 - 5.972y + 1.3694 , \quad y \in [0.6245, 0.6708]$$

$$S_4^{-1}(y) = 8.847y^3 - 19.32y^2 + 15.493y - 3.923, \quad y \in [0.6708, 0.7280]$$

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