A Review on Graphs with Unique Minimum Dominating Sets

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ABSTRACT – A dominating set for a graph G is a subset D of V such that every vertex not in D is adjacent to at least one member of D. This paper deals with some of the graphs having unique minimum dominating sets. We also find a unique minimum dominating sets for block graphs and maximum graphs.

Keywords: Domination sets, Block graphs, Unique Υ set, Unique minimum Domination sets.

1. INTRODUCTION

Graphs can be used to model many types of relations and processes in physical, biological, social and information systems. Many practical problems can be represented by graphs. Emphasizing their application to real-world systems, the term network is sometimes defined to mean a graph in which attributes (e.g. names) are associated with the nodes and/or edges. In computer science, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation, etc. For instance, the link structure of a website can be represented by a directed graph, in which the vertices represent web pages and directed edges represent links from one page to another. A similar approach can be taken to problems in social media, travel, biology, computer chip design, and many other fields. The development of algorithms to handle graphs is therefore of major interest in computer science. The transformation of graphs is often formalized and represented by graph rewrite systems. Complementary to graph transformation systems focusing on rule-based in-memory manipulation of graphs are graph databases geared towards transaction-safe, persistent storing and querying of graph-structured data.

Graph theory is also used to study molecules in chemistry and physics. In condensed matter physics, the three-dimensional structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. In chemistry a graph makes a natural model for a molecule, where vertices represent atoms and edges bonds. This approach is especially used in computer processing of molecular structures, ranging from chemical editors to database searching.

In statistical physics, graphs can represent local connections between interacting parts of a system, as well as the dynamics of a physical process on such systems. Similarly, in computational neuroscience graphs can be used to represent functional connections between brain areas that interact to give rise to various cognitive processes, where the vertices represent different areas of the brain and the edges represent the connections between those areas. Graphs are also used to represent the micro-scale channels of porous media, in which the vertices represent the pores and the edges represent the smaller channels connecting the pores.

2. GRAPHS WITH UNIQUE MINIMUM DOMINATING SETS

Lemma: Let D be a γ-set of a graph G. Suppose for every $x \in D$, $\gamma(D - x) > \gamma(G)$. Then D is the unique γ-set of G.

Proof : Suppose there exists a second γ-set D' of G. If $D \neq D'$, choose $a \in D - D'$. Now D' dominates G, and hence D' certainly dominates $G - a$, so that $|D'| \geq \gamma(G - a)$. However, $\gamma(G - a) > \gamma(G) = |D| = |D'| \geq \gamma(G - a)$, which is a contradiction.

We see that there are three conditions of interest

(i) G has a unique γ-set D.

(ii) G has a γ-set D for which every vertex in D has at least two private neighbours other than itself.

(iii) G has a γ-set D for which every vertex $x \in D$ satisfies $\gamma(G - x) > \gamma(G)$.

As Lemmas 3.2.2 and 3.2.3 show, for all graphs G we have (i) ⇒ (ii) and (iii) ⇒ (i), the converse of those, however, is false, as the following examples sh

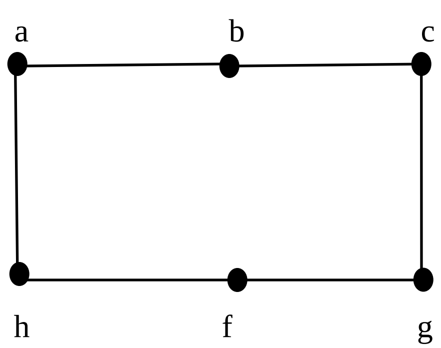

Figure – 1: Connected Graph

C_6 shown above in Figure-1 has a γ-set $D = \{a, d\}$ both vertices in D have two private neighbours; however, D is not a unique γ-set.

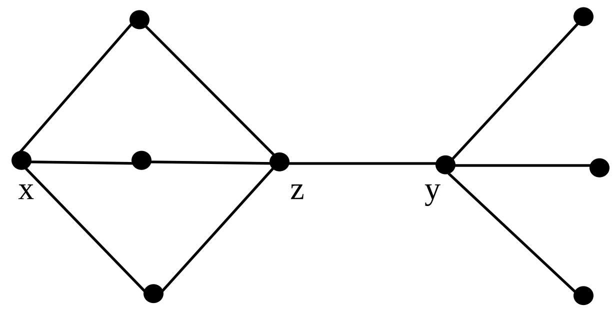

Figure-2: Connected Graph

The graph shown in Figure-2 has the unique γ-set $D = \{x, y\}$. If we delete x,the resultant graph still has domination number 2 as it is dominated by $\{x, y\}$.

Lemma : Let G be a graph which has a unique γ-set D.Then for any $x \in G - D$,$\gamma(G - x) = \gamma(G)$.

Proof : Certainly D dominates $G - x$ and so $\gamma(G - x) \leq \gamma(G)$. If $\gamma(G - x) < \gamma(G)$,then $G - x$ is dominated by some set D' with $|D'| < |D|$. But then $D' \cup \{x\}$ would be a second γ-set for G, different from D, contradicting the uniqueness of D.

Lemma: Let G be a graph with unique γ-set D. Then $\gamma(G - x) \geq \gamma(G)$for all $x \in D$.

Proof : Suppose $\gamma(G - x) < \gamma(G)$for some $x \in D$. Let D' be a γ-set of $G - x$.Then$|D'| < |D|$ and D' dominates all the private neighbours of x with respect to D, other than x. But now the set $D' \cup \{x\}$ is a γ-set of G in which x has only itself as a private neighbours, a contradiction to lemma 3.2.2.

3. MAXIMUM GRAPHS WITH A UNIQUE MINIMUM DOMINATING SET

Theorem: Let G = (V, E) be a graph without isolated vertices with a unique minimum dominating set of cardinality $\gamma \geq 2$ and order $n = 3\gamma$. Then

$$m = |E| \leq \binom{n}{2} - \gamma\left(n + \frac{\gamma - 5}{2}\right)$$
$$= 2\gamma + 2\binom{\gamma}{2}.$$

Proof : Let $D = \{x_1, x_2, \cdots, x_\gamma\}$ be the unique minimum dominating set of G and let $P_i = epn\,(x_i, D, G)$ for $1 \leq i \leq \gamma$. Since $|P_i| \geq 2$ for$1 \leq i \leq \gamma$ and $n = 3\gamma$, we have $|P_i| = 2$ for $1 \leq i \leq \gamma$. Let $P_i = \{p_i^{'}, p_i^{"}\}$for $1 \leq i \leq \gamma$. If there is some $1 \leq i \leq \gamma$such that $p_i^{'}, p_i^{"} \in E$, then $(D\backslash\{x_i\}) \cup \{p_i^{'}\} \neq D$is a minimum dominating set of G, which is a contradiction.

If there are some $1 \leq j < k \leq \gamma$ such that there are two independent edges between P_i and P_j, say $p_i^{'}p_j^{'}, p_i^{"}p_j^{"} \in E$,then $\left(D\backslash\{x_i, x_j\}\right) \cup \{p_i^{'}, p_j^{"}\} \neq D$ is a minimum dominating set of G, which is a contradiction.

If there are some$1 \leq i < j \leq \gamma$ such that $x_i x_j, p_i^{'}p_j^{'}$ and $p_i^{'}p_j^{"} \in E$, then $(D\backslash\{x_j\}) \cup \{p_i^{'}\} \neq D$ is a minimum dominating set of G, which is a contradiction. This implies that for all $1 \leq i < j \leq \gamma$ there are at most two edges between P_i and P_j and if there are two such edges, then they are incident. Furthermore, if $x_i x_j \in E$, thenthere is at most one edge between P_i and P_j . Let v_l for $l \geq 0$ be the number of pairs $\{i, j\}$ with $1 \leq i < j \leq \gamma$ such that there are exactly $l$edges between P_i and P_j. By the above reasonings, we obtain that $v_l = 0$ for all $l \geq 3$ and $m(G[D]) \leq v_0 + v_1$.This implies that $m = |E| = 2\gamma + m(G[D]) + 0.v_0 + 1.v_1 + 2.v_2$

$$\leq 2\gamma + v_0 + v_1 + 0.v_0 + 1.v_1 + 2.v_2$$

$$\leq 2\gamma + 2(v_0 + v_1 + v_2)$$

$$= 2\gamma + 2\binom{\gamma}{2}$$

This completes the proof.

Theorem: If $\gamma \geq 2$,then $\widetilde{m}(n,\gamma) = \binom{n}{2} - \gamma\left(n + \frac{\gamma-5}{2}\right)$.

Proof : We prove that $\widetilde{m}(n,\gamma) \leq \binom{n}{2} - \gamma\left(n + \frac{\gamma-5}{2}\right)$. Therefore, let G be a graph of order n without isolated vertices that has domination number γ and property($*$).

Let $D = \{x_1, x_2, \cdots, x_\gamma\}$ be the unique minimum dominating set and for $1 \leq i \leq \gamma$ let $P_i = epn\,(x_i, D, G)$. As above $|P_i| \geq 2$ for $1 \leq i \leq \gamma$. Let $R = V\backslash\left(D \cup \cup_{i=1}^{\gamma} P_i\right)$. Let $n_0 = |R|$and $n_i = |P_i|$for $1 \leq i \leq \gamma$. We assume $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_\gamma$.

We will estimate the number of edges of G. There are exactly $\sum_{i=1}^{\gamma} n_i$ edges between D and $\cup_{i=1}^{\gamma} P_i$. There are at most $\binom{\gamma}{2} + \binom{n_0}{2} + \gamma n_0$ edges in G $[D \cup R]$. Let $1 \leq i \leq \gamma$. Since there is no vertex $p_i \in P_i$such that $P_i \subseteq \mathrm{N}[p_i, \mathrm{G}]$, there are at most $\binom{n_i}{2} - \left[{}^{n_i}/{}_2\right]$ edges in G$[P_i]$. Since there is no vertex $r_i \in R$ such that $P_i \subseteq \mathrm{N}[r_i, \mathrm{G}]$ there are at most $n_0(n_i - 1)$edges between P_i and R. Now let $1 \leq i < j \leq \gamma$.

Since there is no vertex $p_i \in P_i$such that $P_j \subseteq \mathrm{N}[p_i, \mathrm{G}]$,there are at most $n_i(n_j - 1)$ edges between P_i and P_j .

Furthermore, if $n_i = 2$, then also $n_j = 2$ and it is easy to see that there is at most one edge between P_i and P_j .

Altogether we obtain that $m = |E| \leq f(n_0, n_1, \cdots, n_\gamma)$ for a function f defined as follows

$$f(n_0, n_1, \cdots, n_\gamma) = \sum_{i=1}^{\gamma} n_i + \binom{\gamma}{2} + \binom{n_0}{2} + \gamma n_0 + \sum_{i=1}^{\gamma}\left(\binom{n_i}{2} - \left[\frac{n_i}{2}\right]\right)$$

$$+\sum_{i=1}^{\gamma}(n_0 n_i - n_0) + \sum_{1 \leq i<j \leq \gamma}(n_i n_j - max\{n_i, 3\})$$

$$= \binom{n}{2} - (\gamma - 1)\sum_{i=1}^{\gamma} n_i - \sum_{i=1}^{\gamma}\left[\frac{n_i}{2}\right] - \gamma n_0 - \sum_{i=1}^{\gamma}(\gamma - i)\, max\{n_i, 3\}.$$

Claim: Let $\gamma \geq 2$, $n_i \geq 2$ for $1 \leq i \leq \gamma$ and $n_0 \geq 0$ be integers.

Let $n = \gamma + \sum_{i=0}^{\gamma} n_i$ and let $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_\gamma$. If $\gamma = 2, n_1 = n_2 \geq 4, n_1$ and n_2 are even, then

$$f(n_0, n_1, \cdots, n_\gamma) \leq \binom{n}{2} - \gamma\left(n + \frac{\gamma-5}{2}\right) + 1.$$

otherwise

$$f(n_0, n_1, \cdots, n_\gamma) \leq \binom{n}{2} - \gamma\left(n + \frac{\gamma - 5}{2}\right).$$

Proof :

We claim that, If there is some $1 \leq i \leq \gamma - 1$ such that $n_i \geq 4$ and $n_i > n_{i+1}$, then

$$f(n_0, n_1, \cdots, n_i \cdots, n_\gamma) \leq f(n_0 + 1, n_1, \cdots, n_i - 1, \cdots, n_\gamma) - (\gamma - 1) - \left[\frac{n_i}{2}\right] + \left[\frac{n_i - 1}{2}\right] + \gamma - (\gamma - i)$$

$$\leq f(n_0 + 1, n_1, \cdots, n_i - 1, \cdots, n_\gamma).$$

Similarly, if $\gamma = 2$ and $n_i \geq n_2 + 2$, then $f(n_0, n_1, n_2) < f(n_0 + 2, n_1 - 2, n_2)$ and if $\gamma = 2, n_1 = n_2 + 1$ and n_2 is even, then $f(n_0, n_1, n_2) < f(n_0 + 1, n_1 - 1, n_2)$.

We will consider two special cases.

First, let $n_1 = n_2 = \cdots = n_l = 3$ and $n_{l+1} = n_{l+2} = \cdots = n_\gamma = 2$ for some $0 \leq l \leq \gamma$. We obtain

$$f(n_0, n_1, \cdots, n_\gamma) = \binom{n}{2} - (\gamma - 1)(2\gamma + l) - (\gamma + l) - \gamma(n - (3\gamma + l)) - 3\left(\gamma^2 - \tfrac{1}{2}\gamma(\gamma + 1)\right) = \binom{n}{2} - \gamma\left(n + \frac{\gamma - 5}{2}\right).$$

Now let $n_1 = n_2 = \cdots = n_\gamma \geq 4$.

For $\in = \frac{1}{2}[n_1(mod\ 2)]$ we obtain

$$f(n_0, n_1, \cdots, n_\gamma) = \binom{n}{2} - (\gamma - 1)\gamma\, n_1 - \gamma\left[\frac{n_1}{2}\right] - \gamma n_0 - \left(\gamma^2 - \tfrac{1}{2}\gamma(\gamma + 1)\right)n_1$$

$$= \binom{n}{2} - (\gamma - 1)\gamma\, n_1 - \gamma\frac{n_1}{2} - \gamma \in -\gamma n_0 - \left(\gamma^2 - \tfrac{1}{2}\gamma(\gamma + 1)\right)n_1$$

$$= \binom{n}{2} - \gamma\left(\frac{3}{2}\gamma - 1\right) n_1 - \gamma n_0 - \gamma \in$$

$$= \binom{n}{2} - \gamma\left(\frac{3}{2}\gamma - 1\right) n_1 - \gamma(n - (n_1 + 1)\gamma) - \gamma \in$$

$$= \binom{n}{2} - \gamma\left(\frac{1}{2}\gamma - 1\right) n_1 - \gamma(n - \gamma) - \gamma \in$$

$$\leq \binom{n}{2} - \gamma(2\gamma - 4) - \gamma(n - \gamma) - \gamma \in$$

$$f(n_{0,} n_{1,} \cdots, n_\gamma) = \binom{n}{2} - \gamma(n + \gamma - 4) - \gamma \in$$

If $\gamma = 2$ and $n_1 = n_2 \geq 5$ are odd or if $\gamma \geq 3$, then this implies

$$f(n_{0,} n_{1,} \cdots, n_\gamma) \leq \binom{n}{2} - \gamma\left(n + \left(\gamma - {}^{5}/_{2}\right)\right).$$

If $\gamma = 2$ and $n_1 = n_2$ are even, then this implies $f(n_{0,} n_{1,} \cdots, n_\gamma) \leq \binom{n}{2} - \gamma\left(n + \left(\gamma - {}^{5}/_{2}\right)\right) + 1.$

In view of the above remarks, this completes the proof of the claim. In order to complete the proof of the theorem, it remains to consider the case where$\gamma = 2, n_1 = n_2 \geq 4, n_1$ *and* n_2 are even $m = f(n_{0,} n_{1,} \cdots, n_\gamma)$.

In this case,$G[P_1]$ and $G[P_2]$ are complete graphs in which perfect matchings have been removed and $G[P_1, P_2]$ is a complete bipartite graph in which a perfect matching has been removed.(The graph $G[P_1, P_2]$ has vertex set $P_1 \cup P_2$ and contains all edges of G that join a vertex in $P_1$and a vertex in P_2). If $D' = \{p_1', p_1''\}$ consists of two non-adjacent vertices in P_1 then $(P_1 \cup P_2) \subseteq N[D', G]$ which is a contradiction.

Hence if $\gamma = 2, n_1 = n_2 \geq 4, n_1$ *and* n_2 are even m<=f($n_{0,} n_{1,} \cdots, n_\gamma) - 1$ .In view of the claim, hence the proof.

4. BLOCK GRAPHS WITH UNIQUE MINIMUM DOMINATING SETS

Lemma: Let D be a γ-set of a graph G. If $\gamma(G - x) > \gamma(G)$for every $x \in D$, then D is the unique γ-set of G.

Proof : Let D be a γ-set of the graph G, such that $\gamma(G - x) > \gamma(G)$ for every $x \in D$. Suppose, there is a γ-set D' of G different from D. Then, there is at least one vertex $x \in D - D'$ and D' dominates $G - x$. Hence, $|D'| \geq \gamma(G - x) > \gamma(G)$, which is a contradiction.

Result : Let G be a connected graph with at least one cut vertex. If $B_1, B_2, \cdots, B_t$ are all blocks of G, then the following conditions hold

(i) $|V(B_i) \cap V(B_j)| \leq 1$ for any $1 \leq i < j \leq t$.

(ii) $E(B_i) \cap E(B_j) = \emptyset$ for any $1 \leq i < j \leq t$ and $E(G) = E(B_1) \cup \cdots \cup E(B_t)$.

(iii) If $x \in V(B_i) \cap V(B_j)$ for any $1 \leq i < j \leq t$, then x is a cutvertex of G.

(iv) If x is a cutvertex of G, then x belongs to at least two different blocks of G.

(v) If the vertices a and b do not belong to a common block, then every path from a to b contains a cutvertex $x \neq a$,b of G, such that a and b lie in different *components of* $G - x$.

6. CONCLUSION

This paper deals about "The Review on Graphs With Unique Minimum Dominating Sets", for block graphs and maximum graphs. We investigate some of the structural properties of graphs having a unique gamma set. In particular, three equivalent conditions for this property are given for trees, this leads to a constructive characterization for those trees which have a unique gamma-set.

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