A Review on Domination in Planar Graphs with Small Diameter

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ABSTRACT – Domination and its variations in graphs are now well studied. However, the original domination number of a graph continues to attract attention. Many bounds have been proven and results obtained for special classes of graphs such as cubic graphs and products of graphs. On the other hand, the decision problem to determine the domination number of a graph remains NP-hard even when restricted to cubic graphs or planar graphs of maximum degree 3. In this paper we consider the domination of planar graphs with small diameter.

Keywords: Domination Number, Planar graph, Diameter Graph, Domination set.

1. INTRODUCTION

Domination and its variations in graphs are now well studied. However, the original domination number of a graph continues to attract attention. Many bounds have been proven and results obtained for special classes of graphs such as cubic graphs and products of graphs. On the other hand, the decision problem to determine the domination number of a graph remains NP-hard even when restricted to cubic graphs or planar graphs of maximum degree 3. Hence it is of interest to determine upper bounds on the domination number of a graph.

In this paper we consider the domination of planar graphs with small diameter. It is trivial that a tree of radius 2 and diameter 4 can have arbitrarily large domination number. So the interesting question is what happens when the diameter is 2 or 3.

MacGillivray and Seyffarth “Domination numbers of planar graphs”proved that planar graphs with diameter two or three have bounded domination numbers. In particular, this implies that the domination number of such a graph can be determined in polynomial time. On the other hand, they observed that in general graphs with diameter 2 have unbounded domination number.

We show that there is a unique planar graph of diameter 2 with domination number 3. Hence every planar graph of diameter 2, different from this unique planar graph, has domination number at most 2. We then prove that every planar graph of diameter 3 and of radius 2 has domination number at most 6. We then show that every sufficiently large planar graph of diameter 3 has domination number at most 7.

Theorem : Every planar graph of diameter 2 has domination number at most 2 except for the graph F of Figure 3.1 which has domination number 3.

Proof : To prove Theorem, suppose G is a planar graph of diameter two satisfying γ (G) > 2. If a and b are two vertices in G, then there is always a vertex not dominated by {a, b}. We Shall denote one such vertex by υ_{ab}. Fix an embedding G*of G in the plane.

From the Jordan Closed Curve Theorem, we Know that a cycle C in G* separates the plane into two regions, which we call the sides of C. Vertices of different sides of C are said to be separated by C. The side of C that consists of the unbounded region we call the outside of C, while the side of C that consists of the bounded region we call the inside of C. If C has length n and there are vertices both inside and outside C, then we say that C is a cut-n-cycle. A cut-3-cycle is also called a cut-triangle. Since a cut-set dominates a graph of diameter 2, it follows that G is 3-connected; therefore G has an essentially unique embedding in the plane and so we may speak of cut-cycles of G rather than of G*,lemma3.1.2 establishes the existence of a 4-cycle. Then we show that this cycle is neither both induced and dominating, nor both non-induced and dominating, and therefore not dominating. Finally, we show that it follows that G is isomorphic to F.

2. DOMINATION IN PLANAR GRAPHS WITH SMALL DIAMETER II

The domination number of G, denoted by (G), is the minimum cardinality of a dominating set, while the total domination number of G, denoted by γt(G), is the minimum cardinality of a total dominating set. Domination and its variations in graphs are now well studied. To simplify the notation, if X dominates Y we write $X \succ Y$ while if X totally dominates Y we write $X \succ_t Y$. Further, if a vertex u is adjacent with a vertex υ, we write $u \sim v$, while if u and v are nonadjacent, we write $u \nsim \upsilon$. We denote the eccentricity of a vertex υ in G by $ecc_G(\upsilon)$, or simply $ecc(\upsilon)$ if G is clear from the context. The subgraph induced by a subset $S \subseteq V(G)$ is denoted by G[S].

Theorem : Every planar graph of diameter 3 and of radius 2 has total domination number at most 5.

Proof : The focus is on cut-cycles. Note that in a planar graph of diameter 3,there cannot on both sides of a cut-cycle be vertices not dominated by the cycle. We define a basic cycle as follows.

Let vertex x have eccentricity 2 in G.

Then a basic cycle C is an induced cycle x, $\upsilon_1, \upsilon_2, \ldots, \upsilon_r$, x such that on both sides of the cycle there is a vertex whose neighbors on the cycle are a subset of the two consecutive vertices fartherest from x, specifically $\upsilon(r-1)/2$ and $\upsilon(r+1)/2$ if r is odd, and $\upsilon r/2$ and $\upsilon r/2+1$ if r is even. A special basic cycle is one with the added condition that there is on the dominated side of the cycle a vertex with only one neighbor on the cycle and that neighbor is not x. Our strategy is as follows. In Subsection 3.1 we show the existence of aspecial basic cycle of length 3 or 4 or of a basic 5-cycle in G. Thereafter, in Subsection 4.2 we prove some lemmas about how to totally dominate vertices at distance 2 from two or more vertices, in particular the Divider Lemma.

Basic Cycles Exist

Let G be a plane graph of radius 2 and diameter 3 with central vertex x. We say that it is edge-minimal if for every edge e of G, $diam(G - e) > 3$ or $ecc(x) > 2$ in $G - e$. Clearly, we may assume that G is edge-minimal in proving Theorem 1 (since removing edges can only increase the total domination number).

Theorem : Let G be an edge-minimal plane graph of radius 2 and diameter 3 with central vertex x. Then, $\gamma_t(G) \leq 5$, or there exists a special basic triangle, special basic 4-cycle, or basic 5-cycle.

Proof : Suppose there is neither a special basic cycle of length 3 or 4 nora basic 5-cycle in G. Let $Y = V(G) - N[x]$. Let M be a minimal subset of N(x) that dominates Y . The set M exists since $ecc(x) = 2$. Let $|M| = m$. Since $\gamma_t(G) \leq m + 1$, we may assume m >=5. Let the vertices of M be $n_0, n_1, \ldots, n_{m-1}$ in cyclic order (clockwise) around x in G. Let Y'_ibe the set of vertices of Y whose only neighbor in M is n_i. By the minimality of M, each Yi is nonempty.

Let $Y_0, Y_1, \ldots, Y_{m-1}$be a partition of Y such that $Y_i \subseteq N(n_i)$ for each i. Necessarily, $Y_i \subseteq Y_i$ for each i. We now choose a vertex $y_i \in Y'_i$ for each i. If there is a vertex of Y'_I adjacent to both a vertex of Y_{i-1} and a vertex of Y_{i+1} (where addition is taken modulo m), then this vertex is unique by the planarity of G and we choose this as y_i.

If there is no such vertex of Y_i , then we let $y_i$be any vertex of Y_i adjacent to a vertex of Y_{i-1} or a vertex of Y_{i+1}, if such a vertex exists, failing which we let y_i be any vertex of $Y_{0\,i}$. We say that two neighbors u_1 and u_2 of x are separated if there is a vertex of M between u_1 and u_2 in both directions around x in the embedding of G. We define type-1, type-2 and type-3 edges as follows.

A type-1 edge joins vertices $u_1, u_2 \in N(x)$ such that u_1 and $u_2$are separated. A type-2 edge joins vertices $u_1 \in N(x)$ and $v_2 \in Y$ with $v_2$dominated by a vertex u_2 of M such that u_1 and u_2 are separated. A type-3 edge joins vertices $v_1, v_2 \in Y$ with v_1 and v_2 dominated by vertices u_1 and u_2 of M, respectively, such that u_1 and u_2 are separated.

Theorem: There is no type-1, type-2 or type-3 edge.

Proof : Let e be an edge. Suppose $e = u_1u_2$is a type-1 edge. Then there is a vertex n_i of M inside the cycle C: x, u_1, u_2, x and a vertex n_j of M outside the cycle C. Since the vertices y_i and y_j are not dominated by x, C is a basic triangle. Without loss of generality, C dominates its inside. By assumption, C is non special. That is, every vertex of Y inside C is adjacent to both $u_1$and u_2. By planarity, y_i is the only vertex of Y inside the triangle C, since each such vertex must be adjacent to all of u_1, u_2 and n_j. But then we can remove the edge n_iy_i, contradicting the minimality of G. Hence, G has no type-1 edge.

Suppose $e = u_1v_2$ is a type-2 edge. Then again there is a vertex n_i ofM inside the cycle C: x, u_1, v_2, u_2, x and a vertex n_j of M outside the cycle C with vertices y_i and y_j not dominated by {x, u_2}. Furthermore, since there is no type-1 edge, C is induced and hence a basic 4-cycle. Without loss of generality, C dominates its inside. By assumption, C is non-special. That is, every vertex of Y inside C is adjacent to at least two vertices on the cycle. In particular, since $u_2 \in M$, y_i is adjacent to $u_1$and v_2 (and not to u_2). Hence by planarity, each vertex of Y inside C is adjacent to at most one of u_1 and u_2, and therefore, since C is non-special, is adjacent to v_2. But then we can remove the edge n_iy_i, contradicting the minimality of G. Hence, G has no type-2 edge. If $e = v_1v_2$ is a type-3 edge, then again there is a vertex y_i both insideand outside the cycle C: x, u_1, v_1, v_2, u_2, x not dominated by {x, u_1, u_2}. Furthermore, since there is no type-1 or type-2 edge, C is induced and hence a basic 5-cycle, a contradiction. Hence, G has no type-3 edge.

3. TOTAL DOMINATION IN GRAPHS WITH DIAMETER 2

A hypergraph H = (V, E) is a finite set V = V(H) of elements, called vertices, together with a finite multiset E = E(H) of arbitrary subsets of V, called hyperedges or simply edges. A k-uniform hypergraph is a hypergraph in which every edge has size k. Every (simple) graph is a 2-uniform hypergraph. Thus, graphs are special hypergraphs. A hypergraph H is called an intersecting hypergraph if every two distinct edges of H have a nonempty intersection.

A subset T of vertices in a hypergraph H is a transversal in H if T has a nonempty intersection with every edge of H. A transversal is also called a hitting set in the literature. The transversal number τ (H) of H is the minimum size of a transversal in H. A transversal of H of size τ (H) is called a τ (H) set.

For a graph G = (V, E), we denote by H_G the open neighbourhood hypergraph, abbreviated as ONH, of G; that is, H_G = (V,C) is the hypergraph with vertex set V and with edge set C consisting of the open neighborhoods of vertices of V in G.

Theorem : Let G be a diameter-2 graph of order n. Then the following hold.
(a) γ_t (G) ≤ 1 + δ(G).
(b) If n ≤ 11, then γ_t (G) ≤ 1 +$\sqrt{n}$ with equality if and only if G = G_9.
(c) If G has girth 5, then γ_t (G) = 1 +$\sqrt{n}$ −1.
We next establish an upper bound on the total domination number of a general graph with large minimum degree.

Proof: Let G = (V, E) be a diameter-2 graph of order n. If v is an arbitrary vertex in G, then the diameter-2 constraint implies that N[v] is a TD-set in G. In particular, choosing v to be a vertex of minimum degree proves part (a) of the above Theorem.It is shown that Moore graphs are r-regular and that diameter-2 Moore graphs have an order $n = r^2 + 1$and exist for r = 2, 3, 7, and possibly 57, but for no other degrees.

The Moore graphs for the first three values of r are unique, namely,
(i) the 5-cycle (2-regular graph on n = 5 vertices),
(ii) the Petersen graph (3-regular graph on n = 10 vertices), and
(iii) the Hoffman–Singleton graph (7-regular on n = 50 vertices).

Since G is a diameter-2 graph of girth 5, the graph G is a diameter-2 Moore graph and $n = r^2 + 1$. Hence $r = \sqrt{n} - 1$.

Let D be a γ_t (G)-set. Then $V = \bigcup_{v \in D} N(v)$, implying that |V|≤ $\sum_{v \in D} d_G(v) \leq \Delta(G)$ ≤ (G) · |D|; or equivalently, γ_t(G) =|D|≥ |V|/(G) = n/$\sqrt{n} - 1$. Therefore by Part (a), we have that n/$\sqrt{n} - 1 \leq$ γ_t (G) ≤ 1 +$\sqrt{n} - 1$, or, equivalently, $\sqrt{n} - 1 + 1/\sqrt{n} - 1 \leq$ γ_t (G) ≤ 1 +$\sqrt{n} - 1$. Since both γ_t (G) and n − 1 are integers, γ_t (G) = 1 + $\sqrt{n} -$ 1.Hence the proof.

5. TOTAL DOMINATION IN PLANAR GRAPHS OF DIAMETER TWO

A total dominating set, denoted TDS, of a graph G = (V, E) with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. Every graph without isolated vertices has a TDS, since S D V is such a set. The total domination number of G, denoted by γ_t (G),is the minimum cardinality of a TDS. A TDS of G of cardinality γ_t(G) is called a γ_t(G)-set. The decision problem to determine the domination number and total domination number of a graph remains NP-hardeven when restricted to cubic graphs or planar graphs of maximum degree 3. Hence it is of interest to determine upper bounds on the domination number and total domination number of a graph. A tree of radius 2 and diameter 4 can have arbitrarily large (total) domination number. So the interesting question is what happens when the diameter is 2 or 3.This restriction is reasonable to impose because planar graphs with small diameter are often important in applications. MacGillivray and Seyffarth proved that planar graphs with diameter two or three have bounded domination numbers.

Theorem: If G is a planar graph with diam.(G)= 2, then γ_t(G) ≤ 3.
Proof : Our aim in this paper is to study the problem of characterizing planar graphs with diameter two and total

domination number three. Such a characterization seems difficult to obtain since there are infinitely many such graphs.

we therefore restrict our attention to planar graphs with certain structural properties. We say that a graph G satisfies the domination-cycle property if there is some $\gamma(G)$-set not contained in any induced 5-cycle of G. We characterize the planar graphs with diameter two and total domination number three that satisfy the domination-cycle property.

Notation : For notation and graph theory terminology we in general follow. Specifically, let G = (V, E)be a graph with vertex set V of order n and edge set E. For a set $S \subseteq V$, the subgraph induced by S is denoted by G[S].

The S-external private neighborhood epn(υ, S) of a vertex $v \in S$ is defined by pn(υ, S)= {u ∈ V | N(u) ∩ S={υ}}, and each element of epn(υ, S)is called an S-external private neighbor of υ. The open neighborhood of vertex υ ∈ V is denoted by N(υ)={u ∈ V | u_v ∈ E} while its closed neighborhood is given by N[υ]= N[υ] U{υ}. For a set S ⊆ V, $N(S)=\cup_{\upsilon \in S}N(\upsilon)$ and N[S] U S. If X, Y ⊆ V, then the set X is said to dominate the set Y if

Y⊆ N[X], while X is said to totally dominate the set Y if Y⊆ N[X]. If Y ={υ} and X dominates Y, we simply write that X dominates υ. We note that if X dominates V, then N[X] =V and X is a dominating set of G, and if X totally dominates V, then N(X)= V and X is a total dominating set of G.

For disjoint subsets U and W of V, we let [U,W] denote the set of all edges of G joining a vertex of U and a vertex of W. We denote the degree of a vertex υ in G by $d_{G(\upsilon)}$ or simply by d(υ) if the graph G is clear from the context. For two vertices u and v in a connected graph G, the distance $d_G(u, \upsilon)$ between u and υ is the length of a shortest u-υ path in G.

For a set S⊆V and a vertex v∈V, the distance $d_G(\upsilon, S)$ between υ and S is the minimum distance between v and a vertex of S. If a vertex u is adjacent to a vertex υ, we write u ~v, while if u and v are nonadjacent, we write $u \nsim \upsilon$. If υ is adjacent to no vertex in a set A ⊆ V(G) then we write $\upsilon \nsim A$ and if υ is adjacent to every vertex in A then we write υ ~ A. A plane graph is a planar graph together with an embedding in the plane. From the Jordan Closed Curve Theorem, we know that a cycle C in a plane graph separates the plane into two regions, the interior of C and the exterior of C. If a vertex lies in the interior of C, we simply say that v lies inside C. We denote the set of vertices in the interior and exterior of C by int(C) and ext.(C), respectively. A plane graph divides the plane into regions which we call faces. The unbounded region is called the exterior face and the other regions are called interior faces. If f is a face of a plane graph G, then we can write $f=[u_1, u_2,\ldots, u_k]$ where $u_1, u_2,\ldots, u_k$ are the vertices on the boundary walk of f in clockwise order.

6. CONCLUSION

This paper describes about "A Review On Domination in Planar Graphs With Small Diameter". Domination numbers of planar graphs proved that planar graphs with diameter two or three have bounded domination numbers. This implies that the domination number of such a graph can be determined in polynomial time.

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