# A Review on Rainbow Edge Colouring and Rainbow Domination

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**ABSTRACT** – In graph theory Edge coloured graph which has the distinct coloured edges are well studied.An Edge coloured graph is t-tolerant if it contains no monochromatic star with t+1 edges.In this paper we consider optimal edge coloured complete graphs.We show that in any optimal edge colouring of the complete graph $k_n$ ,Also we prove that in every proper edge colouring of the complete graph $k_n$ .

**Keywords:** Rainbow Cycle, edge coloring , edge chromatic number ,Hamilton cycle.

## 1. INTRODUCTION

An edge-colored graph is rainbow if its edges have distinct colors. Rainbow edge-colored graphs have also been called heterochromatic, polychromatic, or totally multicolored, Within an edge-colored graph G, we consider covering the edges by rainbow matchings or covering the vertices by disjoint rainbow stars.

The existence of a Hamilton cycle with many colors, also the existence of a Hamilton cycle with few colors in any proper edge coloring of a complete graph. A rainbow cycle is a cycle whose all edges have different colors. Given an optimally edge colored complete graph with n vertices, we study the number of colors appearing on its cycles.

The rainbow connection number can be motivated by its interesting interpretation in the area of networking. This new concept comes from the communication of information between agencies of government.

Consider a network G(e.g., a cellular network). To route messages between any two vertices in a pipeline, assign a distinct channel to each link(e.g., a distinct frequency). We need to minimize the number of distinct channels that we use in the network.

The minimum number of distinct channels is called the rainbow connection number and is denoted by rc(G). Let G be an edge-colored graph with n vertices. A rainbow subgraph is a subgraph whose edges have distinct colors. The rainbow edge-chromatic number of G, written $\hat{\aleph}$ (G), is the minimum number of rainbow matchings needed to cover E(G). An edge- colored graph is t-tolerant if it contains no monochromatic star with t+1 edges.

## 2. RAINBOW EDGE COLOURING AND RAINBOW DOMINATION

If G is t-tolerant, then $\aleph$ (G) < t (t +1) n ln n, and examples exist with $\hat{\aleph}$ (G) ≥ t / 2 (n-1). The rainbow domination number, written $\hat{\gamma}$ (G), is the minimum number of disjoint rainbow stars needed to cover V (G). For t-tolerant edge-colored n-vertex graphs, we generalize classical bounds on the domination number: $\hat{\gamma}$ (G) ≤ $\frac{1+ln\,k}{k}$ n

(where k = $\frac{\delta(G)}{t}+1$ ) and $\hat{\gamma}$ (G) ≤ $\frac{t}{t+1}$ n

when G has no isolated vertices.

**Theorem:** There exist infinitely many t-tolerant edge-colored graphs G such that $\hat{\aleph}(G) \geq \frac{t}{2}(|V(G)|-1) = \frac{t}{2}\Delta(G)$.

**Proof**: For t, p ∈ N, start with a proper tp-edge-coloring of $K_{tp}$. Obtain a t-tolerant edge-colored graph G by combining t-tuples s of color classes into single colors.

In G there are only p colors, so $\hat{\alpha}'(G) \leq$ p.

$$\text{Hence } \hat{\aleph}(G) \geq \frac{1}{p}|E(G)$$
$$\geq \frac{t}{2}(tp-1)$$
$$= \frac{t}{2}(|V(G)|-1)$$
$$= \frac{t}{2}\Delta(G).$$

**Theorem:** When n≡ 2 mod 4, there is an edge-colored graph G such that $\hat{\aleph}$ (G) > Δ(G) + 1 and G is a proper n-edge-coloring of $K_{n,n}$.

**Proof:** As noted earlier, proper n-edge-colorings of $K_{n,n}$ correspond to Latin squares of order n. Each rainbow matching corresponds to a partial transversal of the Latin square so $\hat{\aleph}$ (G) is the minimum number of partial tranversals covering the square.

Latin squares of even order need not have transversals. To construct such squares when n ≡2 mod 4, let k = n/2, and let A and B be latin squares of order k, using disjoint sets of k labels in the two squares.

Let C = $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$.Although C is a Latin square of order n, it has no transversal. A transversal must use each of the 2k labels.Since k is odd, some quadrant must contribute at least $\lceil K/2 \rceil$ positions.

Now each of the other three quadrants is limited to $\lfloor K/2 \rfloor$contributions, so a partial transversal has size at most n−1 .

Thus at least $n^2/(n-1)$ partial transversals are needed to cover C. Since (n−1)(n+1) < $n^2$ we have $\hat{\aleph}$ (G) ≥ $n^2/(n-1)$
> (n+1) = Δ(G) + 1.

**Theorem:** Fix t ∈ N and c ∈ R with c > 0. Every t-tolerant edge-colored graph G with average color degree at least c has an edge-colored t-tolerant subgraph H with $\hat{\delta}$ (H) > $\frac{c}{t+1}$.

**Proof** : The claim holds with H = G unless $\widehat{d_G}$(v) ≤ $\frac{c}{t+1}$ for some vertex v. Deleting v decreases the color degree of each neighbor by at most 1, but v may have up to t $\widehat{d_G}$(v) neighbors.Thus deleting v reduces the sum of the color degrees by at most (t + 1) $\widehat{d_G}$(v).

Since

$\sum_{u \in V(G-v)} \widehat{d_{G-v}(u)} \geq \sum_{u \in V(G)} \widehat{d_G}(u) - (t+1)\widehat{d_G}(v) \geq c|V(G)| - c$

deleting v does not reduce the average color degree.

Furthermore, every subgraph of a t-tolerant graph is also t-tolerant. Iteratively deleting vertices with color degree at most c/(t + 1) must end. Since the average color degree never decreases, it ends with a subgraph H having no vertex with color degree at most c/ (t + 1).

## 3. ON RAINBOW CYCLES IN EDGE COLORED COMPLETE GRAPHS

Graph colorings is one of the most important concepts in graph theory. In the present paper we study the existence of a Hamilton cycle with many colors, also the existence of a Hamilton cycle with few colors in any proper edge coloring of a complete graph. A rainbow cycle is a cycle whose all edges have different colors.

Given an optimally edge colored complete graph with n vertices, we study the number of colors appearing on its cycles. We show that there exists a Hamilton cycle with at most $\sqrt{8n}$ colors and a Hamilton cycle with at least n(2/3 − o(1)) colors. A random Hamilton cycle is also shown to have n(1 − 1/e + o(1)) colors on average.

There are examples of optimal edge colorings that have no Hamilton cycle with less than $\log_2$ n colors. Furthermore, in some optimal edge colorings, there is no Hamilton cycle with n − 1 or n colors. We conjecture that there is always a Hamilton cycle with at most O(log n) colors and a Hamilton cycle with at least n − 2 colors.

For every ∈ > 0 and n > $n_0$ (∈), any complete graph $K_n$ whose edges are colored so that no vertex is incident with more than (1− $\frac{1}{\sqrt{2}}$ −∈) edges of the same color, contains a Hamilton cycle in which adjacent edges have distinct colors. Moreover, for every k, 3≤ k ≤ n, any such Kn contains a cycle of length k in which adjacent edges have distinct colors.

Let the edges of the complete graph $K_n$ be colored so that no color is used more than k = k(n) times. This coloring it is called a k-bounded coloring. Clearly, if k = 1, every Hamilton cycle is a rainbow cycle.

**Theorem**:In any optimal edge coloring of the complete graph Kn, there is a Hamilton cycle with at most $\sqrt{8n}$ different colors.

**Proof:**Our proof relies on the following observation: Let $P_1,\dots,P_k$ be k vertex disjoint paths that cover V ($K_n$).

For 1 ≤ j ≤ k, let $v_j$ be an endpoint of $P_j$.

There are $\binom{k}{2}$ edges connecting the $v_j$ 's, and $\aleph'$ ($K_n$) ≤ n colors are used.

We can thus find a set S of at least k(k − 1)/(2n) edges, all of the same color and all connecting the $v_j$ 's Evidently, adding S to $P_1$, …., $P_k$ decreases the number of paths by at least k(k − 1)/(2n) and increases the number of distinct colors that appear on the paths by at most one. In addition, the paths are still vertex-disjoint and cover V ($K_n$).To begin, let Pi be the path of length zero formed by the ith vertex of Kn for 1 ≤ i ≤ n.

Clearly, $P_1$ ,…,$P_n$ cover V ($K_n$) and are vertex-disjoint. Let us set $x_0$ = n.

By the above observation and induction on $i \geq 0$, there are $x_i$ vertex-disjoint paths that use at most i colors and cover V ($K_n$), and

$x_{i+1} \leq x_i - x_i(x_i - 1)/(2n)$.

Clearly, the function $1/(x(x - 1))$ is decreasing in the range $x > 1$. Hence if m is a nonnegative integer and $x_m > 1$, we have

$$m \leq \sum_{i=0}^{m-1} \frac{2n(x_i - x_{i+1})}{x_i\,(x_i - 1)}$$
$$\leq \int_{x_m}^{x0} \frac{2n}{x(x-1)} dx$$
$$\leq \int_{x_m}^{x0} \frac{2n}{(x-1)^2} dx = \frac{2n}{x_m - 1} - \frac{2n}{n-1},$$

which implies that $x_m \leq 1+2n/(m+2)$ for all nonnegative integers m.

Recall that there are $x_m$ vertex-disjoint paths with at most m colors on the edges which cover V($K_n$).We can connect these paths to form a Hamilton cycle by adding $x_m$ edges; the resulting Hamilton cycle has at most $m + x_m$ colors.

For $m = \lceil\sqrt{2n}\rceil - 2$,we have $x_m \leq \sqrt{2n}+1$.

thus atmost $(\lceil\sqrt{2n}\rceil - 2) + (\sqrt{2n} + 1) \leq 2\sqrt{2n}$ colors appear on the Hamilton cycle.

**Theorem:**Given an optimal edge coloring of the complete graph $K_n$, the expected number of different colors that appear on the edges of a random Hamilton cycle of $K_n$ is approximately equal to $(1 - e^{-1})n$, for large enough n.

**Proof**:Let c be an arbitrary color used in the given optimal edge coloring of $K_n$ and let C be the set of edges whose colors are c. The edges in C are a matching of size $\lfloor n/2 \rfloor$. Clearly $K_n$ has $(n - 1)!/2$ Hamilton cycles.

Assume S is a subset of C with size k. We can count the number of Hamilton cycles that contain S by considering the following transformation. For each edge in S, contract its two endpoints into a single vertex.

If H is a Hamilton cycle of $K_n$ that contains S, its transform is a Hamilton cycle of the graph $K_{n-k}$. Furthermore, every Hamilton cycle of $K_{n-k}$ is the transform of exactly $2^k$ Hamilton cycles of $K_n$ that contain S, because the directions of the alignments of the edges of S in H have no impact on the transform of H.

Consequently, there are $2^{k-1}(n - k - 1)!$ Hamilton cycles in $K_n$ that contain S.Thus the probability that a random Hamilton cycle contains S is $2^k (n - k - 1)!/(n - 1)!$.The principle of inclusion and exclusion now implies that the probability of the event that a random Hamilton cycle avoids all edges in C is $P=\sum_{K=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k a_k$

Where , $a_k = \binom{\lfloor \frac{n}{2} \rfloor}{k} \frac{2^k (n-k-1)!}{(n-1)!} = \frac{1}{k!} \prod_{j=0}^{k-1} \frac{2(\lfloor \frac{n}{2} \rfloor - j)}{n-j-1}$.

It is not hard to see that $a_k \leq 1/k!$ for $k \geq 3$. For any real number x, we have $1 + x \leq e^x$; hence for $k \leq n^{1/3}$ we have

$$a_k = \frac{1}{k!} \prod_{j=0}^{k-1} \left(1 + \frac{(n-j-1) - 2(\lfloor \frac{n}{2} \rfloor - j)}{2(\lfloor \frac{n}{2} \rfloor - j)}\right)^{-1}$$
$$\geq \frac{1}{k!} \prod_{j=0}^{k-1} \left(1 + \frac{j}{2(\lfloor \frac{n}{2} \rfloor - k)}\right)^{-1}$$
$$\geq \frac{1}{k!} \prod_{j=0}^{k-1} \exp\left(- \frac{j}{2(\lfloor \frac{n}{2} \rfloor - k)}\right)$$
$$= \frac{1}{k!} exp\left(- \frac{k(k-1)}{4(\lfloor \frac{n}{2} \rfloor - k)}\right)$$
$$\geq \frac{1}{k!} \exp\left(\frac{n^{2/3}}{4\lfloor \frac{n}{2} \rfloor - n^{1/3}}\right)$$
$$= \frac{1}{k!}(1 + o(1)),$$

where o(1) is a function in terms of n and k that becomes arbitrarily small as n gets large.

Splitting the right hand side of Equation into two sums as in

$$p = \sum_{k=0}^{\lfloor n^{1/3} \rfloor} (-1)^k a_k + \sum_{k=\lfloor n^{1/3} \rfloor}^{\lfloor \frac{n}{2} \rfloor} (-1)^k a_k,$$

we note that the Taylor expansion of $e^x$ for $x = -1$ yields

$$\sum_{k=0}^{\lfloor n^{1/3} \rfloor} (-1)^k a_k = \sum_{k=0}^{\lfloor n^{1/3} \rfloor} \frac{(-1)^K}{k!}(1 + o(1))$$

$$= \sum_{k=0}^{\lfloor n^{1/3} \rfloor} \frac{(-1)^K}{k!} + \sum_{k=0}^{\lfloor n^{1/3} \rfloor} \frac{(-1)^k o(1)}{k!}$$
$$=(e^{-1}+o(1))+o(1)= e^{-1}+o(1)$$

On the other hand, we have

$$\left|\sum_{k=\lfloor n^{1/3} \rfloor+1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k a_k\right| \leq \sum_{k=\lfloor n^{1/3} \rfloor+1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} = o(1),$$

since the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ is convergent.

Thus we get $p = e^{-1} + o(1)$. We known $n - 1 \leq \aleph'(Kn) \leq n$, and the color c appears on a random Hamilton cycle with probability $1-p$.

Thus we expect that $\aleph'(Kn)(1-p) = n(1-e^{-1})(1+o(1))$ different colors appear on a random Hamilton cycle on average.

**Theorem:** In any optimal edge coloring of $K_n$, there is a Hamilton cycle with at least $n(2/3 - o(1))$ colors.

**Proof:** Suppose n is even. Without loss of generality assume $\{1, 2, \cdots, n-1\}$ is the set of colors used.

Let A be an $n \times n$ square matrix where $A_{ij} = c(v_iv_j)$ if $1 \le i, j \le n$ and $i \ne j$, and $A_{ii} = n$ if $1 \le i \le n$. Clearly, A is a Latin square.

By a result, we can select $n - O(\log^2 n)$ entries of A that have different values, and are located on distinct rows and columns. Of these entries, only one can be on the diagonal.

Furthermore, if $A_{ij}$ is selected then $A_{ji}$ can not be selected since values appearing on the selected entries should be distinct.

This means that we can select $n - O(\log^2 n) - 1$ edges of $K_n$ such that the selected edges have different colors, and the degree of each vertex is at most two in the graph induced by these edges.

That is, the selected edges consist of vertex-disjoint paths and cycles. Every cycle has at least 3 edges.

Thus we can delete one edge of every cycle to get $(2/3)(n - O(\log^2 n) - 1)$ edges forming vertex-disjoint paths. We can connect these paths to get a Hamilton cycle with at least $(2/3)(n - O(\log^2 n) - 1) = (2/3 - o(1))n$ colors.

When n is odd, $K_n$ is colored using n colors. Moreover, for any of these n colors, exactly one vertex is not an endpoint of an edge of that color. Thus, we can extend the optimal edge coloring of $K_n$ to an optimal edge coloring of $K_{n+1}$. Since n+1 is even, $K_{n+1}$ has a Hamilton cycle with at least $(2/3 - o(1))(n + 1)$ colors. We can now trivially construct a Hamilton cycle for $K_n$ that has $(2/3 - o(1))n$ colors.

## 4. AN EDGE COLOURING PROBLEM FOR GRAPH PRODUCTS

The edges of the Cartesian product of graphs $G \times H$ are to be colored with the condition that all rectangles, i.e, $k_2 \times k_2$ subgraphs, must be colored with four distinct colours. The minimum number of colors in such colorings is determined for all pairs of graphs except when G is 5-chromatic.

A rectangle in the Cartesian product $G \times H$ of two graphs is a four-cycle in the form $k_2 \times k_2$. The term **Good coloring** is used for edge coloring of $G \times H$ such that all rectangles are colored with four distinct colours. We determine rb (G,H), the minimum number of colours needed for a Good coloring of $G \times H$ (rb stands for rainbow). The minimum number colours needed for a Good colouring of $k_m \times k_n$. this number is denoted by
rb ( m,n)and rb (n,n).

**Theorem :** $rb(G,H) \le rb(\aleph(G), \aleph(H))$.

**Proof:** Assume that V(G) is partitioned into k independent sets $A$, and V(H) is partitioned into l independent sets $B_i$ where $k = \aleph(G)$ and $1 = \aleph(H)$ Contract each set $A_i \times B_j$ in $G \times H$ into a single vertex $v_{ij}$ and view this As $M = K_k \times K_l$ by adding all horizontal and vertical edges.

There is a good coloring of M with rb (k, 1) colors. Transfer this coloring to G x H by coloring each edge between $A_i \times B_j$ and $A_i \times B_t$ with the color of $v_{ij}v_{it}$ in $M$, similarly by coloring each edge between $A_i \times B_j$ and $A_i \times B_t$ with the color of $v_{ij}v_{it}$. This is a good coloring of $G \times H$.

**Theorem :** $rb(G,H) \ge \max(\aleph(G), \aleph(H))$.

**Proof:** Assume that $rb(G, H) = t$ and let $\propto$ be a good coloring of $G \times H$ with $t$ colors. Fix an arbitrary edge e of $G$, and for each vertex v in $H$ color $v$ with $\propto(e \times v)$. Since $\propto$ is a good coloring of $G \times H$, we obtain a proper coloring of the graph $H$ with at most $t$ colors. Therefore $\aleph(H) \le t$. A similar argument $\aleph(G) \le t$.

**Theorem:** For each $n \ge 6$ there exist weak decompositions of $K_n$ which are complementary.

**Proof:** Set $m = \lfloor n/2 \rfloor$ and define $A = \{0,1,\ldots, m-3, m-2, m\}$. For odd n, define $B$ as $B = \{2,3,\ldots, m-1, m, m+2\}$, and for even n, define $B$ as $B = \{1,2,\ldots, m-2, m-1, m+1\}$. In both cases $A \cap B = \emptyset$ holds. Therefore defining $G_o$ and $H_o$ as complete graphs with vertex sets A and $B$, respectively, the condition of complementary decompositions is satisfied. Furthermore, if $n \ge 6$ the edges of the complete graphs $G_o + i$ and $H_o + i$ both cover the edges of $k_n$ on [n] because the differences of elements in both A and B give all elements of [n] .

Therefore recursively defining the edge set $E$, for $i = 1,2,\ldots, n-1$ of the graph $G$, as $E_{i+1} = \{\cup pq : p, q \in V(G_o) + i, pq \notin E_j, 0 \le j \le i\}$
(with $Eo = E(Go)$), th e graphs $G$, form a weak decomposition. The graphs $H_i$ can be defined similarly, writing H instead of G.

## 5. CONCLUSION

This paper describes about "A Review On Rainbow edge colouring and rainbow domination" edge coloured graph is tolerant if it contains no monochromatic star. Also every proper edge colouring of the complete graph $k_n$ .

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