A Review on Global Domination Number of a Graph

[1] S. Kalaiselvi [2] R.Jeyamani

[1] Research Scholar, Sakthi College of Arts and Science For Women, Oddanchatram.
[2] Associate Professor, Department of Mathematics, Sakthi College of Arts and Science For Women, Oddanchatram.

ABSTRACT – Adominating set is called a global dominating set if it is a dominating set for a graph G and its complement $\bar{G}$. We investigate some general results for global dominating sets corresponding to the graphs Pn. In the present work we investigate some general results which relate the concept of global domination and duplication of a vertex. In this paper sharp bounds for γ_{gn}, are supplied for graphs whose girth is greater than three. Exact values of this number for paths and cycles are presented as well.

Keywords: Domination, global domination, global sets domination, global neighborhood domination.

1. INTRODUCTION

Graph theory can be defined as the study of graphs. Graphs are mathematical structures used to model pair-wise relations among objects from a certain collection. Graph can be defined as a set V of vertices and set of edges. Where V is collection of $|V| = n$ abstract data types. Vertices can be any abstract data types and can be represented with the points in the plane. These abstract data types are also called nodes. A line segment connecting these nodes is called an edge. Again, more abstractly saying, edge can be abstract data type that shows relation among the nodes.

In this document , I would briefly go over through how and what led to the development of the graph theory which revolutionized the way to many complicated problems to be solved.

The " Konigsberg bridge " problem originated in the city of Konigsberg , formerly in the Germany but , now known as Kaliningrad and part of Russia , located on the river Preger. The city had seven bridges, which connected two island with the main land Via seven bridges. People staying there always wondered whether was there any way to walk over all the bridges once and only once and return to the starting place. The below picture is the map of Konigsberg during Euler's time showing the actual layout of the seven bridges, highlighting the river Preger and the bridges.

2. THE GLOBAL DOMINATION NUMBER OF A GRAPH

Theorem: (i) For a graph G with p vertices , $\gamma_g(G) = p$ if and only iff $G = K_p$ or $\bar{K}_p$.
(ii) $\gamma_g(K_{m,n}) = 2$ for all $m , n \geq 1$
(iii) $\gamma_g(C_4) = 2$, $\gamma_g(C_5) = 3$ and $\gamma_g(C_n) = \{ n / 3\}$ for all $n \geq 6$.
(iv) $\gamma_g(P_n) = 2$ for n =2 , 3 and $\gamma_g(P_n) = \{ n / 3\}$ for all $n \geq 4$.
Proof: We prove only (i) and (ii)- (iv) are obvious.Clearly , $\gamma_g(K_p) = \gamma_g(\bar{K}_p) = p$. Suppose $\gamma_g(G) = p$ and $G = K_p$. $\bar{K}_p$ Then G has at least one edge uv and a vertex w not adjacent to , say v. Then $V - \{v\}$ is a global dominating set and $\gamma_g(G) \leq p - 1$.For some graphs including trees, γ_g is at most equal to γ.

Theorem: Let S be a minimum dominating set of G. If there exists a vertex v in V- S adjacent to only vertices in S, then $\gamma_g \leq \gamma+1$
Proof: This follows since $S \cup \{v\}$ is a global dominating set.

Theorem: For a (p, q) graph G without isolates.
$(2q - p(p-3)) / 2 \leq \gamma_g \leq p - \beta_0 + 1$
Proof : Let S be a minimum global dominating set.
Then every vertex in V–S is not adjacent to at least one vertex in S. This implies $q \leq (p/2) - (p-\gamma_g)$ and the lower bound $(2q - p(p-3)) / 2 \leq \gamma_g \leq p - \beta_0 + 1$ in follows. To establish the upper bound, let B be an independent set with β_0 vertices. Since G has no isolates, V – B is a dominating set of G. Clearly, for $v \in B$,$(V - B) \cup \{v\}$ is a global dominating set of G, and upper bound follows.Since $\alpha_0 + \beta_0 = p$ for any graph of order p without isolates , We have from $(2q - p(p-3)) / 2 \leq \gamma_g \leq p - \beta_0 + 1$.

3. THE GLOBAL SET - DOMINATION NUMBER OF A GRAPH

Theorem: In a tree T with p vertices and e end vertices , that is not a star , the set of non end vertices forms a minimum global set domination-set and $\gamma_{sg} = p - e$
Proof: It is known that the set D of all cut vertices of T forms a γ_s set of T and $\gamma_s = p - e$. Clearly, the sub graph $\langle V(T) -$

$D\rangle$ in $\bar{T}$ is complete. Since $T \neq K_{1,n}$, in $\bar{T}$, each vertex in V(T) – D is adjacent to some vertex in D.
This implies that D is an set domination set of $\bar{T}$ also, and $\gamma_{sg} = p - e$.

Theorem: Let G be a connected graph of order $p \geq 4$. Then, $2 \leq \gamma_{sg} \leq p - 2$.
Proof: Let u and v be adjacent vertices of degree at least two(such vertices clearly exist).Then V – { u , v } is a global s d – set of G .so $\gamma_{sg} \leq p-2$. The bounds in $2 \leq \gamma_{sg} \leq p - 2$ are sharp . The upper bound is attained by paths of length at least 3 and the 5- cycle. All graphs for which the lower bound is attained can be determined.

Theorem: For a graph G of order p , $\gamma_{sg} = 2$ if and only if diam G = diam $\bar{G}$ = 3 and either G or $\bar{G}$ has a bridge which is not an end edge.
Proof: Assume $\gamma_{sg} = 2$. Since diam G $\leq \gamma_s + 1$, We have diam G ≤ 3 and diam $\bar{G} \leq 3$, Now let D = { u , v } be a γ_{sg} - set of G. Suppose u and v are adjacent in G . All vertices in V(G) – D are adjacent to either u or v (but not to both).
If all such vertices are adjacent to only u (~~or~~ v) in G, then $\bar{G}$ is disconnected. Hence, some vertices of V(G) – D are adjacent to u and some are adjacent to v. If $x \in N(u) - \{v\}$ and $y \in N(v)-\{u\}$, then x and y are not adjacent in G , for otherwise, {u ,v } will not be an set domination set in $\bar{G}$. Thus, uv is a bridge in G that is not an end edge, and d(x , y) – 3 – diam G. Also in G , d(u , v) = 3 and hence , diam $\bar{G}$ =3. Conversely, if G has a bridge uv that is not an end edge, and diam G = diam $\bar{G}$ =3 , then every vertex in G is adjacent to u or to v and hence { u , v } is a γ_s – set in G. let $N_G(u)$ be the set of all neighbours of u in G.Then, $N_G(u) = N_G[v]$, Since uv is a bridge in G, every vertex of $N_G(u) - \{v\}$ is adjacent to every vertex of $N_G(v) - \{u\}$ in $\bar{G}$.
Hence {u, v} is an set domination set of $\bar{G}$, and $\gamma_{sg} = 2$.

4. SOME NEW RESULTS ON GLOBAL DOMINATING SET

Lemma: For cycle C_n, let C_n' be the graph obtained by duplication of a vertex x by x' where $x \in V(C_n)$. If S is a dominating set of C_n containing either of the vertices which are adjacent to x, then S is also a dominating set of C_n'.
Proof : If $x \in V(C_n)$ is duplicated by a vertex x' then $V(C_n') = V(C_n) \cup \{x'\}$. Now if S is a dominating set of C_n and $a \in S$ $(a \neq x)$ dominates x then a is adjacent to x' in C_n'.
Thus, a dominates x' in C_n'. That is, S is a dominating set of C_n'.

Theorem: Let C_n' be the graph obtained by duplication of a vertex x of C_n by x'. If S is a global dominating set of C_n containing either of the vertices which are adjacent to x then S is also a global dominating set of C_n'.
Proof: If S is a global dominating set of C_n then S is a dominating set of C_n as well as $\bar{C}_n$. As S is a dominating set of C_n then according to above Lemma , S is also a dominating set of C_n'.
To prove the required result it remains to show that S is a dominating Set of $\bar{C}_n'$. For that if $a \in S$ is adjacent to x in C_n' then it is not adjacent. to x in $\bar{C}_n'$. Now, S being a dominating set of $\bar{C}_n$ $\exists$ a vertex $b \in S$, $a \neq b$ such that b is not adjacent to x in C_n but dominates x in $\bar{C}_n$. As $a \in S$ is adjacent to both x and x' in C_n' implies that it is not adjacent to both x and x' in $\bar{C}_n'$. Moreover, $V(\bar{C}_n') = V(\bar{C}_n) \cup \{x'\}$ and S is a dominating set of $\bar{C}_n$ then above referred vertex $b \in S$ must dominate x' in $\bar{C}_n'$. Hence, S is also a dominating set of $\bar{C}_n$. Thus, S is a dominating set of both C_n' as well as $\bar{C}_n'$. Therefore, S is a global dominating set of C_n'.

Theorem: If S is a γ-set of P_n $(n \geq 6)$ then S is a global dominating set of P_n. Also $\gamma(P_n) = \gamma_g(P_n)$.

Proof For P_n, $n \geq 6$, consider a γ-set, $S = \{v_2, v_5, \ldots, v_{3j+2}\}$ if $n \equiv 0$ or 2(mod 3) and $S = \{v_2, v_5, \ldots, v_{3j+2}\} \cup \{v_{n-1}\}$ if $n \equiv 1 \pmod 3$ where $0 \leq j \leq [n-2/3]$. In P_n, there are two vertices of degree 1 and $(n - 2)$ internal vertices are of degree 2. Now, let $v_i \in S$ and $v_j \in S$ be any two vertices such that $v_j \notin N[v_i]$.Then we claim that these two vertices are sufficient to dominate remaining vertices of $\bar{P}_n$. Since the vertices which are not in $N(v_i)$ must belong to $N(v_j)$, any $S \subset V$ containing v_i and v_j will be a dominating set of $\bar{P}_n$. Thus, S is a dominating set of both P_n as well as $\bar{P}_n$. Hence, S is a global dominating set of P_n. Since, S being a γ-set, it is of minimum cardinality. Therefore, $\gamma(P_n) = \gamma_g(P_n)$ for $n \geq 6$.

Theorem: S is a global dominating set of P_n if and only if it is a global dominating set of P_n'.
Proof: For P_n $(n \geq 4)$, consider the global dominating set $S = \{v_2, v_5, v_8, \ldots, v_{3j+2}\}$ if $n \equiv 0$or2(mod 3) and $S = \{v_2, v_5, v_8, \ldots, v_{3j+2}\} \cup \{v_{n-1}\}$ if $n \equiv 1 \pmod 3$ where $0 \leq j \leq [n-2/3]$. There are three possibilities when duplication of a vertex of P_n takes place. (i) Duplication of a pendant vertex. (ii) Duplication of an internal vertex not belonging to S**.** (iii) Duplication of an internal vertex belonging to S.
Case-I: Either a pendant vertex or an internal vertex of P_n not belonging to S is duplicated.

Here, a duplicated vertex v' is adjacent to a vertex in above referred S and $V(P_n') = V(P_n)\ \{v'\}$.Hence, S is a global dominating set of P_n'

Case-II An internal vertex of P_n belonging to S is duplicated. For P_n ($n \geq 4$), consider the global dominating set $S = \{v_1, v_4, v_7, \ldots, v_{3j+1}\}$if $n \equiv 1$ or 2(mod 3) and $S = \{v_1, v_4, v_7, \ldots, v_{3j+1}\} \cup \{v_n\}$ if $n \equiv 0 \pmod 3$ where $0 \leq j \leq [n-1/3]$. Here the duplicated vertex v' is adjacent to a vertex in above referred S and $V(P_n') = V(P_n) \cup \{v'\}$. Hence S is a global dominating set of P_n'. Conversely, suppose that S is a global dominating set of P_n'. Therefore, S is a dominating set of both as in P_n' and $\overline{P}_n'$. But $V(P_n') = V(P_n) \cup \{v'\}$ and $V(\overline{P}_n') = V(\overline{P}_n') \cup \{v'\}$. Moreover, S being a global dominating set of P_n' a vertex in S which will dominate both the vertices v and v' in P_n' as well as in $\overline{P}_n'$. Hence, S is a dominating set of both P_n and $\overline{P}_n$. Therefore, S is a global dominating set of P_n. Hence, we have proved that S is a global dominating set of P_n if and only if it is a global dominating set of P_n'.

5.GLOBAL NEIGHBORHOOD DOMINATION NUMBER

Theorem: D be a minimum dominating set of G. Then $\gamma_{gn}(G) \leq 1 + \gamma(G)$ iff there is a vertex υ in $V - D$ satisfying:(i)$N(\upsilon) \subset D$, each of the vertices in $N(\upsilon)$is isolated in $< D >$.(ii) $\upsilon_1 \in V - D$ ($\upsilon \neq \upsilon_1$) satisfies (i) then $N(\upsilon) \cap N(\upsilon_1) \neq \emptyset$.

Proof: Assume that $\gamma_{gn}(G) = 1 + \gamma(G)$. Then there is a vertex υ in $V - D$ satisfying (i) and (ii), otherwise $\gamma_{gn}(G) = \gamma(G)$ which is a contradiction. Assume that the converse holds. Then $D \cup \{v\}$ is a gnd – set in G and D is not a gnd – set in G. Thus $D \cup \{v\}$ is a minimum gnd – set in G. Hence $\gamma_{gn}(G) = |D \cup \{\upsilon\}| = \gamma_c(G)+1$.

Theorem:G be a connected graph, the $\gamma(G) \leq \gamma_{gn}(G) \leq \gamma_c(G)$.

Proof: Clearly $\gamma(G) \leq \gamma_{gn}(G)$. Since any connected dominating set for G is a gnd – set for G, $\gamma_{gn}(G) \leq \gamma_c(G)$. Hence $\gamma(G) \leq \gamma_{gn}(G) \leq \gamma_c(G)$.

Theorem: G be a connected graph with $g(G) > 3$, then $\gamma_g(G) \leq \gamma_{gn}(G)$.

Proof: By hypothesis, every gnd – set is a global dominating set in G. Hence $\gamma_g(G) \leq \gamma_{gn}(G)$.

Theorem:(Characterization Result): G be a connected graph. $D \subset V$ is a gnd – set of G iff each vertex in $V - D$ lies on an edge whose end points are totally dominated by the vertices in D.

Proof: Assume that D is a gnd – set for G. Let $\upsilon_1 \in V - D$.Since D is a dominating set for G there is a $\upsilon_2 \in D$ such that $\upsilon_1\upsilon_2 \in E(G)$. Since D is dominating set for G^N there is a $\upsilon_3 \in D$ such that $< \upsilon_1\upsilon_4\upsilon_3 >$ is a path in G for some $\upsilon_4 \in V(G)$. If $\upsilon_4 \neq \upsilon_2$ then υ_1 lies on the edge $\upsilon_1\upsilon_4$, where υ_1 is dominated by υ_2 and υ_4 is dominated by υ_3 ($\upsilon_2, \upsilon_3 \in D$). If $\upsilon_4 = \upsilon_2$, then υ_1 lies on the edge $\upsilon_1\upsilon_2$, where υ_1 is dominated by υ_2 and υ_2 is dominated by υ_3($\upsilon_2, \upsilon_3 \in D$). So in either case υ_1 lies on the edge whose end points are totally dominated by the vertices in D. Assume that the converse holds.

Let $\upsilon_1 \in V - D$. Then by our assumption there is a $\upsilon \in V(G)$, $\upsilon_3, \upsilon_4 \in D$ such that $\upsilon_1\upsilon_3, \upsilon_2\upsilon_4 \in E(G)$.

Case(i): Suppose $\upsilon_2 = \upsilon_3$

Then $< \upsilon_1\upsilon_2\upsilon_4 >$ is a path in G.

$$\Rightarrow \upsilon_1\upsilon_4 \in E(G^N), \upsilon_4 \in D.$$

Case(ii): Suppose $\upsilon_2 \neq \upsilon_3$

Then $< \upsilon_3\upsilon_1\upsilon_2\upsilon_4 >$ is a path in G.

$$\Rightarrow \upsilon_1\upsilon_4 \in E(G^N), \upsilon_4 \in D$$

Therefore υ_1 is dominated by υ_3 in G and by υ_4 in G^N.Since υ_1 is arbitrary, D is a gnd – set of G.

6 CONCLUSION

In This paper deals about "A Review on global domination number of a graph".We discuss some structural properties corresponding to the concept of global dominating sets. Analogous results can be obtained for other graph families and in the context of various types of dominating sets in graphs.

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