A Comparative Study of Boundary Value Problem with Iterative method

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Abstract — *In this paper we give some special equation with B.V.P. further, to solve some Boundary Value Problem. We use iterative method. That is Newton Age iterative method to solve non-liner equation. We show that the compassion b/w the Newton Age method and SOR method using the finite difference formulation with exact solution of boundary value method. The New Age iterative method gives such a manner that the solution retains their accuracy and even order in the vicinity of the singularity. The proposed New-Age iterative method shows the superiority over the corresponding SOR iterative method.*

Keywords—*Partial Differential Equation, principle part, Boundary value problem, New AGE method*

I. INTRODUCTION

A partial differential equation elaborate a relation between an unknown function and its partial derivatives. We separate the family of linear equations for functions in two independent variables into three distinct types: parabolic (e.g., the heat equation), hyperbolic (e.g., the wave equation) and elliptic equations (e.g., the Laplace equation).

A partial differential equation can be define as

$$F\left(x_1,x_2,\dots.x_n;v,v_{x_1},\dots,v_{x_n},v_{x_1x_1},v_{x_1x_2},\dots\dots,v_{x_nx_n},v_{x_1x_1x_1}\dots\right)=0 \quad \dots(1)$$

where $x_i,\dots\dots\dots.x_{x_ix_j}\dots\dots\dots$ are its partial derivatives and the unknown $v=v(x_1,x_2,\dots\dots\dots.x_n)$ is a function of n variables. Equation (1) is linear if F is linear with respect to v and all its derivatives, otherwise it is nonlinear.

2: SPECIFIC EQUATIONS:-

Linear equations:-

2.1 Diffusion or heat equation

$$v_t + D.\Delta v = 0$$

Δ is the Laplace operator and D denotes the thermal properties of the material.

2.2 Laplace Equation:

The second order equation

$$\Delta v = 0$$

Where $v = v(x)$.

It is Non-homogeneous equation if

$$\Delta v = f$$

Then, it is called Poisson Equation.

2.3 Fisher's equation:

$$\frac{\partial v}{\partial t} - D\frac{\partial^2 v}{\partial x^2} = rv(1-v)$$

This is semi linear and second order equation.Now we take second-order linear differential equation for functions in x and y which are two independent variables. The equation of the form

$$\varsigma[v] = Av_{xx} + 2Bv_{xy} + Cv_{yy} + Dv_x + Ev_y + Fv = G \quad \dots(2)$$

Where $v(x,y)$ is the known function and A,B,C,D,E,F,G are functions of x and y. If A,B,C are the functions of x,y,v,v_x,v_y then equation (2) is called quasi-linear PDE.Let us take the coefficients A,B,C do not vanish. The operator

$$\varsigma_0[v] = Av_{xx} + 2Bv_{xy} + Cv_{yy}$$

Where operator ς_0 is called the principal part of ς.We know that much fundamental property of the solutions of equation (1) can be determined by principal part, and it's discriminate. We take $\delta\varsigma = B^2 - 4AC$ of the equation.

If $B^2 - 4AC > 0$ then the equation (1) is called Hyperbolic PDE.

If $B^2 - 4AC = 0$ then the equation (1) is called Parabolic PDE.

If $B^2 - 4AC < 0$ then the equation (1) is called Elliptic PDE.

3.General form of partial differential equation

A general form of PDE

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i\frac{\partial v}{\partial x_i} + b_0(x) = f(x)$$

where $x \in R^n$, $v = v(x)$

II. SOME BOUNDARY VALUE PROBLEMS

1. Dirichlet's boundary value problem:

The differential equation holds in a region δ bounded by A. The solution is known at the boundary A.

Let us take the function f on the boundary A of a region δ, to find a function 'v' such that

$$v_{xx} + v_{yy} = 0, \quad (x,y) \in \delta$$
$$(v(x,y))_A = f(x,y), \quad (x,y) \in A$$

2.Neumann boundary value problem:

On the boundary value problem A, the normal derivative is,

$$\left(\frac{\partial v}{\partial n}\right)_A = g(x,y), \quad (x,y) \in A$$
$$v_{xx} + v_{yy} = 0, \quad (x,y) \in \bar{\delta}$$

where $\bar{\delta} = \delta \cup A$

3. Periodic boundary value problem:

$$v_{xx} + v_{yy} = 0,$$

$(x, y) \in \bar{\delta}$

$v(x+m, y+m) = v(x,y) \quad \forall (x,y)$

where, m is the period and In this case $v(x,y)$ satisfies the periodicity problem.

4.Finite Difference Formulation:

We consider the following boundary value problems with boundary conditions

$v(0) = P \; and \; v(1) = Q$.

Where P and Q are constants.

$$\varepsilon[v(x)] \equiv v''(x) + f(x, v, v') = 0, \qquad 0 < x < 1 \quad \ldots(3)$$

Let us assume that for (1) to have unique solution then $f(x, v, v')$ has continuous derivatives which satisfy the equation (4) for some positive constants T^*, M_* and M^*

$$\left|\frac{\partial f}{\partial \phi}\right| \leq T^*, \; 0 < M_* \leq \frac{\partial f}{\partial \phi} \leq M^* \qquad \ldots(4) \; ,$$

With uniform grid is now placed on the interval $[0,1]$, if $a > 0$ be the mesh spacing with direction x, then $xt = ta$, $t = 0(1)N+1$ and $(N+1)a = 1$. The approx solution of equation (1) is by $v_t(t = 1(1)N)$ which are the solution of the non-linear difference equation with $u_0 = P$ and $u_{N+1} = Q$.

$$-(v_{t-1} - 2v_t - v_{t+1}) + \frac{h^2}{12}[\bar{F}_{t+1} + \bar{F}_{t-1} + 10\bar{\bar{F}}_{t+1}] = 0, t = 1(1)N \qquad \ldots(5)$$

5. The Newton Age Method:

Now we discuss the Newton Age method for the non linear difference equation,

We define,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \\ v_N \end{bmatrix}, \quad \vartheta(v) = \begin{bmatrix} \vartheta_1(v) \\ \vartheta_2(v) \\ \vdots \\ \vdots \\ \vartheta_N(v) \end{bmatrix}$$

The Jacobian of $\vartheta(v)$ can be shown to be the Nth order tri-diagonal matrix

$$Y \equiv \frac{\partial \vartheta(v)}{\partial v} = \begin{bmatrix} 2m_1(v) & n_1(v) & & & 0 \\ l_2(v) & 2m_2(v) & n_2(v) & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & l_N(v) & 2m_N(v) \end{bmatrix} \quad \ldots(6)$$

Where

$$l_t(v) = \frac{\partial \vartheta_t}{\partial v_{t-1}}, \quad t = 2(1)N$$

$$2m_t(v) = \frac{\partial \vartheta_t}{\partial v_t}, \; t = 1(1)N \qquad \ldots(7)$$

$$n_t(v) = \frac{\partial \vartheta_t}{\partial v_{t+1}}, \quad t = 1(1)N-1$$

For computing $\vartheta_t(v), l_t(v), m_t(v), n_t(v)$. We Use $v_0 = P$ and $v_{N+1} = Q$

We define

$$v^{(s+1)} = v^s + \Delta v^s \qquad s = 0,1,2,3, \ldots \ldots \ldots(8)$$

Where Δv^s is the solution of the system

$$Y \Delta v^s = -\vartheta(v^s) \qquad s = 0,1,2,3 \ldots \ldots \qquad \ldots(9)$$

Here we use the AGE iterative algorithm within inner loop to solve above equation (9)

Consider

$$Y = Y_1 + Y_2 \qquad \ldots(10)$$

$$Y_1 = \begin{bmatrix} \begin{bmatrix} m_1 & n_1 \\ l_1 & m_2 \end{bmatrix} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \begin{bmatrix} m_{N-1} & n_{N-1} \\ l_N & m_N \end{bmatrix} \end{bmatrix}_{N \times N}$$

$$Y_2 = \begin{bmatrix} [m_1] & & & & 0 \\ & \begin{bmatrix} m_2 & n_3 \\ l_3 & m_3 \end{bmatrix} & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \begin{bmatrix} m_{N-1} & n_{N-1} \\ l_N & m_N \end{bmatrix} \end{bmatrix}_{N \times N}$$

Taking N as even.We derive two steps alternating direction implicit methods as:

$$(Y_1 + \varphi_1 I)\Delta v^{(s+1/2)} = -\vartheta(v^s) - (Y_2 - \varphi_1 I)\Delta v^s \qquad \ldots(11a)$$

$$(Y_2 + \varphi_2 I)\Delta v^{(s+1)} = -\vartheta(v^s) - (Y_1 - \varphi_2 I)\Delta v^{(s+1/2)} \qquad \ldots(11b)$$

Since the matrices $(Y_1 + \varphi_1 I)$ and $(Y_2 + \varphi_2 I)$ consist of simple $(2 * 2)$ sub matrices then they can be easily inverted by inspection to give the Newton AGE method.

$$\Delta v^{(s+1/2)} = (Y_1 + \varphi_1 I)^{-1} - [\vartheta(v^s) - (Y_2 - \varphi_1 I)\Delta v^{(s+1)} \qquad \ldots(12a)$$

$$\Delta v^{(s+1)} = (Y_2 + \varphi_2 I)^{-1} - [\vartheta(v^s) - (Y_1 - \varphi_2 I)\Delta v^{(s+1/2)} \qquad \ldots(12b)$$

Where

$$(Y_1 + \varphi_1 I) = \begin{bmatrix} \left[\begin{bmatrix} m_1 & n_1 \\ -l_2 & \beta_1 \end{bmatrix} / \Delta_1\right] & & & 0 \\ & \ddots & & \\ & & \ddots & \\ & & & \left[\begin{bmatrix} \beta_{N-1} & -n_{N-2} \\ -l_{N-1} & \beta_{N-2} \end{bmatrix} / \Delta_{N-2}\right] \\ 0 & & & & [1/\beta N] \end{bmatrix} \qquad \ldots(13)$$

$$(Y_2 + \varphi_2 I) = \begin{bmatrix} [1/\gamma_1] & & & & 0 \\ & \left[\begin{bmatrix} \gamma_2 & -n_2 \\ -l_2 & \gamma_2 \end{bmatrix} / \gamma_2\right] & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \dfrac{\left(\begin{bmatrix} \gamma_N & -n_{N-1} \\ -l_N & \gamma_{N-1} \end{bmatrix}\right)}{\Delta_{N-1}} \end{bmatrix} \quad \ldots(14)$$

……(14)

Where $\gamma_t = m_t + \varphi_1, \ t = 1(1)N$ and $\Delta_t = \gamma_t \ \gamma_{t+1} - n_t l_{t+1}$, $t = 2(2)N-1$ and $\beta_t = m_t + \varphi_2, Y = 1(1)N$ and $\Delta_t = \beta_t \beta_{t+1} - n_t l_{t+1}, \ t = 1(2)N-2.$

Now using simple determination of the product of matrix $(Y_1 + \varphi_1 I)^{-1}(Y_2 + \varphi_1 I)$ and $(Y_2 + \varphi_1 I)^{-1}(Y_1 - \varphi_2 I)$ for parallel computing. We can expressed as 3 terms recurrence relations.

III.COMPARATIVE RESULT WITH NUMERICAL EXPERIMENT

We are using the non singular problem with B.C. for computational calculation. We have also exact solution of the problems which are using comparison. In all cases, we have considered $u^0 = 0$.Considered only 3-inner iteration for comparing the Newton Age iteration method with the corresponding Newton SOR iteration method.It is observed that the New Age method is quite effective in approximation the closed form solution of B.V.P.

Problem6.1: Non- Linear singular problem

$u''(x) + \ln x u'(x) + u = 2 + 2x \ln x + x^2$
With the boundary conditions, $u(0) = 0, u'(0) = 0$
The exact solution is given by $u(x) = x^2$. The root mean square (RMS) errors between Newton-AGE and Newton- SOR methods are tabulated in Table-1 .
Sol:

		$Newton - SOR$		$Newton - AGE$		
$S.No.$	N	ω	$iter$	ω	$iter$	$RMS\ error$
1	10	1.320	08	0.382	07	$7.2920e(-04)$
2	20	1.500	15	0.290	15	$1.3752e(-06)$
3	30	1.550	37	0.293	34	$4.4566e(-07)$

Table-1

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