Congruence Lattices of Uniform Lattices

I.Athal [*], .M.Karthivel [**]

[*]*Assistant Professor, Department of Mathematics, PSNA College Of Engineering and Technology, Dindigul, Tamilnadu, India.*
[**]*Assistant Professor, Department of Mathematics, PSNA College Of Engineering and Technology, Dindigul, Tamilnadu, India.*

1.1 INTRODUCTION

In this chapter we prove that every finite distributive lattice D can be represented as the congruence lattice of finite uniform lattice L. Infact we prove that "For any finite distributive lattice D, there exists a finite uniform lattice L such that the congruence lattice of L is isomorphic to D, and L satisfies the properties (P) and (Q) where

(P) Every join-irreducible congruence of L

we introduce a very simple kind of chopped lattices. we prove that the ideal lattice of this chopped lattice is uniform.

is of the form q (0,p), for a suitable atom p of L.

(Q) If q_1, q_2 ,…………., q_n Œ J (ConL)

are pairwise incomparable, then L contains atoms p_1, p_2,………, p_n that generate an ideal isomorphic to B_n and satisfy q_i=q (0,p_i), for all i £ n.

To prove this result, we introduce a new lattice construction which is described . Then we find the congruences on this new lattice

NOTATION:

B_n will denote the Boolean algebra with 2^n elements. For a bounded lattice A with bounds 0 and 1, A^- will denote the lattice A – {0,1}

We start with the definition of uniform lattices.

PROOF OF THE MAIN RESULT

THEOREM : 1.2

For any finite distributive lattice D, there exists a finite uniform lattice L such that the congruence lattice of L is isomorphic to D and L satisfies the properties (P) and (Q) where

(P) : Every join-irreducible congruence of L is of the form q(0,p), for a suitable atom p of L.

(Q) : If $q_1,q_2,\ldots,q_n$ Œ J(ConL) are pairwise incomparable, then L contains atoms $p_1,p_2,\ldots,p_n$ that generate an ideal isomorphic to B_n and satisfy q_i=q $(0,p_i)$, for all i£n.

Proof :-

We prove the result using induction on n, where n is the number of join-irreducible elements.

Let D be a finite distributive lattice with n join-irreducible elements.

If n = 1, then D @ B_1, so there is a lattice L=B_1 that satisfies the theorem 1.2.

Let us assume that, for all finite distributive lattices with fewer than n join-irreducible elements, there exists a lattice L satisfying theorem 2.6.1 and properties (P) and (Q).

Assume that D has n join-irreducible elements.

Let q be a minimal element of J(D).

Let $q_1,q_2,\ldots,q_k(k\geq 0)$ be all upper bounds of q in J(D).

Let D_1 be a distributive lattice with $J(D_1)$=J(D)-{q}.

By induction assumption there exists a lattice L_1 satisfying Con L_1@D_1 and (P) and (Q).

If k = 0, then D @ B_1 x D_1 and L = B_1 x L_1, obviously satisfies all the requirements of the theorem and so the proof is over.

So, assume $k \geq 1$

The congruences of L_1 corresponding to the q_i's are pairwise incomparable and therefore can be written in the form $q(0,p_i)$ and the p_i's generate an ideal I_1 isomorphic to B_k.

The lattice $N(B_2,B_k)$ also contains an ideal $(B_k)_*$ isomorphic to B_k.

Identifying I_1 and $(B_k)_*$, We get the chopped lattice K and the lattice L=IdK.

By lemma IdK is uniform. That is L is uniform.

Let q be a join-irreducible congruence of L.

Then we can write q as q(a,b) where a is covered by b.

By lemma., it follows that we can assume that either a, b Œ L_1, or a, b Œ $N(B_2,B_k)$

In either case, there exists an atom q in L_1 or q in $N(B_2, B_k)$ so that

q (a,b)= q(0,q) in L_1 or q(a,b)=q(0,q) in $N(B_2, B_k)$.

Obviously, q is an atom of Land q(a,b)=q(0,q) in L verifying (P)for L.

Let $q_1, q_2,\ldots,q_t$ be pairwise in-comparable join-irreducible congruences of L.

To verify condition (Q), we have to find atoms $p_1,p_2,\ldots,p_t$ of L satisfying $q_i = q(0,p_i)$ for all $i £ t$ and such that $p_1,p_2,\ldots,p_t$ generate an ideal of L isomorphic to B_t.

Let p denote an atom in $N(B_2,B_k) - I_1$

Infact, there are two atoms but they generate the same congruence q(0,p).

If q(0,p) is not one of $q_1,q_2,\ldots\ldots,q_t$ then clearly we can find $p_1,p_2, \ldots, p_t$ in L_1 as required and $p_1,p_2, \ldots\ldots,p_t$ also serves in L.

If q(0,p) is one of $q_1,q_2,\ldots\ldots,q_t$ say $q(0,p) = q_t$, then let $p_1,p_2,\ldots\ldots,p_{t-1}$ be the set of atoms establishing (Q) for $q_1,q_2,..,q_{t-1}$ in L_1 and therefore in L.

Then $p_1,p_2,\ldots\ldots,p_{t-1},p$ represent the congruences $q_1,q_2,\ldots\ldots,q_t$ and they generate an ideal isomorphic to B_t by lemma .

Therefore L satisfies (Q).

It is clear from this discussion that J(ConK) has exactly one more element than J(ConL_1), namely, q(0,p).

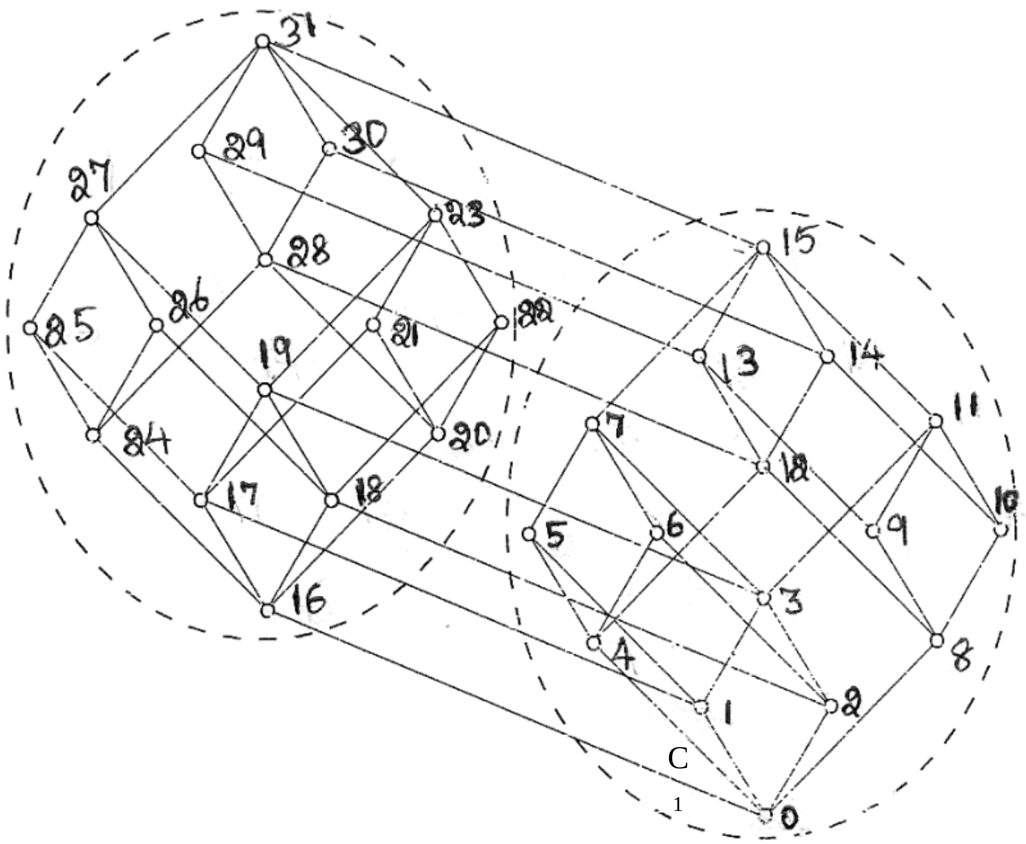
This join-irreducible congruence relates to the join-irreducible congruences of ConL, exactly as q relates to the join-irreducible elements of D.

Therefore D @ ConL.

Hence the theorem.

EXAMPLE :

The uniform construction for the four-element chain is



This lattice has four congruences. C_o has 32 blocks. C_0 is a null congruence

C_1 has 8 blocks.

C_1={{0,1,2,3},{4,5,6,7},{8,9,10,11},{12,13 14,15},(16,17,18,19},{20,21,22,23}, {24,25,26,27},{28,29,30,31} }.

C_2 has 2 blocks.

C_2={{0,1,2,3,4,5,6,7,8,9,10,1

1,

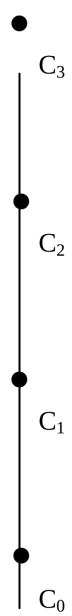
12,13,14,15},{16,17,18,19,20,21,22,23,2
4,

25,26,27,28,29,30,31}

}. C_3 has 1 block.

C_3 is all
congruence.

The congruence lattice of this lattice is



∴

Every finite distributive lattice D can be represented

as the congruence lattice of a finite uniform lattice L.

REFERENCES

1) G.Gratzer, H.Lakser, and E.T.Schmidt, Congruence lattices of small planar lattices. Proc. Amer. Math. Soc. 123(1995), 2619-2623.
2) G.Gratzer, General Lattice Theory, Second edition, Birkhauser Verlag, Basel, 2010.
3) G.Gratzer and E.T.Schmidt, Congruence-preserving extensions of finitelattices in to sectionally complemented lattice, Proc. Amer.Math.Soc.127(1999), 1903-1915.
4) G.Gratzer and F.Wehrung, Proper congruence-preserving extensions of lattices, Acta Math. Hungar. 85(1999),175-185.
5) G.Gratzer and E.T.Schmidt, Regular congruence-preserving extensions. Algebra Universalis 46(2008), 119-130.
6) G.Gratzer, E.T.Schmidt, and K.Thomsen, Congruence lattices of uniform lattices. Houston. J.Math.29 (2011).
7) G.Gratzer and E.T.Schmidt, Finite lattices with isoform congruences. Tatra.Mt.Math.Publ 27(2009), 111-124.
8) G.Gratzer and E.T.Schmidt, Finite lattices and congruences. Algebra Universalis.
9) R.Freese-UACALC program-Sec:http://WWW. Math. hawaii. edu/~ralph/UACALC/.