Applications of Mathematics in Engineering and Sciences

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Abstract

Mathematics is the back bone of all Engineering branches and Sciences. The students might have studied the Mathematical subjects such as Linear Algebra and Calculus but during the course of Engineering they find it difficult to relate themto the Engineering problems. The present paper addresses the applications of Mathematics in particular Laplace Transforms, Fourier series, Differential and Partial Differential equations in the Mechanical, Electrical and communication, Computer Sciences.This may givebetter understanding of the applications of Mathematics to the students and motivate them in applying these concepts in their respective areas.

Key words: Laplace transforms, Fourier series, discrete mathematics, Ordinary and Partial differential equations.

Introduction: The solutions of physical problems have been a challenge to the scientists and engineers. The use of Laplace Transforms simplifies the analysis of electronic circuits and solutions of linear differential equations. Laplace transform converts the system transform function or differential equation into s-domain, using s-domain to convert input functions, finding an output function algebraically combining input and transfer functions, using partial fractions to reduce the output function to simpler components and conversion of output equation back to time domain. Laplace transform is used to solve the problems of time invariant systems such as electrical circuits, harmonics, oscillations, mechanical system, control theory and optical devices. Laplace transform simplifies the process of analysing the behaviour of a dynamic or synthesizing a new system (H.K.Dass)

It was observed by many researchers and electrical engineers that through Fourier series noise in the signal can be identified and this noise can be eliminated by Inverse Fast Fourier Transfrom. Matt Hollingsworth, Duhamel, Piron and Etcheto to name a few. The Inverse Fourier Transform has wide applications in Digital Signal Processing which cleans, transform or amplifies a signal. Fourier transform is used to decomposing the signals into its constituent frequencies and its oscillatory functions. It represents signal in a frequency domain and transforms one complex value function of a real variable into another. A representation of nonperiodic function as an integral over a continuous range of frequencies can be obtained by using Fourier transform, which involves conversion of a Fourier series into a double infinite series of complex exponentials [K.A.Stroud]

In analysing a system, we come across Time Invariant, Linear Differential Equations of second and higher orders, which are difficult to solve by using the methods of ordinary differential equations. By applying Laplace transformations these equations result in an algebraic equation in terms of the transform function of the unknown quantity.

Limit of the current I in an L, R, C circuit: An impulsive voltage $E\delta(t)$ is applied to a circuit consist of an inductance L, a resistance R and a condenser C in a series with zero initial conditions (B.S.Grewal). This kind of problem appears in the field of **Electrical and Electronic Engineering.** We find the limit of I as $t \to 0$.

Using the Kirchhoff laws of voltage, the equation of the circuit governing the current I is written as

$$L\frac{dI}{dx} + RI + \frac{1}{C}\int_0^t I dt = E\delta(t) \text{ where I = 0, when } t = 0 \qquad (1)$$

Applying Laplace transform on both sides of (1), we get

$$L\left\{L\frac{dI}{dx} + RI + \frac{1}{C}\int_0^t I dt\right\} = L[E\delta(t)] \qquad (2)$$

$$\Rightarrow L\{s\,L(I) - I(0)\} + RL(I) + \frac{1}{Cs}L(I) = E \qquad (3)$$

$$\because L\left[\frac{dy}{dx}\right] = sL[y] - y(0) \;\; and \; L[\int f(t)dt] = \frac{L[f(t)]}{s}$$

$$\Rightarrow \left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right)L[I] = \frac{E}{L}s$$

$$\Rightarrow (s^2 + 2as + a^2 + b^2)L[I] = \frac{E}{L}s \;\; where \; \frac{R}{L} = 2a \; and \; \frac{1}{CL} = a^2 + b^2 \qquad (4)$$

$$\Rightarrow L[I] = \frac{E}{L(s^2+2as+a^2+b^2)}s \qquad (5)$$

Equation (5) can be solved by reducing the right hand side into partial fraction

$$\Rightarrow L[I] = \frac{E}{L}\left\{\frac{s+a-a}{(s+a)^2+b^2}\right\} = \frac{E}{L}\left\{\frac{s+a}{(s+a)^2+b^2} - \frac{a}{(s+a)^2+b^2}\right\} \qquad (6)$$

Applying the Inverse Laplace transform on both sides of (6), we get

$$I = \frac{E}{L}\{e^{-at}(\cos(bt) - \sin(bt))\} \qquad (7)$$

by First shifting theorem of Laplace transforms and (7) gives the current in the circuit at any time t.

Taking the limit $t \to 0$ on both sides of (7), we get $I = \frac{E}{L}$

Though the initial current I = 0, a large current will develop instantaneously due to the impulse voltage applied at t = 0.

Displacement of any point in a vibrating string: A string of length l is stretched and fastened at two end points. String started vibrating by displacing the string in the form of

$y = a\sin(\frac{\pi x}{l})$and released at time $t = 0$. We can find the displacement of any point at a distance x from one end at any time t (B.S.Grewal) This type of problem we come across in the field of **mechanical vibratory systems** and electrical engineering.

The governing equation of the vibrating string is $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ (1)

Since the two ends of the string have been fixed, for all time we get the boundary conditions

$$y(0,t) = 0 \; and \; y(l,t) = 0 \qquad (2)$$

As the string is at rest when time $t = 0$, initial transverse velocity at any point of the string is,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad (3)$$

and the initial displacement is in the form $y(x,0) = a\sin(\frac{\pi x}{l})$ (4) Solving (1),by separation of variables technique displacement at any time t, at any point x is given by

$$y(x,t) = (A\cos mx + B\sin mx)(C\cos cmt + D\sin cmt) \quad (5)$$

By (2), $y(0,t) = A(C\cos cmt + D\sin cmt) = 0 \Rightarrow A = 0\ for\ all\ t$ (6)

Hence, $y(x,t) = (B\sin mx)(C\cos cmt + D\sin cmt)$ (7)

$$\Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = B\sin mx(Dcm) = 0, by\ (3)$$

$$\Rightarrow BDcm = 0$$

If $B = 0$, (7) will give $y(x,t) = 0$ which is a trivial solution. Therefore, $D = 0$. (8)

Hence, (7) becomes $y(x,t) = BC\sin mx\cos cmt$ (9)

By (2), $y(l,t) = BC\sin ml\cos cmt = 0\ for\ all\ t$

Since B and C $\neq 0$, $\sin(ml) = 0 \Rightarrow ml = n\pi \Rightarrow m = \frac{n\pi}{l}, where\ n\ is\ an\ integer$ (10)

Therefore, the solution of the vibrating string (1), with the boundary and initial conditions (2) and (3) respectively is

$$y(x,t) = BC\sin(\frac{n\pi x}{l})\cos(\frac{cn\pi t}{l}) \quad (11)$$

Using the condition (4), $y(x,0) = BC\sin\left(\frac{n\pi x}{l}\right) = a\sin\left(\frac{\pi x}{l}\right) \Rightarrow BC = a\ and\ n = 1$

Hence the required solution of (1) is $y(x,t) = a\ \sin(\frac{\pi x}{l})\cos(\frac{c\pi t}{l})$ (12)

We can observe that (12) is a stationary sine wave of varying amplitude whose frequency is $c/2l$ and the motion of the each point $y(x,t)$ of the string is simple harmonic with period $2l/c$.

Adding all solutions of the form (11), we get $\sum_{n=1}^{\infty} y(x,t) = \sum_{n=1}^{\infty} BC\sin(\frac{n\pi x}{l})\cos(\frac{cn\pi t}{l})$

which is Fourier series of $y(x,0) = a\sin(\frac{\pi x}{l})$, $0 < x < l$.

In analysing the signal systems, it will be useful to approximate a periodic function with period T, as a **Fourier Series**, which is a sum of sine and cosine terms. We can find the spectrum of a periodic signal,

$$f(t) = \begin{cases} -k, & for\ -\pi < t < 0 \\ k, & for\ 0 < t < \pi \end{cases} \quad (1)$$

through its Fourier Series representation. Such type of problems we can come across in the Electrical communications.

The given signal $f(t)$ is periodic with period T = 2π, which can be approximated as a Fourier Series,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cos\ nt + b_n Sin\ nt \quad (2)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$ (3)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) Cos\ nt\ dt \quad (4)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) Sin\ nt\ dt \quad (5)$$

For the given $f(t)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\int_{-\pi}^{0} -k\ dt + \frac{1}{\pi}\int_{0}^{\pi} k\ dt = \frac{2k\pi}{k} = 2k \quad (6)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)Cos\,nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\,Cos\,nt\,dt + \int_{0}^{\pi} k\,Cos\,nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-ksin\ nt}{n}\right)_{-\pi}^{0} + \left(\frac{ksin\ nt}{n}\right)_{0}^{\pi}\right] = 0 \quad (7)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)Sin\,nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\,Sin\,nt\,dt + \int_{0}^{\pi} k\,Sin\,nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-kCos\ nt}{n}\right)_{-\pi}^{0} + \left(\frac{kCos\ nt}{n}\right)_{0}^{\pi}\right] = \begin{cases} 0, & for\ n\ even \\ \frac{2k}{\pi(2n-1)}, & othewise \end{cases} \quad (8)$$

Therefore, the Fourier Series representation of the given signal $f(t)$ is

$$f(t) = \frac{k}{2} + \sum_{n=1}^{\infty} \frac{2k}{\pi(2n-1)} Sin\,(2n-1)t \quad (9)$$

$$= \frac{k}{2} + \frac{2k}{\pi} Sin\,t + \frac{2k}{3\pi} Sin\,3t + \frac{2k}{5\pi} Sin\,5t + \frac{2k}{7\pi} Sin\,7t + \cdots \quad (10)$$

The spectrum of this signal is $\frac{k}{2}, \frac{2k}{\pi}, \frac{2k}{3\pi}, \frac{2k}{5\pi}$, ... exists at the frequencies ω= $0, t, 2t, 3t, \ldots$

Conclusions:

The Laplace Transforms play an important in solving the Differential Equations that come across in the Engineering branches. Using Fourier Series we can analyse the spectrum of a signal and a non period function can be expressed as a Fourier series will be useful in the communication engineering. The boundary value problems can be solved by using Fourier Series.

References:

1. H.K.Das., Advanced Engineering Mathematics ,S.Chand and company limited, New Delhi, 2009.
2. K.A.Stroud., Advanced Engineering Mathematics, Mac Williams, 2003.
3. B.S.Grewal, Higher Engineering Mathematics, Khanna Publications, 40th edition.