Case Study of the Group$GL(n,\mathbb{Z})$

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Abstract: *In this Article, we have discussed some of the properties of the infinite non-abelian group* $GL(n,\mathbb{Z}), n \geq 2$. *Such as the number of elements of order 2, number of subgroups of order 2 in this group and embedding of the* $GL(n,\mathbb{Z})$ *in* $GL(m,\mathbb{Z})\forall\ m \geq n$, *Moreover for every finite group* G, *there exists* $k \in \mathbb{N}$ *such that* $GL(k,\mathbb{Z})$ *has a subgroup isomorphic to the group* G.

Keywords: *Infinite non-abelian group,* $GL(n,\mathbb{Z})$

Notations: $GL(n,\mathbb{Z}) = \{[a_{ij}]_{n\times n} : a_{ij} \in \mathbb{Z}\ \&\det([a_{ij}]_{n\times n}) = \pm 1\}$

Theorem1: $GL(n,\mathbb{Z})$ can be embedded in $GL(m,\mathbb{Z})\forall\ m \geq n$.

Proof: If we prove this theorem for $m = n + 1$, then we are done.

Let us define a mapping $\varphi : GL(n,\mathbb{Z}) \longrightarrow GL(n+1,\mathbb{Z})$

Such that $$\varphi\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & \dots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & & a_{nn} & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$

Such that determinant of A and B is ± 1 , where a_{ij} , $b_{ij} \in \mathbb{Z}$.

Then $A.B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} . \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$

$= \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$

Where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots\ a_{in}b_{nj}$

$$\Rightarrow \varphi(A.B) = \begin{bmatrix} c_{11} & \dots & c_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ c_{n1} & & c_{nn} & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \quad \cdots(1)$$

$$\varphi(A).\varphi(B) = \begin{bmatrix} a_{11} & \dots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & & a_{nn} & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} . \begin{bmatrix} b_{11} & \dots & b_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ b_{n1} & & b_{nn} & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & \cdots & c_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ c_{n1} & & c_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \qquad \cdots (2)$$

Now from (1) and (2) one can easily say that φ is homomorphism.

Now consider $ker\varphi$,

$$Ker\varphi = \{ A \in GL(n,\mathbb{Z}) : \varphi(A) = I_{(n+1)\times(n+1)} \} \quad where\ I\ is\ identity\ matrix$$

Clearly $Ker\varphi = \{I_{n\times n}\}$

Hence φ is injective homomorphism.

So, by fundamental theorem of isomorphism one can easily conclude that $GL(n,\mathbb{Z})$ can be embedded in $GL(m,\mathbb{Z})$ for all m $\geq n$.

Theorem 2: $GL(n,\mathbb{Z})$ is non-abelian infinite group $n \geq 2$.

Proof: Consider $GL(2,\mathbb{Z}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}\ and \det(A) = \pm 1 \right\}$

Consider $A_c = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ $such\ that\ c \in \mathbb{Z}$

And let H= $\{A_c : c \in \mathbb{Z}\}$

Clearly H is subset of $GL(2,\mathbb{Z})$.

And H has infinite elements.

Hence $GL(2,\mathbb{Z})$ is infinite group.

Now consider two matrices $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \in GL(2,\mathbb{Z})$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

And

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 1 & 3 \end{bmatrix}$$

Hence, $GL(2,\mathbb{Z})$ is non-abelian.

And by a direct application theorem 1, one can conclude that $GL(n,\mathbb{Z})$ is infinite and non-abelian$\forall\ n \geq 2$

Theorem 3: Number of elements of order 2 in $GL(n,\mathbb{Z})$ is infinite and hence number of subgroups of order 2 is infinite.

Proof: Consider $GL(2,\mathbb{Z}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}\ and \det(A) = \pm 1 \right\}$

Let A $\in GL(2,\mathbb{Z})$, and order of A is 2.

Then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $for\ some\ a, b, c, d\ \in \mathbb{Z}\ and\ A^2 = I$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2 + bc & ab + bd \\ ca + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^2 + bc = 1, ab + bd = 0, ca + cd = 0, bc + d^2 = 1$$

Since $a^2 + bc = 1$, then:

Case (i):$a^2 = 1, bc = 0$

$$\Rightarrow a = \pm 1\ and\ either\ b\ or\ c = 0$$

Subcase (i): If we take $a = 1\ and\ b = 0$

Then $\begin{bmatrix} 1 & 0 \\ c + cd & d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow c + cd = 0$$

$$\Rightarrow c(1 + d) = 0$$

$$\Rightarrow either\ c = 0\ or\ d = -1$$
$$\Rightarrow if\ c \neq 0\ then\ d = -1$$

Let $c \neq 0$

Then our case is $a = 1, b = 0, c \neq 0, d = -1$

$\Rightarrow$ $\begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}$has order 2 if $c \neq 0\ and\ c \in \mathbb{Z}$

Clearly c has infinite choices.

⇒ Number of elements of order 2 in $GL(2, \mathbb{Z})$ is infinite.

And since number of elements of order 2 = Number of subgroups of order 2 in any group.

⇒ Number of subgroups of order 2 in $GL(2, \mathbb{Z})$ is infinite.

Also by a direct application of theorem 1, one can easily conclude that this theorem is valid for every value of n.

Theorem 4:Every finite group can be embedded in S_n for some$n \in \mathbb{N}$

Proof: Let G be any group and A(G) be the group of all permutations of set G.

For any $a \in G$, define a map $f_a: G \longrightarrow G$ such that $f_a(x) = ax$

Then as $x = y \Rightarrow ax = ay$

$$\Rightarrow f_a(x) = f_a(y)$$

Hence, f_a is well defined

Clearly f_a is one-one.

Also for any $y \in G$, Since $f_a(a^{-1}y) = y$.

$\Rightarrow f_a$ is onto.

And hence f_a is permutation.

Let K be set of all such permutations

Clearly K is subgroup of A(G).

Now define a mapping $\varphi : G \longrightarrow K$ such that $\varphi(a) = f_a$

Clearly φ is well-defined and one-one map.

And consider the following equation

$$\varphi(a.b) = f_{ab} = f_a o f_b = \varphi(a).\varphi(b)$$

Which shows that φ is homomorphism

Obviously, φ is onto homomorphism

$\Rightarrow \varphi$ is isomorphism.

And hence the theorem.

Theorem 5: S_n is isomorphic to some subgroup of $GL(n, \mathbb{Z})$ for all $n \in \mathbb{N}$.

Proof: Let S_n be the permutation group on n symbols.

Define $\varphi : S_n \longrightarrow GL(n, \mathbb{Z})$ such that: $\varphi(\sigma) = [\sigma]_{n\times n} \ \forall \sigma \in S_n$

Where $[\sigma]_{n\times n}$ is permutation matrix obtained by σ i.e. if $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix}$ then

$$[\sigma]_{n\times n} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Where R_i is $R_{\beta_i}^{th}$ row of identity matrix.

Clearly φ is a homomorphism.

Now consider the kernel of this homomorphism.

$\ker \varphi = \{\sigma : \varphi(\sigma) = I_{n\times n}\} \Rightarrow i = \beta_i \ \forall i$

$\Rightarrow \ker \varphi$ is trivial.

Hence the homomorphism is injective.

$\Rightarrow$ S_n is isomorphic to some subgroup of $GL(n, \mathbb{Z})$ for all $n \in \mathbb{N}$.

Theorem 6: For every finite group G, there exists $k \in \mathbb{N}$ such that $GL(k, \mathbb{Z})$has a subgroup isomorphic to the group G.

Proof: It is an obvious observation of theorem 4 and theorem 5.

CONCLUSION

$GL(n, \mathbb{Z})$is non-abelian infinite group having infinite number of elements of order 2 as well as subgroups of order 2. Also $GL(n, \mathbb{Z})$ can be embedded in $GL(m, \mathbb{Z}) \forall\ m \geq n$, Moreover for every finite group G, there exists $k \in \mathbb{N}$ such that $GL(k, \mathbb{Z})$has a subgroupisomorphic to the group G.

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