

A Review on Lower Bounds for the Domination Number

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ABSTRACT – We prove several Lower bounds on the domination number of simple connected graph. In this paper we prove that $(2k+1) \gamma_k(T) \geq |V| + 2k - kn_1$ for each tree. $T=(V,E)$ with n_1 leafs, and we characterize the class of tree that satisfy the equality $(2k+1) \gamma_k(T) \geq |V| + 2k - kn_1$.

Keywords: Domination, Distance domination number, Tree.

1. INTRODUCTION

Domination in graphs has been studied extensively in recent years. The study of domination in graphs originated around 1850 with the problems of placing minimum number of queens on an $n \times n$ chessboard so as to cover or dominate every square. With very few exceptions these problems still remain unsolved today. The theory of domination in graphs introduced by Ore and Berge is an emerging area of research in graph theory today

Berge presents the problem of five queens, namely, place five queens on the chess board so that every square is covered by at least one queen. The solution to these problems are nothing but dominating sets in the graph, whose vertices are the queens of the chessboard and vertices u, v are adjacent if a queen move from u to v in one move. This leads to domination in graphs.

2. APPLICATION OF GRAPH THEORY

Graph theoretical concepts are widely used to study and model various applications, in different areas. They include, study of molecules, construction of bonds in chemistry and the study of atoms. Similarly, graph theory is used in sociology for example to measure actors prestige or to explore diffusion mechanisms.

Graph theory is used in biology and conservation efforts where a vertex represents regions where certain species exist and the edges represent migration path or movement between the regions. this information is important when looking at breeding patterns or tracking the spread of disease, parasites and to study the impact of migration that affect other species. Graph theoretical concepts are widely used in Operations Research. For example, the travelling salesman problem, the shortest spanning tree in a weighted graph, obtaining an optimal match of jobs and men and locating the shortest path between two vertices in a graph. It is also used in modeling transport networks, activity networks and theory of games.

The network activity is used to solve large number of combinatorial problems. The most popular and successful applications of networks in OR is the planning and scheduling of large complicated projects. The best well known problems are PERT(project Evaluation Review Technique)and CPM(Critical Path Method). Next, Game theory is applied to the problems in engineering, economics and war science to find optimal way to perform certain tasks in competitive environments to represent the method of finite game a digraph is used. Here, the vertices represent the positions and the edges represent the moves. Everything in our world is linked cities are linked by street, rail and flight networks. Pages on the internet are linked by hyperlinks. The different components of an electric circuit or computer chip are connected and the paths of disease outbreaks form a network. Scientists, engineers and many others want to analyze, understand and optimize these networks. And this can be done using graph theory. For example, mathematicians can apply graph theory to road networks, trying to find a way to reduce traffic

congestion. An idea which, if successful, could save millions every year which are lost due to time spent on the road as well as mitigating the enormous environmental impact. It could also make life safer by allowing emergency services to travel faster and avoid car accidents in the first place. These intelligent transportation systems could work by collecting location data from smart phones of motorists and telling them where and how fast to drive in order to reduce overall congestion.

Graph theory is already utilized on flight networks. Airlines want to connect countless cities in the most efficient way, moving the most passengers with the fewest possible trips a problem very similar to the travelling Salesman. At the same time, air traffic controllers need to make sure hundreds of planes are at the right place at the right time and don't crash an enormous task that would be almost impossible without computers and graph theory. One area where speed and the best connections are of crucial importance is the design of computer chips. Integrated circuits (ICs) consist of millions of transistors which need to be connected. Although the distances are only a few millimeters, it is important to optimize these countless connections to improve the performance of the chip. Graph theory also plays an important role in the evolution of animals and languages, crowd control and the spread of diseases.

3. LOWER BOUNDS FOR THE DOMINATION NUMBER

Several lower bounds on the domination number of simple connected graphs. Among these are the following: the domination number is at least two-thirds of the radius of the graph, three times the domination number is at least two more than the number of cut-vertices in the graph, and the domination number of a tree is at least as large as the minimum order of a maximal matching.

Theorem: Let G be a connected graph with $n > 1$ and diameter d . Then,

$$\gamma \geq \frac{d+1}{3}$$

Proof : Since the diameter can actually equal the radius, it is sometimes twice as good as a lower bound on the domination number (take cycles for

instance). Moreover, it is similar to the well known result that the independence number is at least the radius – originally a conjecture of Graffiti and proven independently several times. In addition, it is that the total domination number (that is, the cardinality of a set of minimum order having the property that every vertex in the graph is adjacent to a vertex in the set) is at least the radius.

Theorem: Let G be a connected graph with $n > 1$.

Then, $\gamma \geq \frac{2}{3}r$ Moreover, this bound is sharp.

Proof : Let D be a minimum dominating set of G . Form a spanning tree T of G , as prescribed in statement, so that D is also a minimum dominating set of T . Since $r(G) \leq r(T)$, $2r(T) - 1 \leq d(T)$ (because T is a tree) and $\gamma(T) = \gamma(G)$, we can apply to T and obtain the following chain of inequalities:

$$2r(G) - 1 \leq 2r(T) - 1 \leq d(T) \leq 3\gamma(T) - 1 = 3\gamma(G) - 1.$$

Equality holds in the bound above for cycles with orders congruent to 0 modulo 6. On the other hand, the tree obtained by amalgamating a pendant vertex to each vertex of a path has radius about $n/3$ while it has domination number of $\frac{n}{2}$ – thus showing that the difference between these two expressions can be made arbitrarily large.

Theorem 3.1.3

For any connected graph G with x cut-vertices,

$$\gamma \geq \frac{x+2}{3}$$

Moreover, this bound is sharp.

Proof

Let D be a minimum dominating set of G . Form a spanning tree T of G , as prescribed in statement. So that D is also a minimum dominating set of T . Let $x(T)$ denote the number of cut-vertices of T and note that $x(T) \geq x$, since any cut-vertex of G is also a cut-vertex of T . Now, applying Theorem to T we find,

$$\gamma(G) = \gamma(T) \geq \frac{n-1(T)+2}{3} = \frac{x(T)+2}{3} \geq \frac{x+2}{3}$$

If T is a tree such that the distance between any two leaves is congruent to 2 modulo 3. Since for trees, the number of cut-vertices is exactly $n - 1$, equality

holding in The sufficient condition for equality holding in the above Theorem. An example of a graph here equality holds in this Theorem that is not necessarily a tree is a graph with a cut-vertex of degree $n - 1$. Since cycles have no cut-vertices, the difference between the expressions in Theorem can be made arbitrarily large.

4. LOWER BOUNDS ON THE DISTANCE DOMINATION NUMBER OF A GRAPH

Let $k \geq 1$ be an integer and let G be a graph. In 1975, Meir and Moon introduced the concept of a distance k -dominating set (called a “ k -covering”) in a graph. A set S is a k -dominating set of G if every vertex is within distance k from some vertex of S ; that is, for every vertex v of G , we have $d(v, S) \leq k$. The k -domination number of G , denoted $\gamma_k(G)$, is the minimum cardinality of a k -dominating set of G . When $k = 1$, the 1-domination number of G is precisely the domination number of G ; that is, $\gamma_1(G) = \gamma(G)$.

Lemma: For $k \geq 1$, every connected graph G has a spanning tree T such that $\gamma_k(T) = \gamma_k(G)$.

Proof : Let $S = \{V_1, \dots, V_\ell\}$ be a minimum k -dominating set of G . Thus, $|S| = \ell = \gamma_k(G)$. We now partition the vertex set $V(G)$ into ℓ sets V_1, \dots, V_ℓ as follows.

Initially, we let $V_i = \{V_i\}$ for all $i \in [\ell]$. We then consider sequentially the vertices not in S . For each vertex $v \in V(G) \setminus S$, we select a vertex $V_i \in S$ at minimum distance from v in G and add the vertex v to the set V_i . We note that if $v \in V(G) \setminus S$ and $v \in V_i$ for some $i \in [\ell]$, then $d_G(v, V_i) = d_G(v, S)$, although the vertex V_i is not necessarily the unique vertex of S at minimum distance from v in G . Further, since S is a k -dominating set of G , we note that $d_G(v, V_i) \leq k$. For each $i \in [\ell]$, let T_i be a spanning tree of $G[V_i]$ that is distance preserving from the vertex v_i ; that is, $V(T_i) = V_i$ and for every vertex $v \in V(T_i)$, We have $d_{T_i}(v, V_i) = d_G(v, V_i)$. We now let T be the spanning tree of G obtained from the disjoint union of the ℓ trees T_1, \dots, T_ℓ by adding $\ell - 1$ edges of G . We remark that these added $\ell - 1$ edges exist as G is connected. We now consider an arbitrary vertex, v say, of G . The vertex $v \in V_i$ for some $i \in [\ell]$. Thus $d_T(v, V_i) \leq d_{T_i}(v, V_i) = d_G(v, V_i) = d_G(v, S) \leq k$. Therefore, the set S is a

k -dominating set of T , and so $\gamma_k(T) \leq |S| = \gamma_k(G)$. However, by Observation , $\gamma_k(G) \leq \gamma_k(T)$.

Lemma :

Let G be a connected graph that contains a cycle, and let C be a shortest cycle in G . If v is a vertex of G outside C that k -dominates at least $2k$ vertices of C , then there exist two vertices $u, w \in V(C)$ that are both k -dominated by v and such that a shortest (u, v) -path does not contain w and a shortest (v, w) -path does not contain u .

Proof : Since v is not on C , it has a distance of at least 1 to every vertex of C . Let u be a vertex of C at minimum distance from v in G . Let Q be the set of vertices on C that are k -dominated by v in G . Thus, $Q \subseteq V(C)$ and, by assumption, $|Q| \geq 2k$. Among all vertices in Q , let $w \in Q$ be chosen to have maximum distance from u on the cycle C . Since there are $2k - 1$ vertices within distance $k - 1$ from u on C , the vertex w has distance at least k from u on the cycle C . Let P_u be a shortest (u, v) -path and let P_w be a shortest (v, w) -path in G . If $w \in V(P_u)$, then $d_G(v, w) < d_G(v, u)$, contradicting our choice of the vertex u . Therefore $w \notin V(P_u)$. Suppose that $u \in V(P_w)$. Since C is a shortest cycle in G , the distance between u and w on C is the same as the distance between u and w in G . Thus, $d_G(u, w) = d_C(u, w)$, implying that $d_G(v, w) = d_G(v, u) + d_G(u, w) \geq 1 + d_G(u, w) = 1 + d_C(u, w) \geq 1 + k$, a contradiction. Therefore, $u \notin V(P_w)$.

4.1 Lower bound

We provide various lower bounds on the k -domination number for general graphs.

Theorem: For $k \geq 1$, if G is a connected graph with diameter d , then $\gamma_k(G) \geq \frac{d+1}{2k+1}$

Proof : Let $P : u_0 u_1 \dots u_d$ be a diametral path in G , joining two peripheral vertices $u = u_0$ and $v = u_d$ of G . Thus, P has length $\text{diam}(G) = d$. We show that every vertex of G k -dominates at most $2k+1$ vertices of P . Suppose, to the contrary, that there exists a vertex $q \in V(G)$ that k -dominates at least $2k+2$ vertices of P . (Possibly, vertex $q \in V(P)$).

Let Q be the set of vertices on the path P that are k -dominated by the vertex q in G . By

supposition, $|Q| \geq 2k + 2$. Let i and j be the smallest and largest integers, respectively, such that $u_i \in Q$ and $u_j \in Q$. We note that $Q \subseteq \{u_i, u_{i+1}, \dots, u_j\}$. Thus, $2k + 2 \leq |Q| \leq j - i + 1$. Since P is a shortest (u, v) -path in G , we therefore note that vertex $d_G(u_i, u_j) = d_P(u_i, u_j) = j - i \geq 2k + 1$.

Let P_i be a shortest (u, q) -path in G and let P_j be a shortest (q, v) -path in G . Since the vertex q k -dominates both u_i and u_j in G , both paths P_u and P_v have length at most k . Therefore, the (u_i, u_j) -path obtained by following the path P_i from u_i to q , and then proceeding along the path P_j from q to u_j , has length at most $2k$, implying that $d_G(u_i, u_j) \leq 2k$, a contradiction.

Therefore, every vertex of G k -dominates at most $2k + 1$ vertices of P .

Let S be a minimum k -dominating set of G . Thus, $|S| = \gamma_K(G)$. Each vertex of S k -dominates at most $2k + 1$ vertices of P , and so S k -dominates at most $|S|(2k + 1)$ vertices of P . However, since S is a k -dominating set of G , every vertex of P is k -dominated the set S , and so S k -dominates $|V(P)| = d + 1$ vertices of P . Therefore, $|S|(2k + 1) \geq d + 1$, or, equivalently, $\gamma_K(G) = (d + 1)/(2k + 1)$. That the lower bound of Theorem 4.3.3 is tight may be seen by taking G to be path, $v_1 v_2 \dots v_n$, of order $n = \ell(2k + 1)$ for some $\ell \geq 1$.

Let that the $d = \text{diam}(G)$, and so $d = n - 1 = \ell(2k + 1) - 1$. By Theorem such that $\gamma_K(G) \geq \frac{(d+1)}{(2K+1)} = \ell$. The set

$$S = \bigcup_{i=0}^{\ell-1} \{v_{k+1+i(2k+1)}\}$$

is a k -dominating set of G , and so $\gamma_K(G) \leq |S| = \ell$.

Consequently, $\gamma_K(G) = \ell = \frac{(d+1)}{(2K+1)}$.

5. A LOWER BOUND FOR THE DISTANCE K-DOMINATION NUMBER OF TREES

A subset D of vertices of a graph $G = (V, E)$ is a distance k -dominating set for G if the distance between every vertex of $V - D$ and D is at most k . The minimum size of a distance k -dominating set of

G is called the distance k -domination number $\gamma_K(G)$ of G . In this paper we prove that $(2k + 1) \gamma_K(T) \geq |V| + 2k - kn_1$ for each tree $T = (V, E)$ with n_1 leaves, and we characterize the class of trees that satisfy the equality

$$(2k + 1) \gamma_K(T) = |V| + 2k - kn_1.$$

Lemma: Let T be a tree with $\gamma_K(T) > 1$. Then there exists an edge uv in T such that

$$\gamma_K(T) = \gamma_K(T_u) + \gamma_K(T_v).$$

Proof :

Let $P = v_0 v_1 \dots v_l$ be a longest path in T . Since $\gamma_K(T) > 1$, we have $1 \geq 2k + 1$. Now let D be a minimum distance k -dominating set of T such that (1) $v_K \in D$ and (2) $\sum_{x \in D} d(x, p)$ is minimal.

For $1 \leq i \leq l - 1$ let T_i be the component of $T - \{v_{i-1} v_i, v_i v_{i+1}\}$ that contains the vertex v_i . Note that condition (2) implies that all vertices $x \in V(T_i) \cap D$ satisfy the inequality $d(v_i, x) \leq i - k$ for $i \geq k$.

Let the $0 \leq p \leq k$ be the greatest integer such that v_K has at least one private k -neighbor in T_{K+p} .

We will now show that $d(v_{K+p}, v) \leq k - p$ for all vertices $v \in V(T_{K+p})$, i.e., $V(T_{K+p}) \in N^k[v_K]$.

Let the theorem of the $y \in PN^k[v_K, D] \cap V(T_{K+p})$ be a private k -neighbor of v_K in T_{K+p} and suppose that $z \in V(T_{K+p}) - N_K[v_K]$ is not a k -neighbor of v_K .

Then $d(v_{K+p}, y) \leq k - p$ and $k - p + 1 \leq d(v_{K+p}, z) \leq k + p$ (the latter inequality holds because P is a longest path in T). In addition, there exists a vertex $v_K \neq x \in D$ such that $Z \in N^k[x]$. y is also a k -neighbor of x .

Suppose first that $x \notin V(T_{K+p})$. Since $d(x, v_{K+p}) \leq p$ and $d(v_{K+p}, y) \leq k - p$, it follows that $d(x, y) \leq d(x, v_{K+p}) + d(v_{K+p}, y) \leq k$, a contradiction. Suppose second that $x \in V(T_{K+p})$, i.e., $x \in V(T_j)$ for an integer of $1 \leq j \leq l - 1$.

Then $d(x, y) = d(x, v_j) + d(v_j, v_{K+p}) + d(v_{K+p}, y) \leq d(x, v_j) + d(v_j, v_{K+p}) + d(v_{K+p}, z) = d(x, z)$ and thus, since $z \in N_K[x]$, we conclude that $y \in N^k[x]$, a contradiction.

Let us now remove the edge $uv = v_{k+p}v_{k+p+1}$. We shall show now that $\{v_k\}$ is a distance k -dominating set of T_u and that $D - v_k$ is a distance k -dominating set of T_v which completes the proof.

Since v_k has no private k -neighbor in T_v , it is immediate that $D - v_k$ is a distance k -dominating set of T_v . Now assume that there exists a vertex $y \in V(T_u)$ that is no k -neighbor of v_k .

Then $y \in V(T_{k+q})$ for an integer $1 \leq q \leq p-1$ and $d(v_{k+q}, y) \geq k - q + 1$.

Let $v_k \neq x \in D$ be a k -neighbor of y . We shall now conclude a contradiction to the assumption that v_k has a private k -neighbor in T_{K+p} . Let $x \in V(T_{K+j})$ for an integer $1 \leq j \leq 1-p$ and let z be an arbitrary vertex of T_{K+p} .

Then

$$\begin{aligned} d(x, z) &\leq d(x, v_{K+j}) + d(v_{K+j}, v_{K+p}) + \\ d(v_{K+p}, z) &\leq j + (p - j) + \\ (k-p) &= k \end{aligned}$$

6. CONCLUSION

In this paper, "A review on Lower Bounds for the Domination Number" can make an in depth study in graphs and its related works. We also discussed about the properties of the Lower bounds for the domination number. We also arrived out Lower bounds on The distance domination number of a Graph. Also we preliminary Lemmas of Lower bound on the distance domination and we obtain certain direct Product graph on connection with other lower bounds on the distance domination.

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