A Note on Weak Soft Structures

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Abstract**—***We continue the study weak soft structures and the properties of some soft sets over X with respect to a weak soft structure on X. Also, we introduce the structures* $\alpha(\tilde{w})$, $\pi(\tilde{w})$, $\sigma(\tilde{w})$, $\beta(\tilde{w})$,$\rho(\tilde{w})$,$r(\tilde{w})$*and study properties of them.*

Keywords —*Soft set, Soft topology, Weak soft structure*

I. INTRODUCTION

The concept of soft sets was initiated by Molodtsov[6] in 1999 as a completely new approach for modelling vagueness and uncertainty. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. Maji et al. [5] presented some new definitions on soft sets such as a subset, the complement of a soft set. Also Ali et al. [1] gave some new operations on soft sets. Later Ali et al. [2] studied some important properties associated with these new operations. Research works on soft sets are progressing rapidly in recent years.

Shabir and Naz[8] introduced the soft topological spaces which are defined over an initial universe with a fixed set of parameters. Later many authors got important results in soft topological spaces.

Császár[3] defined the concept of weak structures and showed that these structures can replace general topologies, generalized topologies or minimal structures.Ekici [4] and Navaneethakrishnan et al. [7] continued to study weak structures and their properties. Then Zakari et al. [9] defined the soft weak structures and discuss some of its properties. Also, they investigated some new separation axioms and compactness in it.

In this paper, we continue investigating the some properties of weak soft structures which are defined over an initial universe with a fixed set of parameters. Also, the structures $\alpha(\tilde{w})$, $\pi(\tilde{w})$, $\sigma(\tilde{w})$, $\beta(\tilde{w})$, $\rho(\tilde{w})$, $r(\tilde{w})$ and properties of them are introduced.

II. PRELIMINARIES

Let X be an initial universe set and E be the set of all possible parameters with respect to X. Parameters are often attributes, characteristics or properties of the objects in X. Let P(X) denote the power set of X. Then a soft set over X is defined as follows.

Definition 2.1 [6]:

A pair (F,A) is called a soft set over X where $A \subseteq E$ and

$F: A \rightarrow P(X)$ is a set valued mapping. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $\forall \varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F,A). It is worth noting that $F(\varepsilon)$ may be arbitrary. Some of them may be empty, and some may have nonempty intersection.

Definition 2.2 [5]:

A soft set (F,A) over X is saidto be anullsoft set denotedby Φ ifforall $e \in A$, $F(e)=\emptyset$. A soft set (F,A) over X is saidto be an absolutesoft set denotedby$\tilde{A}$ if for all $e \in A$, $F(e)=X$.

Definition 2.3 [8]:

Let Y be a nonemptysubset of X, then $\tilde{Y}$ denotesthesoft set (Y,E) over X forwhich $Y(e)=Y$, forall $e \in E$. In particular, (X,E) will be denotedby$\tilde{X}$.

Definition 2.4 [5]:

Fortwosoftsets (F,A) and (G,B) over X, we say that (F,A) is a softsubset of (G,B) if $A \subseteq B$ and $F(e) \subseteq G(e)$ forall $e \in A$. We write $(F,A) \sqsubseteq (G,B)$. (F,A) is said to be a soft super set of (G,B), if (G,B) is a soft subset of (F,A). Wedenote it by $(G,B) \sqsubseteq (F,A)$. Then (F,A) and (G,B) aresaidto be softequalif (F,A) is a softsubset of (G,B) and (G,B) is a softsubset of (F,A).

Definition 2.5 [5]:

Theunion of twosoftsets of (F,A) and (G,B) over X is thesoft set (H,C), where $C=A \cup B$ and for all $e \in C$, $H(e)=F(e)$ if $e \in A \setminus B$, $H(e)=G(e)$ if $e \in B \setminus A$, $H(e)=F(e) \cup G(e)$ if $e \in A \cap B$. We write $(F,A) \sqcup (G,B)=(H,C)$.

Definition 2.6 [5]:

Theintersection (H,C) oftwosoftsets (F,A) and (G,B) over X, denotedby $(F,A) \sqcap (G,B)$, is defined as $C=A \cap B$ and $H(e)=F(e) \cap G(e)$ for all $e \in C$.

Definition 2.7 [8]:

Thedifference (H,E) of twosoftsets (F,E) and (G,E) over X, denotedby $(F,E) \setminus (G,E)$, is defined as $H(e)=F(e) \setminus G(e)$ forall $e \in E$.

Definition 2.8 [8]:

Therelativecomplement of a soft set (F,E) is denotedby $(F,E)^c$ and is defined by $(F,E)^c=(F^c,E)$ where F^c:E→P(X) is a mappinggivenby$F^c(e)$=X\F(e) forall e∈E.

Definition 2.9 [8]:

Let (F,E) be a soft set over X and x∈X. x∈(F,E) read as x belongstothesoft set (F,E) whenever x∈F(e) for all e∈E. For any x∈X, x∉(F,E), ifx∉F(e) forsome e∈E.

Allthesoftsetsaretakenwithparameter set E.

Definition 2.10 [8]:

Let$\tilde{\tau}$ be the collection of soft sets over X, then $\tilde{\tau}$ is said to be a soft topology on X if

(1) Φ, $\tilde{X}$ belong to $\tilde{\tau}$

(2) theunion of anynumber of softsets in $\tilde{\tau}$ belongs to $\tilde{\tau}$

(3) theintersection of anytwosoftsets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

Thetriplet (X,$\tilde{\tau}$,E) is called a soft topological space over X.

III. WEAK SOFT STRUCTURES

Definition 3.1 [9]:

Let$\tilde{w}$ be the collection of soft sets over X, then $\tilde{w}$ is said to be a weak soft structure on X if Φ∈ $\tilde{w}$.

Thetriplet (X,$\tilde{w}$,E) is called a weak soft space over X.Themembers of $\tilde{w}$ is called a soft $\tilde{w}$-open set in X. A soft set (F,E) over X is called a soft $\tilde{w}$-closed set in X, if itsrelativecomplement$(F,E)^c$ belongs to $\tilde{w}$.

Clearly, eachsofttopology is weaksoftstructure.

Definition 3.2 [9]:

Let (X,$\tilde{w}$,E) be a weak soft space over X and (F,E) be a soft set over X. Then, $i_{\tilde{w}}$(F,E) is called the union of all soft $\tilde{w}$-open sets contained in (F,E) and $c_{\tilde{w}}$(F,E) is called theintersection of allsoft $\tilde{w}$ -closed sets containing (F,E).

Theorem3.1 [9]:

Let (X,$\tilde{w}$,E) bea weaksoftspaceover X and (F,E), (G,E) aresoftsetsover X. Thenthefollowingpropertieshold:

(1) $i_{\tilde{w}}$(F,E)⊑(F,E)⊑$c_{\tilde{w}}$(F,E).

(2) (F,E)⊑(G,E) implies $i_{\tilde{w}}$(F,E)⊑$i_{\tilde{w}}$(G,E) and $c_{\tilde{w}}$(F,E)⊑$c_{\tilde{w}}$(G,E).

(3)$i_{\tilde{w}}i_{\tilde{w}}$(F,E)=$i_{\tilde{w}}$(F,E) and$c_{\tilde{w}}c_{\tilde{w}}$(F,E)=$c_{\tilde{w}}$(F,E).

(4) $i_{\tilde{w}}(F,E)^c$ = $(c_{\tilde{w}}(F,E))^c$ and $c_{\tilde{w}}(F,E)^c=(i_{\tilde{w}}(F,E))^c$.

Theorem3.2 [9]:

Let (X,$\tilde{w}$,E) bea weaksoftspaceover X, (F,E) be a soft set over X and x∈X. Then,

(1) x∈ $i_{\tilde{w}}$(F,E) if and only if there exists a soft $\tilde{w}$-open set (H,E)⊑(F,E) suchthat x∈(H,E),

(2) x ∈ $c_{\tilde{w}}$ (F,E) if and only if (H,E)⊓(F,E)≠Φwhenever x∈(H,E)∈ $\tilde{w}$.

Proposition3.1 [9]:

Let (X,$\tilde{w}$,E) bea weaksoftspaceover X. If (F,E)∈ $\tilde{w}$, then (F,E)=$i_{\tilde{w}}$(F,E) and if (F,E) is soft $\tilde{w}$-closed, then (F,E)=$c_{\tilde{w}}$(F,E).

Proposition3.2:

Let (X,$\tilde{w}$,E) bea weaksoftspaceover X and (F,E) be a soft set over X. Then

$c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}$(F,E)=$c_{\tilde{w}}i_{\tilde{w}}$(F,E) and

$i_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$(F,E)=$i_{\tilde{w}}c_{\tilde{w}}$(F,E).

Proof:ByTheorem3.1, wehave$i_{\tilde{w}}$(F,E)⊑$c_{\tilde{w}}i_{\tilde{w}}$(F,E), then $i_{\tilde{w}}i_{\tilde{w}}$ (F,E)= $i_{\tilde{w}}$ (F,E)⊑ $i_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}$ (F,E) and then $c_{\tilde{w}}i_{\tilde{w}}$ (F,E)⊑ $c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}$ (F,E) ... (i). Also we have $i_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}$ (F,E)⊑ $c_{\tilde{w}}i_{\tilde{w}}$ (F,E), then $c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}$ (F,E)⊑$c_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}$(F,E)=$c_{\tilde{w}}i_{\tilde{w}}$(F,E) ... (ii). From (i) and (ii), we obtain $c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}$(F,E)=$c_{\tilde{w}}i_{\tilde{w}}$(F,E). The second part is obtained in a similar way.

Theorem3.3:

Let (X,$\tilde{w}$,E) bea weaksoftspaceover X and (F,E), (G,E) be softsetsover X. Thenthefollowingpropertieshold:

(1) $c_{\tilde{w}}$(F,E)⊔$c_{\tilde{w}}$(G,E)⊑$c_{\tilde{w}}$((F,E)⊔(G,E)).

(2) $i_{\tilde{w}}$((F,E)⊓(G,E))⊑$i_{\tilde{w}}$(F,E)⊓$i_{\tilde{w}}$(G,E).

Proof:Theproof is obvious.

Remark3.1:

Let (X, $\tilde{w}$,E) bea weaksoftspaceover X. Foranysoftsets (F,E) and (G,E) over X, $i_{\tilde{w}}$ ((F,E)⊓(G,E))= $i_{\tilde{w}}$ (F,E)⊓ $i_{\tilde{w}}$ (G,E) and $c_{\tilde{w}}$((F,E)⊔(G,E))=$c_{\tilde{w}}$(F,E)⊔$c_{\tilde{w}}$(G,E) are not true in general as shown in the following example.

Example3.1:

Let$X=\{x_1,x_2,x_3,x_4,x_5\}$, $E=\{e_1,e_2\}$ and $\tilde{w}$ =$\{\Phi,(F_1,E),(F_2,E),(F_3,E)\}$ where $(F_1,E),(F_2,E),(F_3,E)$ be softsetsover X, defined as follows

$F_1(e_1)=\{x_1,x_2\}$, $F_1(e_2)=\{x_3,x_4\}$,
$F_2(e_1)=\{x_2,x_3\}$, $F_2(e_2)=\{x_4,x_5\}$,
$F_3(e_1)=\{x_1,x_3\}$, $F_3(e_2)=\{x_3,x_5\}$.

Then$\tilde{w}$ defines a weak soft structure on X and hence (X,$\tilde{w}$,E) is a weak soft space over X.

Let (G_1,E) and (G_2,E) be defined as follows:

$G_1(e_1)=\{x_1,x_2,x_3\}$, $G_1(e_2)=\{x_2,x_3,x_5\}$,
$G_2(e_1)=\{x_2,x_3,x_4\}$, $G_2(e_2)=\{x_1,x_4,x_5\}$.

Then $i_{\widetilde{w}}(G_1,E)=\{(e_1,\{x_1,x_3\}),(e_2,\{x_3,x_5\})\}$, $i_{\widetilde{w}}(G_2,E)=\{(e_1,\{x_2,x_3\}),(e_2,\{x_4,x_5\})\}$ and so $i_{\widetilde{w}}(G_1,E)\sqcap i_{\widetilde{w}}(G_2,E)=\{(e_1,\{x_3\}),(e_2,\{x_5\})\}$. Since $(G_1,E)\sqcap(G_2,E)=\{(e_1,\{x_2,x_3\}),(e_2,\{x_5\})\}$, we get $i_{\widetilde{w}}((G_1,E)\sqcap(G_2,E))=\Phi$. Hence, we obtain $i_{\widetilde{w}}(G_1,E)\sqcap i_{\widetilde{w}}(G_2,E)\neq i_{\widetilde{w}}((G_1,E)\sqcap(G_2,E))$.

Let (H_1,E) and (H_2,E) be defined as follows:

$H_1(e_1)=\{x_4,x_5\}$, $H_1(e_2)=\{x_2\}$,
$H_2(e_1)=\{x_2\}$, $H_2(e_2)=\{x_1,x_2\}$.

Then $c_{\widetilde{w}}(H_1,E)=\{(e_1,\{x_4,x_5\}),(e_2,\{x_1,x_2\})\}$, $c_{\widetilde{w}}(H_2,E)=\{(e_1,\{x_2,x_4,x_5\}),(e_2,\{x_1,x_2,x_5\})\}$ and so $c_{\widetilde{w}}(H_1,E)\sqcup c_{\widetilde{w}}(H_2,E)=\{(e_1,\{x_2,x_4,x_5\}),(e_2,\{x_1,x_2,x_5\})\}$. Since $(H_1,E)\sqcup(H_2,E)=\{(e_1,\{x_2,x_4,x_5\}),(e_2,\{x_1,x_2,x_5\})\}$, we get $c_{\widetilde{w}}((H_1,E)\sqcup(H_2,E))=\widetilde{X}$. Hence, we obtain $c_{\widetilde{w}}(H_1,E)\sqcup c_{\widetilde{w}}(H_2,E)\neq c_{\widetilde{w}}((H_1,E)\sqcup(H_2,E))$.

Theorem3.4:
Let $(X,\widetilde{w},E)$ be a weak soft space over X such that $\widetilde{w}$ is closed under finite intersection and (F,E), (G,E) are soft sets over X. Then the following properties hold:

(1) $c_{\widetilde{w}}(F,E)\sqcup c_{\widetilde{w}}(G,E)=c_{\widetilde{w}}((F,E)\sqcup(G,E))$.
(2) $i_{\widetilde{w}}((F,E)\sqcap(G,E))=i_{\widetilde{w}}(F,E)\sqcap i_{\widetilde{w}}(G,E)$.

Proof:
(1) By Theorem 3.3, $c_{\widetilde{w}}(F,E)\sqcup c_{\widetilde{w}}(G,E)\sqsubseteq c_{\widetilde{w}}((F,E)\sqcup(G,E))$. Suppose $x\notin c_{\widetilde{w}}(F,E)\sqcup c_{\widetilde{w}}(G,E)$. Then $x\notin c_{\widetilde{w}}(F,E)$ and $x\notin c_{\widetilde{w}}(G,E)$. Then there exist $(H,E), (K,E)\in\widetilde{w}$ containing x such that $(H,E)\sqcap(F,E)=\Phi$ and $(K,E)\sqcap(G,E)=\Phi$. If $x\in(H,E)\sqcap(K,E)\in\widetilde{w}$ such that $((H,E)\sqcap(K,E))\sqcap((F,E)\sqcup(G,E))$ $=(((H,E)\sqcap(K,E))\sqcap(F,E))\sqcup(((H,E)\sqcap(K,E))\sqcap(G,E))$ $\sqsubseteq((H,E)\sqcap(F,E))\sqcup((K,E)\sqcap(G,E))=\Phi$ and so $x\notin c_{\widetilde{w}}((F,E)\sqcup(G,E))$.
Hence $c_{\widetilde{w}}((F,E)\sqcup(G,E))\sqsubseteq c_{\widetilde{w}}(F,E)\sqcup c_{\widetilde{w}}(G,E)$ and so $c_{\widetilde{w}}(F,E)\sqcup c_{\widetilde{w}}(G,E)=c_{\widetilde{w}}((F,E)\sqcup(G,E))$.

(2) The proof follows from (1).

Definition 3.3 [9]:
Let $(X,\widetilde{w},E)$ be a weak soft space over X and (F,E) be a soft set over X. If $c_{\widetilde{w}}(F,E)=\widetilde{X}$, then (F,E) is called a $\widetilde{w}$-dense soft set.

Theorem3.5:
Let $(X,\widetilde{w},E)$ be a weak soft space over X such that $\widetilde{w}$ is closed under finite intersection and (F,E) is a soft set over X. Then the following properties hold:

(1) $(G,E)\sqcap c_{\widetilde{w}}(F,E)\sqsubseteq c_{\widetilde{w}}((G,E)\sqcap(F,E))$ for every $(G,E)\in\widetilde{w}$.
(2) $c_{\widetilde{w}}((G,E)\sqcap c_{\widetilde{w}}(F,E))=c_{\widetilde{w}}((G,E)\sqcap(F,E))$ for every $(G,E)\in\widetilde{w}$.
(3) $c_{\widetilde{w}}(G,E)=c_{\widetilde{w}}((G,E)\sqcap(F,E))$ for every $(G,E)\in\widetilde{w}$ and (F,E) is a $\widetilde{w}$-dense soft set.

Proof:
(1) Let $x\in((G,E)\sqcap c_{\widetilde{w}}(F,E))$. Then $x\in(G,E)$ and $x\in c_{\widetilde{w}}(F,E)$. If $x\in(H,E)\in\widetilde{w}$, then $x\in((H,E)\sqcap(G,E))\in\widetilde{w}$ and so $((H,E)\sqcap(G,E))\sqcap(F,E)\neq\Phi$ which implies that $(H,E)\sqcap((G,E)\sqcap(F,E))\neq\Phi$. Hence $x\in c_{\widetilde{w}}((G,E)\sqcap(F,E))$ which implies that $(G,E)\sqcap c_{\widetilde{w}}(F,E)\sqsubseteq c_{\widetilde{w}}((G,E)\sqcap(F,E))$.

(2) From (1), $(G,E)\sqcap c_{\widetilde{w}}(F,E)\sqsubseteq c_{\widetilde{w}}((G,E)\sqcap(F,E))$ and so $c_{\widetilde{w}}((G,E)\sqcap c_{\widetilde{w}}(F,E))\sqsubseteq c_{\widetilde{w}}((G,E)\sqcap(F,E))$.
But $(G,E)\sqcap(F,E)\sqsubseteq(G,E)\sqcap c_{\widetilde{w}}(F,E)\sqsubseteq c_{\widetilde{w}}((G,E)\sqcap c_{\widetilde{w}}(F,E))$ and so $c_{\widetilde{w}}((G,E)\sqcap(F,E))\sqsubseteq c_{\widetilde{w}}((G,E)\sqcap c_{\widetilde{w}}(F,E))$. Hence, we obtain $c_{\widetilde{w}}((G,E)\sqcap c_{\widetilde{w}}(F,E))=c_{\widetilde{w}}((G,E)\sqcap(F,E))$.

(3) The proof follows from (2).

IV. THE SOFT STRUCTURES $\alpha(\widetilde{w}), \sigma(\widetilde{w}), \pi(\widetilde{w}), \rho(\widetilde{w}), \beta(\widetilde{w}), r(\widetilde{w})$

Definition4.1:
Let $(X,\widetilde{w},E)$ be a weak soft space over X and (F,E) be a soft set over X. Then

(1) (F,E) is called a soft α-$\widetilde{w}$-open set in X if $(F,E)\sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$,
(2) (F,E) is called a soft σ-$\widetilde{w}$-open set in X if $(F,E)\sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$,
(3) (F,E) is called a soft π-$\widetilde{w}$-open set in X if $(F,E)\sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$,
(4) (F,E) is called a soft ρ-$\widetilde{w}$-open set in X if $(F,E)\sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E)\sqcup i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$,
(5) (F,E) is called a soft β-$\widetilde{w}$-open set in X if $(F,E)\sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$,
(6) (F,E) is called a soft r-$\widetilde{w}$-open set in X if $(F,E)=i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$.

We will denote the family of all soft α-$\widetilde{w}$-open (soft σ-$\widetilde{w}$-open, soft π-$\widetilde{w}$-open, soft ρ-$\widetilde{w}$-open, soft β-$\widetilde{w}$-open, soft r-$\widetilde{w}$-open) sets of a weak soft space $(X,\widetilde{w},E)$ by $\alpha(\widetilde{w})$, $\sigma(\widetilde{w})$, $\pi(\widetilde{w})$, $\rho(\widetilde{w})$, $\beta(\widetilde{w})$ and $r(\widetilde{w})$.

The relative complement of a soft α-$\widetilde{w}$-open (resp. soft σ-$\widetilde{w}$-open, soft π-$\widetilde{w}$-open, soft ρ-$\widetilde{w}$-open, soft β-$\widetilde{w}$-open, soft r-$\widetilde{w}$-open) set is called a soft α-$\widetilde{w}$-closed (resp. soft σ-$\widetilde{w}$-closed, soft π-$\widetilde{w}$-closed, soft ρ-$\widetilde{w}$-closed, soft β-$\widetilde{w}$-closed, soft r-$\widetilde{w}$-closed) set.

Theorem4.1:

Let $(X,\widetilde{w},E)$ bea weak soft space over X. Then we have $\widetilde{w} \sqsubseteq \alpha(\widetilde{w}) \sqsubseteq \sigma(\widetilde{w}) \sqsubseteq \rho(\widetilde{w}) \sqsubseteq \beta(\widetilde{w})$ and $\alpha(\widetilde{w}) \sqsubseteq \pi(\widetilde{w}) \sqsubseteq \rho(\widetilde{w})$.

Proof:

Let $(X,\widetilde{w},E)$ bea weak soft space over X and $(F,E) \in \widetilde{w}$. By Proposition 3.1 $(F,E)=i_{\widetilde{w}}(F,E)$, then $i_{\widetilde{w}}(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$ by Theorem 3.1. Hence $(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$. Since (F,E) is soft $\widetilde{w}$-open, we have $(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$. Thus $(F,E) \in \alpha(\widetilde{w})$.

Similarly $i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$ by Theorem 3.1. We have $(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$ since $(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$. So $\alpha(\widetilde{w}) \sqsubseteq \sigma(\widetilde{w})$. Cearly $(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$ implies $(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E) \sqcup i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$ so that $\sigma(\widetilde{w}) \sqsubseteq \rho(\widetilde{w})$.

$c_{\widetilde{w}}i_{\widetilde{w}}(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$ and $i_{\widetilde{w}}c_{\widetilde{w}}(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$ by Theorem 3.1. Then we have $c_{\widetilde{w}}i_{\widetilde{w}}(F,E) \sqcup i_{\widetilde{w}}c_{\widetilde{w}}(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$. Hence $\rho(\widetilde{w}) \sqsubseteq \beta(\widetilde{w})$.

Similarly $i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$ by Theorem 3.1, so $\alpha(\widetilde{w}) \sqsubseteq \pi(\widetilde{w})$. Also $i_{\widetilde{w}}c_{\widetilde{w}}(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E) \sqcup i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$ and thus we obtain $\pi(\widetilde{w}) \sqsubseteq \rho(\widetilde{w})$.

Theorem4.2:

Let $(X,\widetilde{w},E)$ be a weak soft space over X and (F,E) be a soft set over X. Then $(F,E) \in r(\widetilde{w})$ if and only if $(F,E) \in \alpha(\widetilde{w})$ and $(F,E)^c \in \beta(\widetilde{w})$.

Proof:

Let $(F,E) \in r(\widetilde{w})$. Hence we have $(F,E)=i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$. From Theorem 3.1, $i_{\widetilde{w}}(F,E)=i_{\widetilde{w}}i_{\widetilde{w}}c_{\widetilde{w}}(F,E)=i_{\widetilde{w}}c_{\widetilde{w}}(F,E)=(F,E)$.Then we have $(F,E)=i_{\widetilde{w}}(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$. It follows that $(F,E)=i_{\widetilde{w}}(F,E)=i_{\widetilde{w}}i_{\widetilde{w}}(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$.Hence, $(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$ and we obtain $(F,E) \in \alpha(\widetilde{w})$.

On the other hand, since $(F,E)=i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$, $(F,E)^c=(i_{\widetilde{w}}c_{\widetilde{w}}(F,E))^c$. From Theorem 3.1, we get $(F,E)^c=c_{\widetilde{w}}i_{\widetilde{w}}(F,E)^c$ and $c_{\widetilde{w}}(F,E)^c=c_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)^c$. From Theorem 3.1, we have $c_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)^c = c_{\widetilde{w}}i_{\widetilde{w}}(F,E)^c = (F,E)^c$ .Also $c_{\widetilde{w}}i_{\widetilde{w}}(F,E)^c \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}c_{\widetilde{w}}(F,E)^c$. This implies that $(F,E)^c=c_{\widetilde{w}}(F,E)^c=c_{\widetilde{w}}i_{\widetilde{w}}(F,E)^c \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}c_{\widetilde{w}}(F,E)^c$.Thus, $(F,E)^c \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}c_{\widetilde{w}}(F,E)^c$ and so $(F,E)^c \in \beta(\widetilde{w})$.

Conversely, let $(F,E) \in \alpha(\widetilde{w})$ and $(F,E)^c \in \beta(\widetilde{w})$. That is $(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$ and $i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E) \sqsubseteq (F,E)$. Hence $(F,E)=i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$ and we obtain $(F,E) \in r(\widetilde{w})$ by Proposition 3.2.

Theorem4.3:

Let $(X,\widetilde{w},E)$ be a weak soft space over X and (F,E) be a soft set over X. Then $(F,E) \in r(\widetilde{w})$ if and only if $(F,E) \in \pi(\widetilde{w})$ and $(F,E)^c \in \sigma(\widetilde{w})$.

Proof:

Let $(F,E) \in \pi(\widetilde{w})$ and $(F,E)^c \in \sigma(\widetilde{w})$. We get $(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$ and $i_{\widetilde{w}}c_{\widetilde{w}}(F,E) \sqsubseteq (F,E)$. Thus $(F,E)=i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$ and we obtain $(F,E) \in r(\widetilde{w})$.

The converse is obvious from the fact that $(F,E)=i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$.

Theorem4.4:

Let $(X,\widetilde{w},E)$ be a weak soft space over X and (F,E) be a soft set over X. Then $(F,E) \in \pi(\widetilde{w})$ if and only if there exists a $(G,E) \in r(\widetilde{w})$ such that $(F,E) \sqsubseteq (G,E)$ and $c_{\widetilde{w}}(F,E)=c_{\widetilde{w}}(G,E)$.

Proof:

Let$(F,E) \in \pi(\widetilde{w})$. Hence we get $(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$. If we take $(G,E)=i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$, then we obtain $(G,E) \in r(\widetilde{w})$ by Proposition 3.2 and also $(F,E) \sqsubseteq (G,E)$ and $c_{\widetilde{w}}(F,E)=c_{\widetilde{w}}(G,E)$.

Let $(G,E) \in r(\widetilde{w})$ such that $(F,E) \sqsubseteq (G,E)$ and $c_{\widetilde{w}}(F,E)=c_{\widetilde{w}}(G,E)$. Then $i_{\widetilde{w}}c_{\widetilde{w}}(F,E)=i_{\widetilde{w}}c_{\widetilde{w}}(G,E)=(G,E)$ by Theorem 3.1 and so $(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}(F,E)$. Hence we obtain $(F,E) \in \pi(\widetilde{w})$.

Theorem4.5:

Let$(X,\widetilde{w},E)$ bea weak soft space over X and (F,E) be a soft set over X. If (F,E) is both soft $\widetilde{w}$-open and soft $\widetilde{w}$-closed, then $(F,E) \in \alpha(\widetilde{w})$ and $(F,E)^c \in \pi(\widetilde{w})$.

Proof:

Let(F,E) be soft $\widetilde{w}$-open and soft $\widetilde{w}$-closed. Then $(F,E)=i_{\widetilde{w}}(F,E)$ and $(F,E)=c_{\widetilde{w}}(F,E)$ by Proposition 3.1. We have $(F,E)=i_{\widetilde{w}}(F,E) \sqsubseteq c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$. From Theorem 3.1, we obtain $(F,E)=i_{\widetilde{w}}(F,E)=i_{\widetilde{w}}i_{\widetilde{w}}(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$. Thus, $(F,E) \in \alpha(\widetilde{w})$ since $(F,E) \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}i_{\widetilde{w}}(F,E)$. On the other hand, since $(F,E)=i_{\widetilde{w}}(F,E)$ and $(F,E)=c_{\widetilde{w}}(F,E)$, then$(F,E)^c=(i_{\widetilde{w}}(F,E))^c=c_{\widetilde{w}}(F,E)^c$ and $(F,E)^c=(c_{\widetilde{w}}(F,E))^c=i_{\widetilde{w}}(F,E)^c$ by Theorem 3.1. Then $(F,E)^c=i_{\widetilde{w}}(F,E)^c=i_{\widetilde{w}}i_{\widetilde{w}}(F,E)^c \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}(F,E)^c$ by Theorem 3.1. Hence $(F,E)^c \sqsubseteq i_{\widetilde{w}}c_{\widetilde{w}}(F,E)^c$ and we have $(F,E)^c \in \pi(\widetilde{w})$.

Remark4.1:

The following example shows that the converse of Theorem 4.5 is not true in general.

Example4.1:

Let $X=\{x_1,x_2,x_3,x_4\}$, $E=\{e_1,e_2\}$ and $\widetilde{w}=\{\Phi,(F_1,E),(F_2,E),(F_3,E),(F_4,E)\}$ where $(F_1,E),(F_2,E),(F_3,E),(F_4,E)$ be soft sets over X, defined as follows

$F_1(e_1)=\{x_4\}$, $F_1(e_2)=\{x_1,x_2,x_3\}$,
$F_2(e_1)=\{x_1,x_2\}$, $F_2(e_2)=\{x_2,x_3\}$,
$F_3(e_1)=\{x_2,x_3\}$, $F_3(e_2)=\{x_3,x_4\}$,
$F_4(e_1)=\{x_1,x_2,x_4\}$, $F_4(e_2)=\{x_1\}$.

Then $\tilde{w}$ defines a weak soft structure on X and hence (X,$\tilde{w}$,E) is a weak soft space over X. Let (G,E) be a soft set over X such that (G,E)={(e_1,{x_1,x_2,x_3}),(e_2,{x_2,x_3,x_4})}. Then $i_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}$ (G,E)= $\tilde{X}$. Since (G,E)⊑ $i_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}$ (G,E), (G,E) ∈ α($\tilde{w}$). Also $i_{\tilde{w}}c_{\tilde{w}}(G,E)^c$ = $\tilde{X}$. Since $(G,E)^c$ ⊑ $i_{\tilde{w}}c_{\tilde{w}}(G,E)^c$, $(G,E)^c$ ∈π($\tilde{w}$). But (G,E) is neither soft $\tilde{w}$-open nor soft $\tilde{w}$-closed.

Theorem4.6:
Let (X,$\tilde{w}$,E) bea weak soft space over X and (F,E) be a soft set over X. If there exists a soft $\tilde{w}$-open set (G,E) such that (G,E)⊑(F,E)⊑ $c_{\tilde{w}}$ (G,E), then (F,E) ∈σ($\tilde{w}$).

Proof:
Let(G,E) be a soft $\tilde{w}$ -open set such that (G,E)⊑(F,E)⊑ $c_{\tilde{w}}$ (G,E). Since (G,E)⊑(F,E), then $i_{\tilde{w}}$(G,E)=(G,E)⊑$i_{\tilde{w}}$(F,E) and then $c_{\tilde{w}}$(G,E)⊑$c_{\tilde{w}}i_{\tilde{w}}$(F,E) by Theorem 3.1. Hence we obtain (F,E)⊑$c_{\tilde{w}}i_{\tilde{w}}$(F,E) and so (F,E) ∈σ($\tilde{w}$).

Theorem4.7:
Let (X, $\tilde{w}$,E) bea weak soft space over X.If (F,E)⊑(G,E)⊑ $c_{\tilde{w}}$ (F,E) and (F,E) ∈ β($\tilde{w}$), then (G,E) ∈β($\tilde{w}$).

Proof:
Let(F,E)⊑(G,E)⊑ $c_{\tilde{w}}$ (F,E) and (F,E) ∈β($\tilde{w}$). We have (F,E)⊑$c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$(F,E). Since (G,E)⊑$c_{\tilde{w}}$(F,E), then (G,E)⊑ $c_{\tilde{w}}$ (F,E)⊑ $c_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$ (F,E)= $c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$ (F,E)⊑ $c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$ (G,E) by Theorem 3.1. Thus, we obtain (G,E) ∈β($\tilde{w}$).

Theorem4.8:
Let (X,$\tilde{w}$,E) be a weak soft space over X and (F,E) be a soft set over X. If (F,E) ∈π($\tilde{w}$), then (F,E) is the intersection of a soft set (G,E) ∈r($\tilde{w}$) and a $\tilde{w}$-dense soft set (H,E).

Proof:
Let(F,E) ∈π($\tilde{w}$). From Theorem 4.4, there exists a (G,E) ∈ r($\tilde{w}$) such that (F,E)⊑(G,E) and $c_{\tilde{w}}$(F,E)=$c_{\tilde{w}}$(G,E). If we take (H,E)=(F,E)⊔$(G,E)^c$, then by Theorem 3.3 we obtain $c_{\tilde{w}}$((G,E)⊔$(G,E)^c$)=$\tilde{X}$⊑$c_{\tilde{w}}$(G,E)⊔$c_{\tilde{w}}(G,E)^c$=$c_{\tilde{w}}$(F,E)⊔ $c_{\tilde{w}}(G,E)^c$ ⊑ $c_{\tilde{w}}$((F,E)⊔$(G,E)^c$)=$c_{\tilde{w}}$(H,E). Thus, (H,E) is a $\tilde{w}$-dense soft set and so (F,E)=(G,E)⊓(H,E).

Theorem4.9:
Let (X,$\tilde{w}$,E) be a weak soft space over X and (H,E) be a soft set over X. If (H,E) ∈ β($\tilde{w}$), then (H,E)=(F,E)⊓(G,E) such that (F,E) ∈σ($\tilde{w}$) and (G,E) is a $\tilde{w}$-dense soft set.

Proof:
Let(H,E) ∈β($\tilde{w}$). Then (H,E)⊑$c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$(H,E). We obtain $c_{\tilde{w}}$ (H,E)⊑ $c_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$ (H,E)= $c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$ (H,E) by Theorem 3.1. Moreover, $i_{\tilde{w}}c_{\tilde{w}}$ (H,E)⊑ $c_{\tilde{w}}$ (H,E) and then $c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$ (H,E)⊑ $c_{\tilde{w}}c_{\tilde{w}}$ (H,E)= $c_{\tilde{w}}$ (H,E). And so $c_{\tilde{w}}$ (H,E)= $c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$ (H,E). This implies that (F,E)= $c_{\tilde{w}}$ (H,E) ∈ σ($\tilde{w}$). If we take (G,E)=(H,E)⊔$(c_{\tilde{w}}(H,E))^c$, then (G,E) is a $\tilde{w}$-dense soft set and (H,E)=(F,E)⊓(G,E).

Remark4.2:
The following example shows that the converse of Theorem 4.9 is not true in general.

Example4.2:
Let X={x_1,x_2,x_3} and E={e_1,e_2} and $\tilde{w}$ ={Φ,(F_1,E),(F_2,E),(F_3,E),(F_4,E),(F_5,E),(F_6,E),(F_7,E)} where (F_1,E),(F_2,E),(F_3,E),(F_4,E),(F_5,E),(F_6,E),(F_7,E) be soft sets over X, defined as follows

$F_1(e_1)$={x_1,x_2}, $F_1(e_2)$={x_1,x_2},
$F_2(e_1)$={x_2}, $F_2(e_2)$={x_1,x_3},
$F_3(e_1)$={x_2,x_3}, $F_3(e_2)$={x_1},
$F_4(e_1)$={x_2}, $F_4(e_2)$={x_1},
$F_5(e_1)$={x_1,x_2}, $F_5(e_2)$=X,
$F_6(e_1)$=X, $F_6(e_2)$={x_1,x_2},
$F_7(e_1)$={x_2,x_3}, $F_7(e_2)$={x_1,x_3}.

Then $\tilde{w}$ defines a weak soft structure on X and hence (X,$\tilde{w}$,E) is a weak soft space over X.Let (F,E) and (G,E) be defined as follows:

F(e_1)={x_2,x_3}, F(e_2)={x_1,x_2}.
G(e_1)={x_3}, G(e_2)={x_2,x_3}.

Then $c_{\tilde{w}}i_{\tilde{w}}$ (F,E)= $\tilde{X}$. Since (F,E)⊑ $c_{\tilde{w}}i_{\tilde{w}}$ (F,E), (F,E) ∈σ($\tilde{w}$). Also, since $c_{\tilde{w}}$(G,E)=$\tilde{X}$, (G,E) is a $\tilde{w}$-dense soft set. Then (F,E)⊓(G,E)=(H,E)={(e_1,{x_3}),(e_2,{x_2})}. Since $c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$(H,E)=Φ, (H,E) is not a soft subset of $c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$(H,E). Therefore (H,E)∉β($\tilde{w}$).

Theorem 4.10:
Let (X,$\tilde{w}$,E) be a weak soft space and (F,E) be a soft set over X. Then the following hold:

(1) If (F,E) is both soft $\tilde{w}$-open and soft $\tilde{w}$-closed, then (F,E) ∈r($\tilde{w}$) and $(F,E)^c$ ∈r($\tilde{w}$).
(2) If (F,E) is both soft $\tilde{w}$-open and soft $\tilde{w}$-closed, then (F,E) ∈α($\tilde{w}$) and $(F,E)^c$ ∈α($\tilde{w}$).
(3) (F,E) ∈σ($\tilde{w}$) if and only if $c_{\tilde{w}}$(F,E)=$c_{\tilde{w}}i_{\tilde{w}}$(F,E).

Proof:
(1) The proof is clear from Proposition 3.1.

(2) By (1), it follows that (F,E)=$i_{\tilde{w}}c_{\tilde{w}}i_{\tilde{w}}$(F,E)=$c_{\tilde{w}}i_{\tilde{w}}c_{\tilde{w}}$(F,E) and so the proof follows from Theorem 4.1.

(3) (F,E) ∈σ($\tilde{w}$) if and only if (F,E)⊑$c_{\tilde{w}}i_{\tilde{w}}$(F,E) if and only if $c_{\tilde{w}}$(F,E)⊑$c_{\tilde{w}}i_{\tilde{w}}$(F,E)⊑$c_{\tilde{w}}$(F,E) if and only if $c_{\tilde{w}}$(F,E)=$c_{\tilde{w}}i_{\tilde{w}}$(F,E).

Remark4.3:

The following example shows that the converse of (2) of Theorem 4.10 is not true.

Example4.3:

Let $X=\{x_1,x_2,x_3\}$ and $E=\{e_1,e_2\}$ and $\tilde{w}=\{\Phi,(F_1,E),(F_2,E),(F_3,E)\}$ where $(F_1,E),(F_2,E),(F_3,E)$ be soft sets over X, defined as follows

$F_1(e_1)=\{x_1\}$, $F_1(e_2)=\{x_1\}$,
$F_2(e_1)=\{x_2\}$, $F_2(e_2)=\{x_2\}$,
$F_3(e_1)=\{x_1,x_2\}$, $F_3(e_2)=\{x_1,x_2\}$.

Then $\tilde{w}$ defines a weak soft structure on X and hence $(X,\tilde{w},E)$ is a weak soft space over X.

Let (G,E) be a soft set over X such that $(G,E)=\{(e_1,\{x_1,x_3\}), (e_2,\{x_1,x_3\})\}$. Then $(G,E) \in \alpha(\tilde{w})$ and $(G,E)^c \in \alpha(\tilde{w})$. But, it is clear that $(G,E) \notin \tilde{w}$.

Theorem 4.11:

Let $(X,\tilde{w},E)$ bea weak soft space such that $\tilde{X} \in \tilde{w}$. Then $i_{\tilde{w}}(F,E) \neq \Phi$ for every nonempty $(F,E) \in \sigma(\tilde{w})$.

Proof:

Let $\Phi \neq (F,E) \in \sigma(\tilde{w})$. If $i_{\tilde{w}}(F,E)=\Phi$, then $c_{\tilde{w}} i_{\tilde{w}}(F,E)=\Phi$, since $\tilde{X} \in \tilde{w}$. Therefore, $(F,E)=\Phi$ and this is a contradiction.

Remark4.3:

The following example shows that $(F,E) \in \sigma(\tilde{w})$ does not imply that $i_{\tilde{w}}(F,E) \neq \Phi$.

Example4.4:

Let$X=\{x_1,x_2,x_3,x_4,x_5\}$ and $E=\{e_1,e_2\}$. Let us take the weak soft structure $\tilde{w}$ on X in Example 3.1 and (H,E) be a soft set over X such that $(H,E)=\{(e_1,\{x_2,x_3\}), (e_2,\{x_5\})\}$. Then $(H,E) \in \sigma(\tilde{w})$ but $i_{\tilde{w}}(H,E)=\Phi$.

REFERENCES

[1] M.I. Ali, F. Feng, X.Y. Liu, W.K. Min and M.Shabir, "On some new operations in soft set theory", Computers and Mathematics with Applications, vol. 57, pp. 1547-1553, 2009.

[2] M.I. Ali, M. Shabir and M. Naz, "Algebraic structures of soft sets associated with new operations", Computers and Mathematics with Applications, vol. 61, pp. 2647-2654, 2011.

[3] A. Császár, "Weak structures", Acta Math. Hungar., vol. 131, pp. 193-195, 2011.

[4] E. Ekici, "On Weak Structures due to Császár", Acta Math. Hungar., vol. 134 (4), pp. 565-570, 2012.

[5] P.K. Maji, R.Biswas and A.R. Roy, "Soft set theory", Computers and Mathematics with Applications, vol. 45, pp. 555-562, 2003.

[6] D. Molodtsov, "Soft set theory-first results", Computers and Mathematics with Applications, vol. 37, pp. 19-31, 1999.

[7] M. Navaneethakrishnan and S. Thamaraiselvi, "On Weak Structures of Császár", vol. 137 (3), pp. 224-229, 2012.

[8] M. Shabir and M. Naz, "On soft topological spaces", Computers and Mathematics with Applications, vol. 61, pp. 1786-1799, 2011.

[9] A.H. Zakari, A.Ghareeb and S.Omran, "On soft weak structures", Soft Computing, DOI 10.1007/s00500-016-2136-8, 2016.