Nilpotency in Frattini Subgroups

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Abstract—*In 1970, Menegazzo gave a complete description of the structure of soluble IM-groups, i.e., groups in which every subgroup can be obtained as intersection of maximal subgroups. The work is devoted to the structure of finite nilpotent algebras. Relations between nilpotency and Frattini subgroups are given in this paper.*

Keywords: *Maximal subgroup, Frattini Subgroups, Nilpotent*

Introduction

Let G be a group. If G has no maximal subgroups, then the Frattini subgroup of G denoted by Frat G to be G itself. If G has maximal subgroups then the intersection of all maximal subgroups of Gis sait to be the Frat G of G. Since the intersection of an empty family of subset X is X itself, so Frat G is the common subgroup of all its maximal subgroups of G. Thus if M denotes the family of all maximal subgroups of G, then $Frat\ G = \cap_{K \in M} K$ .The Frattini subgroup of a group G is the intersection of the maximal subgroups of G; if G has no maximal subgroups $Frat\, G = G.$

Let G be a group and $\phi \in Aut(G)$. Let K be a subgroup of G. Then K is maximal if and only if $\phi(K)$ is maximal. In fact ϕ induces a bijective map from M to itself given by $K\ \alpha\ \ \phi(K)$. Further, since ϕ is bijective it preserves intersection and so $\phi(Frat\, G) = Frat\, G$ .Hence the Frattini subgroup is a characteristic subgroup.The Frattini subgroup has the important property that it is thecollection of all non generators of group G if $G = \langle g, X \rangle$ always implies that $G = \langle X \rangle$ when X is a subset of G. In this paper we establish the following results

General Properties of Frattini Subgroup

Theorem 2.1:

In any group G the Frattini subgroups equals the set of non generators of G.

Proof:

Let $g \in FratG$ and suppose that $G = \langle g, X \rangle$ but $G \neq \langle X \rangle$. Then $g \in \langle X \rangle$ so there exists a subgroup M which is maximal subject to $\langle X \rangle$ is a subgroup of M and $g \notin M$. Now if M is a proper subgroup of H and H is a subgroup of G, then $g \in H$ and H = G.

Consequently M is maximal in G. But $g \in Frat\ G$ which is a subgroup of M and consequently $G = \langle g, X \rangle = M$, a contradiction. Hence g is a nongenerator.

Conversely suppose that g is a nongenerator which does not belongs to Frat G, so that $g \notin M$ for some maximal subgroup M of G. Then $M \neq \langle g, M \rangle$, whence $G = \langle g, M \rangle$. But this implies that G = M since g is non generator.

Theorem2.2 :

Let G be a finite group.

(i)If $N < G, H$ is a subgroup of G and N is a subgroup of Frat H, then N is a subgroup of Frat G.

(ii)If $K < G,$ then Frat K is a subgroup of G.

(iii)If $N < G,$ then $(Frat\, G)N / N$ is a subgroup of $Frat(G / N)$ with equality if N is a subgroup of Frat G.

(iv)If A is an abelian normal subgroup of G such that $(Frat\, G) \cap A = 1$ there is a subgroup H such that G = HA and $H \cap A = 1.$

Proof:

(i) If N is not a subgroup of Frat G then N is also not a subgroup of M for some maximal M and G = MN. Hence $H = H \cap (MN) = (H \cap M)N.$ A minimal normal subgroup of a nilpotent group is contained in the center. It follows that $H = H \cap M$ and H isa subgroup of M; but this gives the contradiction N is a subgroup of M.

(ii) Apply (i) with N = Frat K and H = K

(iii) This follows at once from the definition.

(iv) Choose H is a subgroup of G minimal subject to G = HA. Now $H \cap A < H$ and also $H \cap A < A$ since A is abelian: therefore $H \cap A < HA = G.$ If $H \cap A$ is a subgroup of Frat H then (i) shows that $H \cap A$is a subgroup of $(Frat\, G) \cap A = 1.$ Then we can assume that $H \cap A$ is not a subgroup for some M maximal in H in which case $H = M(H \cap A)$ and G = HA = MA, in contradiction to the minimality.

Theorem 2.3 :
Let G be a group.
(i) If Frat G is a subgroup of $K < G$, where K is finite and K/Frat G is nilpotent, then K is nilpotent. In particular Frat G is always nilpotent if it is finite.
(ii)Let $FFrat\,G$ be defined by $FFrat\,G / Frat\,G = Fit(G / Frat\,G)$. If G is finite, then $FFrat\,G = Fit\,G$ also $FFrat\,G / Frat\,G$ is the product of all the minimal normal subgroups of G/Frat G.
Proof:
(i) Let P be a Sylow p−subgroup of K; It is enough to prove that $P < G$. Let F = Frat G and H = PF is a subgroup of K. Since H/F is a Sylow p−subgroup of K/F and K/F is nilpotent, H/F is characteristic in K/F whenence $H < G$. Since K is a finite normal subgroup of group G and P is Sylow p−subgroup of K, then $G = N_G(P)H = N_G(P)F$, which shows that $G = N_G P$ and $P < G$.
(ii) Taking H to be FFrat G in (i) we deduce that H is nilpotent andH is a subgroup of Fit G. But the opposite inclusion is obviously true,so K = Fit G.
We can assume that Frat G = 1. Write L = Fit G. In a nilpotentgroup every maximal subgroup of L is normal and has prime index.Hence L' is a subgroup of Frat L and Frat L is a subgroup of Frat G =1 and L is abelian. Denote by N the product of all the abelian minimal normal subgroups of G; then certainly N is a subgroup of L. Since Nis abelian normal subgroup and $(Frat\,G) \cap N = 1$ so there exists asubgroup K such that G = KN and $K \cap N = 1$. Now $K \cap L < K$ and $K \cap L < L$ since L is abelian.Thus $K \cap L < KL = G$. Since $(K \cap L) \cap N = 1$, the normal subgroup $K \cap L = 1$ can not contain aminimal normal subgroup of G; we conclude that $K \cap L = 1$ and $L = L \cap (KN) = N$..

Remark:
If a maximal subgroup M of a group G is normal, then G/M hasprime order and G' is a subgroup of M. Thus $M < G$ if and only if G' is a subgroup of M. All maximal subgroups of G are normal if andonly if $G' \in M$. All maximal subgroups of G are normal if and only if G' is a subgroup of $Frat\ G$. So G is nilpotent.

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