A Survey on Energy of Some Graphs

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Abstracts: *In this paper we are studied on energy of graphs, hyper graphs, co energetic graphs , Spectrum of graphs and some results related to energy of graphs, hyper graphs etc.,*

I. Introduction

Our survey is on the energy of a graph, spectral moments and energy of graphs, second stage spectrum and energy of graph, distance energy of graph.

Let G be a graph and A be a adjacency matrix of G denoted by A(G) [defn. 3].the Eigen values of A which are the zeros of $|\lambda I - A|$ are called the Eigen values of G and form its spectrum denoted by spec(G)[3].the energy of a graph G is the sum of the absolute values of its Eigen values, and is denoted by E(G)[def12]. The totally disconnected graph ${k_n}^c$ has zero energy while the complete graph k_n has energy 2(n-1) [n-vertices] [I Gut man 7].But it was disproved in [150:H.B.walikar et al].Graphs for which the energy is greater than 2(n-1) are called hyper energetic graphs. If E(G)≤ 2(n-1),then G is called non-hyper energetic graphs.

In theoretical chemistry, the π -electron energy of a conjugated carbon molecule, computed using the Huckel molecular orbital (HMO) model[7] coincides with the energy as defined ,hence result in graph energy assume special significance .

In the next part we going to discuss the nature of the energy of the graph k_n-H, where H is the Hamiltonian cycle of G and then it is shown that there exists an infinite number of values of n for which k-regular graphs exists whose energies are arbitrarily small compare to the known sharp bound $k + \sqrt{k(n-1)(n+1)}$ for the energy of k-regular graphs on n-vertices and the existence of eqi-energetic graphs not having the same spectrum is established.

II. Circulant Graphs

Lemma 1: [R Balakrishnan 2] If C is a circulant matrix of order n with first row $a_1, a_{2,}\ldots\ldots\ldots a_n$ then determinant of c given by detC=$\pi(a_1 + a_2\, w^j + a_3\, w^{2j} \ldots\ldots\ldots a_n w^{(n-1)j}$) where w is a primitive n^{th} root of unity.

Note: circulant graphs have been used in the study of graphs of decomposition problems [1, 9]

III. Energy Bounds

Let G be a graph with n-vertices and m-edges,

$$E(G) \le \frac{2m}{n} + \sqrt{\frac{n-1}{2m - \frac{(2m)^2}{n}}} = B_1 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

While if G is K-regular

$$E(G) \le K + \sqrt{K(n-1)(n-K)} = B_2 \ldots\ldots\ldots\ldots (2)$$

Since K=$\frac{2m}{n}$ for a K-regular graph. The bound B_2 is an immediate consequences of the bound B_1 .if k=3 the upper bound B_2=3+$\sqrt{2(n-1)(n-3)}$ but $B_2 \le$ 2(n-1) is equivalent to $(n-1)^2 \ge 0$ which is true. Hence all cubic graphs are non-hyper energetic.

Theorem: [R.Balakrishnan 10] For each$\epsilon \ge 0$ there exists infinitlymany n for each of which there exists a k-regular graph G of order 'n' with k$< n-1$ and $\frac{E(G)}{B_2} < \epsilon$

This theorem proves that there are $\emptyset(n)$-regular graphs of order n for infinitly many n whose energies are much smaller compare to the bounds B_2

IV. Energetic Graphs

Definition: Two graphs of the same order are called equienergetic (resp: co-spectral) if they have the same energy naturally two co-spectral graphs are equienergetic.

Definition: the tensor product of two graphs $G_1 and\ G_2$ is the graph $G_1 \otimes G_2$ with vertex set V$(G_1)X$ V(G_2) and in which the vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 v_1 \in E(G_1) and\ u_2 v_2 \in E(G_2)$

Lemma: [R.Balakrishnan 2] If A is a matrix of order r with spectrum $\{\lambda_1, \lambda_2 \dots \dots .. \lambda_r\}$ and B is a matrix of order with spectrum $\{\mu_1, \mu_2, \dots \dots \dots . \mu_s\}$ then the spectrum A$(G_1 \otimes G_2)$ is (λ_i, μ_j) where $1 \leq i \leq r\ and\ 1 \leq j \leq s$

Theorem: [R.Blakrishnan 2] There exists (non-isomorphic) equienergetic graphs that are not co-spectral

V. Graph Energy

Let G be a simple graph with n vertices and m edges and let A=(a_{ij}) be the adjacency matrix for G. The Eigen values $\lambda_1, \lambda_2, \dots \dots \dots \lambda_n$ of A, assumed that they are in non-increasing order. Since A is symmetric matrix with trace zero, these Eigen values are real, with sum is zero. Thus $\lambda_1 \geq \lambda_2 \dots \dots \dots \dots . \geq \lambda_n$

$\lambda_1 + \lambda_2 + \cdots \dots \dots \dots . + \lambda_n$ =0

Since the energy of a graph is not affected by isolated vertices, we assume, throughout that graphs have no isolated vertices implying that m$\geq \frac{n}{2}$. If a graph is not connected its energy is the sum of the energies of its connected components. Thus there is no loss in generality for assuming that graphs are connected [Although we don't this blanket assumption]

The concept of graph energy aroses in chemistry where certain numerical quantities such as the heat formation of a hydrocarbon, are related to total π-electron energy that can be calculated as the energy of an appropriate molecular graph [see [12] where possible, we cite here this survey paper by Gut man in place of specific references]

VI. Hyper Energetic Graphs

The complete graph k_n has eigen values n-1 and -1 at one time it was thought that the complete graph k_n had the largest energy among all n vertex graphs G_1 that is E(G)$\leq 2(n-1)$ with equality if and only if G=k_n

Godsil in the early 1980's constructed an example of graph on n vertices whose energy exceeds 2(n - 1).Now graph G whose energy satisfies E(G)> $2(n-1)$ are called hyper energetic. The simple construction of a family of hyper energetic graphs is due to Walikar Ramane and Hampiholi.

The line graph L(k_n) of k_n is hyper energetic for n≥ 5

Since the line graph of k_n has $\frac{n(n-1)}{2}$ vertices it doesn't furnish a hyper energetic graph for all n. Gut man and Zhang constructed hyper energetic graph of all orders n≥ 9 by removing edges forming a star from k_n. An example of a hyper energetic graph is also know for n=8.

Using Gaussian sums Shaplinski [4] gave constructions of circulant graphs with high energy.

VII. Spectral moments and Energy of graphs

Let G be a simple graph with n-vertices and m edges. The eigen values of G are denoted by $\lambda_1, \lambda_2, \dots \dots \dots \lambda_n$ and are assumed to be labeled in a non-decreasing manner i.e $\lambda_1 \geq \lambda_2 \dots \dots \dots \dots . \geq \lambda_n$

The basic properties of graph, the Eigen values can be found in the book D.Covetkoviet al[6].

The energy of a graph G is denoted by E=e(G)= $\sum_{i=1}^{n} |\lambda_i|$

The graph energy concept has a chemical motivation. Namely for graph which in the huckel molecular orbital theory represent the carbon atom skeleton of some conjugated hydrocarbons E is related to the total π-electron energy. The total π-electron energy is a linear function of e.

For a non-negative integer k, the k^{th} spectral moment of the graph G is defined as

$M_k = M_k$ (G)= $\sum_{i=1}^{n} |\lambda_i|^k$

Note that M_k is equal to the number of closed walks of length K in G in [30]

Because both the energy and spectral moments are symmetric functions of graph Eigen values. If there exists relation between them.

Theorem1: [In Pefa et al [7]] Let G be a bipartite graph with at least one edge and let r ,s, t even partition integers such that 4r=s+t+2. Then E(G) ≥ $M_r(G)^2\ [M_s(G)\quad M_t(G)]^{-1/2}$

Theorem 1 : [Bo Zhouetal] Let G be graph with at least one edge and let r,s,t be non negative real number such that 4r=s+t+2 then E(G)≥ $M_r{}^*(G)^2$ $[M_s{}^*(G)\quad M_T{}^*(G)]^{-1/2}$ where $M_k{}^*=\epsilon$

Evidently theorem1 is a special case of theorem1a for G. Being bipartite graph and for s and t being even + ve integers.

Lemma: Let a_1,$a_{2,}$……….a_n be +ve real numbers n>1 and r,s,t be non negative real numbers such that 4r=s+t+2 then $[\sum_{i=1}^{n} a_i{}^r]^4 \leq [\sum_{i=1}^{n} a_i]^2$ [$\sum_{i=1}^{n}(a_i)^s \sum_{i=1}^{n}(a_i)^t$

If (s,t)≠(1,1) then the above inequality holds if and only if a_1=$a_{2,}$=……….=a_n If s=t=1 (consequently r=1) then result holds trivially.

On Second Stage Spectrum And Energy of a Graph

Let G be a graph with vertex set V(G)={ v_1,$v_{2,}$……….v_n}. The distance between the vertices v_i and $v_{j,}$ where v_i, $v_{j,}$ ∈ V(G) be equal to the length [equal number of edges] of a shortest path starting at v_i and ending at $v_{j,}$(vice versa) [1].

In inorganic chemistry [11] there is a concept called second electron affinity. It is the energy supplied to an x^{-1}(G) ion to form x^{-2}(G) ion i.e., to form a second stage ion form the original ion.

This concept is motivated us to define the second stage matrix A_2(G) of a graph G, which is the symmetric ' n x n' matrix whose entry is unity if the vertices v_i& $v_{j,}$ are at a distance two and zero otherwise. As $A_2(G)$ is symmetric (0,1) matrix with zero diagonal, it may be viewed as the adjacency matrix of some graph G^* that in [1] was named derived graph of G .

Example: Let $K_n, P_{n,} S_n\ and\ C_n$ be respectively. The n-vertex complete graph path, star and cycle and $K_{a,b}$ be complete bipartite graph on a+b vertices. Let $\bar{G}$ denotes the complement of the graph G. Then $(K_n)^+ \equiv \bar{K}_n$ $(P_n)^+ \equiv P_{\frac{n}{2}} \cup P_{\frac{n}{2}}$

$(S_n)^+ \equiv K_{n-1} \cup K_1 (K_{a,b})^+ \equiv K_a \cup K_b$ and

$$(S_n)^+ \equiv \begin{cases} \bar{K}_3 & if\ n=3 \\ K_1 \cup K_2 & if\ n=4 \\ C_{\frac{n}{2}} \cup C_{\frac{n}{2}} & if\ n\ is\ even\ \&\&\ and\ n\geq 6 \\ C_n & if\ n\ is\ odd\ and\ n\geq 5 \end{cases}$$

Since $G_1 \cup G_2 \equiv (G_1)^+ \cup (G_2)^+$ the derived graph of a disconnected graph is necessarily disconnected. However in numerous cases (e.g for all bipartite graphs) the derived graph of a connected graph is also disconnected. In [1] also upper bounds for the largest Eigen values of G^+ were established.

Lemma 1.1 [P.J.Davis 8]

Let A=$\begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$ be a 2X2 block symmetric matrix. Then the eigen values of A are those of $A_0 + A_1$ together with those of $A_0 - A_1$

Lemma1.2 [D.Cvetkovial 5] Let L(G) denote the line graph of the graph G if G is r-regular and connected, r≥3 with spec(G)=($r_1\ \lambda_1, \lambda_2, \dots\dots\dots \lambda_n$) then spec.L(G) = $\begin{bmatrix} 2r-2 & \lambda_2+r-2 \dots\dots\dots & \lambda_n+r-2 & -2 \\ 1 & 1 & 1 & \frac{n(r-2)}{2} \end{bmatrix}$

Lemma1.4: [D.Cvetkovic et al 5] Let G be connected r-reguar graph with spectrum ($r_1\ \lambda_1, \lambda_2, \dots\dots\dots \lambda_n$) then spec.($\bar{G}$) ={n-r-1 ,$-(\lambda_2+1)$,……-(λ_n+1)}

Lemma 1.5: [D.Cvetkovic et al 5] For every p≥ 3 there exists a pair of 4-regular non-cospectral graphs on n-vertices.

Graphs of Diameter Two

The diameter of a graph is the maximum distance between its vertices. If the diameter of the graph G is two, then any pair of non-adjacent vertices is at distance two and is thus connected in G^+, consequently $G^+ \cong \bar{G}$

Theorem1: [S.K Ayyaswamy etal [1]] Let G be an r-regular graph of diameter 2 and let its spectrum be $(r_1\ \lambda_2, \lambda_3, \dots \dots \dots \lambda_n)$. Then spec$(G^+)$ ={n-r-1 ,$-(\lambda_2+1)$,……-($\lambda_n+1)$}

Theorem2: [S.K Ayyaswamy etal [1]] Let G be an r-regular graph of diameter 2 and let its spectrum be $(r_1\ \lambda_2, \lambda_3, \dots \dots \dots \lambda_n)$. Then spec$\{(GXK_2)^+\}$ =

$$\begin{bmatrix} 3n-2(r+2) & -(\lambda_1+1)\dots\dots\dots-n & 0 \\ 1 & 1 \qquad\qquad 1 & n-1 \end{bmatrix}$$

Where i=1,2,…..n

Theorem3: [S.K Ayyaswamy etal[1]] Let G be (n,m)- graph of diameter 2, then

$$\sqrt{2m^+ + n(n-1)\Delta^{\frac{2}{n}}} \le E(G^+) \le \sqrt{2nm^+}$$

$$2\sqrt{m^+} \le E(G^+) \le 2m^+$$

E$(G^+) \le \frac{2m^+}{n} + \sqrt{(n-1)[2m^+ - (\frac{2m^+}{n})^2]}$ where Δ={det $A_n(G)$}

Theorem3: [S.K Ayyaswamy etal [1]] For i=1,2,…… Let G_i be an r_i-regular graph with n_i vertices and spectrum $(r_{i1}\ \lambda_{i2}, \lambda_{i3}, \dots \dots \dots \lambda_{in})$. Then spec$((G_1 \Delta G_2)^+)$ consists of eigen values -$\lambda_{i,j}$-1 for i=1,2.. and j=2,3,… n_i and two more eigen values n_1-r_1-1 , n_2-r_2-1

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