On general Eulerian integral of certain products of A-functions

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ABSTRACT
The object of this paper is to establish an general Eulerian integral involving the product of three multivariable A-functions defined by Gautam et al [1] and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular case concerning the multivariable H-function.

Keywords: Eulerian integral, multivariable A-function, Lauricella function of several variables, multivariable H-function, generalized hypergeometric function.

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1. Introduction

In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of three multivariable A-functions defined by Gautam et al [1] and a generalized hypergeometric function with general arguments. The A-function of several variables is an extension of multivariable H-function defined by Srivastava et al [6].
The serie representation of the multivariable A-function is given by Gautam [1] as

$$A[u_1,\cdots,u_v]=A^{0,\lambda:(\alpha',\beta');\cdots;(\alpha^{(v)},\beta^{(v)})}_{A,C:(M',N');\cdots;(M^{(v)},N^{(v)})}\left(\begin{array}{c} \mathrm{u}_1 \\ \cdot \\ \cdot \\ \mathrm{u}_v \end{array}\middle| \begin{array}{c} [(\mathrm{g}_j);\gamma',\cdots,\gamma^{(v)}]_{1,A}: \\ \cdots \\ [(\mathrm{f}_j);\xi',\cdots,\xi^{(v)}]_{1,C}: \end{array}\right.$$

$$\left.\begin{array}{c}(\mathrm{q}^{(1)},\eta^{(1)})_{1,M^{(1)}};\cdots;(q^{(v)},\eta^{(v)})_{1,M^{(v)}} \\ \cdots \\ \cdots \\ (\mathrm{p}^{(1)},\epsilon^{(1)})_{1,N^{(1)}};\cdots;(p^{(v)},\epsilon^{(v)})_{1,N^{(v)}}\end{array}\right)=\sum_{G_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{v}\phi_i u_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon_{G_i}^{(i)}g_i!} \tag{1.1}$$

where

$$\phi_1=\frac{\prod_{j=1}^{\lambda}\Gamma\left(1-g_j+\sum_{i=1}^{v}\gamma_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\lambda'+1}^{A}\Gamma\left(g_j-\sum_{i=1}^{v}\gamma_j^{(i)}U_i\right)\prod_{j=1}^{C}\Gamma\left(1-f_j+\sum_{i=1}^{v}\xi_j^{(i)}\eta_{G_i,g_i}\right)} \tag{1.2}$$

$$\phi_i=\frac{\prod_{j=1,j\neq m_i}^{\alpha^{(i)}}\Gamma\left(p_j^{(i)}-\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=1}^{\beta^{(i)}}\Gamma\left(1-q_j^{(i)}+\eta_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\alpha^{(i)}+1}^{N^{(i)}}\Gamma\left(1-p_j^{(i)}+\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=\beta^{(i)}+1}^{M^{(i)}}\Gamma\left(q_j^{(i)}-\eta_j^{(i)}\eta_{G_i,g_i}\right)},i=1,\cdots,v \tag{1.3}$$

and $\eta_{G_i,g_i}=\dfrac{p_{G_i}^{(i)}+g_i}{\epsilon_{G_i}^{(i)}},i=1,\cdots,v$ (1.4)

which is valid under the following conditions : $\epsilon_{m_i}^{(i)}[p_j^{(i)}+p_i']\neq\epsilon_j^{(i)}[p_{m_i}+g_i]$ (1.5)
and

$$u_i \neq 0, \sum_{j=1}^{A} \gamma_j^{(i)} - \sum_{j=1}^{C} \xi_j^{(i)} + \sum_{j=1}^{M^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{N^{(i)}} \epsilon_j^{(i)} < 0, i = 1, \cdots, v \tag{1.6}$$

Here $\lambda, A, C, \alpha_i, \beta_i, M_i, N_i \in \mathbb{N}^*; i = 1, \cdots, v ; f_j, g_j, p_j^{(i)}, q_j^{(i)}, \gamma_j^{(i)}, \xi_j^{(i)}, \eta_j^{(i)}, \epsilon_j^{(i)} \in \mathbb{C}$

The A-function is defined and represented in the following manner.

$$A(z'_1, \cdots, z'_s) = A^{m',n':m'_1,n'_1;\cdots;m'_s,n'_s}_{p',q':p'_1,q'_1;\cdots;p'_s,q'_s} \left(\begin{array}{c|c} z'_1 & (a'_j; A'^{(1)}_j, \cdots, A'^{(s)}_j)_{1,p'} : \\ \cdot & \\ \cdot & \\ \cdot & \\ z'_s & (b'_j; B'^{(1)}_j, \cdots, B'^{(s)}_j)_{1,q'} : \end{array} \right.$$

$$\left. \begin{array}{l} (c'^{(1)}_j, C'^{(1)}_j)_{1,p'_1}; \cdots ; (c'^{(s)}_j, C'^{(s)}_j)_{1,p'_s} \\ \\ (d'^{(1)}_j, D'^{(1)}_j)_{1,q'_1}; \cdots ; (d'^{(s)}_j, D'^{(s)}_j)_{1,q'_s} \end{array} \right) \tag{1.7}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L'_1} \cdots \int_{L'_s} \phi(t_1, \cdots, t_s) \prod_{i=1}^{s} \zeta_i(t_i) z_i'^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.8}$$

where $\phi(t_1, \cdots, t_s), \zeta_i(t_i), i = 1, \cdots, s$ are given by :

$$\phi(t_1, \cdots, t_s) = \frac{\prod_{j=1}^{m'} \Gamma(b'_j - \sum_{i=1}^{s} B'^{(i)}_j t_i) \prod_{j=1}^{n'} \Gamma(1 - a'_j + \sum_{i=1}^{s} A'^{(i)}_j t_j)}{\prod_{j=n'+1}^{p'} \Gamma(a'_j - \sum_{i=1}^{s} A^{(i)}_j t_j) \prod_{j=m'+1}^{q'} \Gamma(1 - b'_j + \sum_{i=1}^{s} B'^{(i)}_j t_j)} \tag{1.9}$$

and

$$\zeta_i(t_i) = \frac{\prod_{j=1}^{n'_i} \Gamma(1 - c'^{(i)}_j + C'^{(i)}_j t_i) \prod_{j=1}^{m'_i} \Gamma(d'^{(i)}_j - D'^{(i)}_j t_i)}{\prod_{j=n'_i+1}^{p'_i} \Gamma(c'^{(i)}_j - C'^{(i)}_j t_i) \prod_{j=m'_i+1}^{q'_i} \Gamma(1 - d'^{(i)}_j + D'^{(i)}_j t_i)} \tag{1.10}$$

Here $m', n', p', m'_i, n'_i, p'_i, c'_i \in \mathbb{N}^*; i = 1, \cdots, s ; a'_j, b'_j, c'^{(i)}_j, d'^{(i)}_j, A'^{(i)}_j, B'^{(i)}_j, C'^{(i)}_j, D'^{(i)}_j \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i) z'_k| < \frac{1}{2}\eta_k \pi, \xi^* = 0, \eta_i > 0 \tag{1.11}$$

$$\Omega_i = \prod_{j=1}^{p'} \{A'^{(i)}_j\}^{A'^{(i)}_j} \prod_{j=1}^{q'} \{B'^{(i)}_j\}^{-B'^{(i)}_j} \prod_{j=1}^{q'_i} \{D'^{(i)}_j\}^{D'^{(i)}_j} \prod_{j=1}^{p'_i} \{C'^{(i)}_j\}^{-C'^{(i)}_j}; i = 1, \cdots, s \tag{1.12}$$

$$\xi_i^* = Im\left(\sum_{j=1}^{p'} A'^{(i)}_j - \sum_{j=1}^{q'} B'^{(i)}_j + \sum_{j=1}^{q'_i} D'^{(i)}_j - \sum_{j=1}^{p'_i} C'^{(i)}_j\right); i = 1, \cdots, s \tag{1.13}$$

$$\eta_i = Re\left(\sum_{j=1}^{n'} A_j'^{(i)} - \sum_{j=n'+1}^{p'} A_j'^{(i)} + \sum_{j=1}^{m'} B_j'^{(i)} - \sum_{j=m'+1}^{q'} B_j'^{(i)} + \sum_{j=1}^{m_i'} D_j'^{(i)} - \sum_{j=m_i'+1}^{q_i'} D_j'^{(i)} + \sum_{j=1}^{n_i'} C_j'^{(i)} - \sum_{j=n_i'+1}^{p_i} C_j'^{(i)}\right)$$

$$i = 1, \cdots, s \tag{1.14}$$

Consider the second multivariable A-function.

$$A(z_1'', \cdots, z_u'') = A_{p'',q'':p_1'',q_1'';\cdots;p_u'',q_u''}^{m'',n'':m_1'',n_1'';\cdots;m_u'',n_u''}\left(\begin{array}{c|l} z"_1 & (a"_j; A_j''^{(1)}, \cdots, A_j''^{(u)})_{1,p''} : \\ \cdot & \\ \cdot & \\ \cdot & \\ z"_u & (b"_j; B_j''^{(1)}, \cdots, B_j''^{(u)})_{1,q''} : \end{array}\right.$$

$$\left.\begin{array}{l} (c"_j^{(1)}, C_j''^{(1)})_{1,p_1''}; \cdots; (c_j''^{(u)}, C_j''^{(u)})_{1,p_u''} \\ \\ (d"_j^{(1)}, D_j''^{(1)})_{1,q_1''}; \cdots; (d_j''^{(u)}, D_j''^{(u)})_{1,q_u''} \end{array}\right) \tag{1.15}$$

$$= \frac{1}{(2\pi\omega)^u}\int_{L_1''} \cdots \int_{L_u''} \phi'(x_1, \cdots, x_u) \prod_{i=1}^{u} \theta_i'(x_i) z_i''^{x_i} \mathrm{d}x_1 \cdots \mathrm{d}x_r \tag{1.16}$$

where $\phi'(x_1, \cdots, x_u), \theta_i'(x_i), i = 1, \cdots, u$ are given by :

$$\phi'(x_1, \cdots, x_u) = \frac{\prod_{j=1}^{m''} \Gamma(b_j'' - \sum_{i=1}^{u} B_j''^{(i)} x_i) \prod_{j=1}^{n''} \Gamma(1 - a_j'' + \sum_{i=1}^{u} A_j''^{(i)} x_j)}{\prod_{j=n''+1}^{p''} \Gamma(a_j'' - \sum_{i=1}^{u} A_j''^{(i)} x_j) \prod_{j=m''+1}^{q''} \Gamma\left(1 - b_j'' + \sum_{i=1}^{u} B_j''^{(i)} x_j\right)} \tag{1.17}$$

and

$$\theta_i'(x_i) = \frac{\prod_{j=1}^{n_i''} \Gamma(1 - c_j''^{(i)} + C_j''^{(i)} x_i) \prod_{j=1}^{m_i''} \Gamma(d_j''^{(i)} - D_j''^{(i)} x_i)}{\prod_{j=n_i''+1}^{p_i''} \Gamma(c_j''^{(i)} - C_j''^{(i)} x_i) \prod_{j=m_i''+1}^{q_i''} \Gamma(1 - d_j''^{(i)} + D_j''^{(i)} x_i)} \tag{1.18}$$

Here $m'', n'', p'', m_i'', n_i'', p_i'', c_i'' \in \mathbb{N}^*; i = 1, \cdots, u$; $a_j'', b_j'', c_j''^{(i)}, d_j''^{(i)}, A_j''^{(i)}, B_j''^{(i)}, C_j''^{(i)}, D_j''^{(i)} \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i') z_k''| < \frac{1}{2}\eta_k' \pi, \xi'^* = 0, \eta_i' > 0 \tag{1.19}$$

$$\Omega_i' = \prod_{j=1}^{p''} \{A_j''^{(i)}\}^{A_j''^{(i)}} \prod_{j=1}^{q''} \{B_j''^{(i)}\}^{-B_j''^{(i)}} \prod_{j=1}^{q_i''} \{D_j''^{(i)}\}^{D_j''^{(i)}} \prod_{j=1}^{p_i''} \{C_j''^{(i)}\}^{-C_j''^{(i)}}; i = 1, \cdots, u \tag{1.20}$$

$$\xi_i'^* = Im\left(\sum_{j=1}^{p''} A_j''^{(i)} - \sum_{j=1}^{q''} B_j''^{(i)} + \sum_{j=1}^{q_i''} D_j''^{(i)} - \sum_{j=1}^{p_i''} C_j''^{(i)}\right); i = 1, \cdots, u \tag{1.21}$$

$$\eta_i' = Re\left(\sum_{j=1}^{n''} A_j''^{(i)} - \sum_{j=n''+1}^{p''} A_j''^{(i)} + \sum_{j=1}^{m''} B_j''^{(i)} - \sum_{j=m''+1}^{q''} B_j''^{(i)} + \sum_{j=1}^{m_i''} D_j''^{(i)} - \sum_{j=m_i''+1}^{q_i''} D_j''^{(i)} + \sum_{j=1}^{n_i''} C_j''^{(i)} - \sum_{j=n_i''+1}^{p_i''} C_j''^{(i)}\right)$$

$$i = 1, \cdots, u \tag{1.22}$$

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [3 ,page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\, {}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,r$. The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$

The Lauricella function $F_D^{(k)}$ is defined as

$$F_D^{(k)}\left[a,b_1,\cdots,b_k;c;x_1,\cdots,x_k\right] = \frac{\Gamma(c)}{\Gamma(a)\prod_{j=1}^{k}\Gamma(b_j)}\frac{1}{(2\pi\omega)^k}\int_{L_1}\cdots\int_{L_k}\frac{\Gamma\left(a+\sum_{j=1}^{k}\zeta_j\right)\Gamma(b_1+\zeta_1),\cdots,\Gamma(b_k+\zeta_k)}{\Gamma\left(c+\sum_{j=1}^{k}\zeta_j\right)}$$

$$\prod_{j=1}^{k}\Gamma(-\zeta_j)(-x_j)^{\zeta_i}\,\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_k \tag{2.2}$$

where $max\left[|arg(-x_1)|,\cdots,|arg(-x_k)|\right] < \pi, c \neq 0,-1,-2,\cdots$.

We first establish the formula

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha,-\sigma_1,\cdots,-\sigma_k;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right] \tag{2.3}$$

where $a,b\in\mathbb{R}(a<b),\alpha,\beta,f_i,g_i,\sigma_i\in\mathbb{C},(i=1,\cdots,k)$; $min(Re(\alpha),Re(\beta))>0$ and

$$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\} < 1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava et al ([5], page 60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \tag{2.4}$$

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [4, page 454].

3. Eulerian integral

Let

$$X=m'_1,n'_1;\cdots;m'_s,n'_s;m''_1,n''_1;\cdots;m''_u,n''_u;1,0;\cdots;1,0;1,0;\cdots;1,0 \tag{3.1}$$

$$Y=p'_1,q'_1;\cdots;p'_s,q'_s;p''_1,q''_1;\cdots;p''_u,q''_u;0,1;\cdots;0,1;0,1;\cdots;0,1 \tag{3.2}$$

$$A=(a'_j;A_j'^{(1)},\cdots,A_j'^{(s)},0,\cdots,0,0,\cdots,0,0,\cdots,0)_{1,p'} \tag{3.3}$$

$$B=(b'_j;B_j'^{(1)},\cdots,B_j'^{(s)},0\cdots,0,0\cdots,0,0,\cdots,0)_{1,q'} \tag{3.4}$$

$$A'=(a''_j;0,\cdots,0,A_j''^{(1)},\cdots,A_j''^{(u)},0,\cdots,0,0,\cdots,0)_{1,p''} \tag{3.5}$$

$$B'=(b''_j;0,\cdots,0,B_j''^{(1)},\cdots,B_j''^{(u)},0,\cdots,0,0,\cdots,0)_{1,q''} \tag{3.6}$$

$$\mathrm{C}=(\mathrm{c'}_j^{(1)},C_j'^{(1)})_{1,p'_1};\cdots;(c_j'^{(s)},C_j'^{(s)})_{1,p'_s};(c_j''^{(1)},C_j''^{(1)})_{1,p''_1};\cdots;(c_j''^{(u)},C_j''^{(u)})_{1,p''_u}$$

$$(1,0);\cdots;(1,0);(1,0);\cdots;(1,0) \tag{3.7}$$

$$D=(\mathrm{d'}_j^{(1)},D_j'^{(1)})_{1,q'_1};\cdots;(d_j'^{(s)},D_j'^{(s)})_{1,q'_s};(\mathrm{d''}_j^{(1)},D_j''^{(1)})_{1,q''_1};\cdots;(d_j''^{(u)},D_j''^{(u)})_{1,q''_u};$$

$$(0,1);\cdots;(0,1);(0,1);\cdots;(0,1) \tag{3.8}$$

$$K_1=(1-\alpha-\sum_{i=1}^{v}\eta_{G_i,g_i}(\mu_i+\mu'_i);\mu_1,\cdots,\mu_s,\mu'_1,\cdots,\mu'_u,1,\cdots,1,\upsilon_1,\cdots,\upsilon_l) \tag{3.9}$$

$$K_2=(1-\beta-\sum_{i=1}^{v}\eta_{G_i,g_i}(\rho_i+\rho'_i);\rho_1,\cdots,\rho_s,\rho'_1,\cdots,\rho'_u,0,\cdots,0,\tau_1,\cdots,\tau_l) \tag{3.10}$$

$$K_P=[1-A_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1,\cdots,1]_{1,P} \tag{3.11}$$

$$K_j=[1+\sigma_j-\sum_{i=1}^{v}\eta_{G_i,g_i}(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(s)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(u)},0\cdots,1,\cdots,0,\zeta'_j,\cdots,\zeta_j^{(l)}]_{1,k} \tag{3.12}$$

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$$L_1=(1-\alpha-\beta-\sum_{i=1}^{v}\eta_{G_i,g_i}(\mu_i+\mu_i'+\rho_i+\rho_i');\mu_1+\rho_1,\cdots,\mu_s+\rho_s,\mu_1'+\rho_1',\cdots,\mu_u'+\rho_u',$$

$$1,\cdots,1,\upsilon_1+\tau_1,\cdots,\upsilon_l+\tau_l) \tag{3.13}$$

$$L_Q=[1-B_j;0,\cdots,0,0,\cdots,0,0,\cdots,0,1\cdots,1]_{1,Q} \tag{3.14}$$

$$L_j=[1+\sigma_j-\sum_{i=1}^{v}\eta_{G_i,g_i}(\lambda_i^{(j)}+\lambda_i'^{(j)});\lambda_j^{(1)},\cdots,\lambda_j^{(s)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(u)},0\cdots,0,\zeta_j',\cdots,\zeta_j^{(l)}]_{1,k} \tag{3.15}$$

$$A_1=A,A';B_1=B,B' \tag{3.16}$$

We have the following result

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}A\begin{pmatrix}\mathrm{z"'}_1(t-a)^{\mu_1+\mu_1'}(b-t)^{\rho_1+\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}-\lambda_j'^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z"'}_v(t-a)^{\mu_v+\mu_v'}(b-t)^{\rho_v+\rho_v'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(v)}-\lambda_j'^{(v)}}\end{pmatrix}$$

$$A\begin{pmatrix}\mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(s)}}\end{pmatrix}$$

$$A\begin{pmatrix}\mathrm{z'}_1(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z'}_u(t-a)^{\mu_u'}(b-t)^{\rho_u'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}}\end{pmatrix}$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z_i''(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=(b-a)^{\alpha+\beta-1}$$

$$=P_1\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\sum_{g_1,\cdots,g_v=0}^{\infty}\sum_{M_1=0}^{\alpha^{(1)}}\cdots\sum_{M_v=0}^{\alpha^{(v)}}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u}z''^{R_k}B_uB_{u,v}$$

$$A^{m'+m'',n'+n''+l+k+2;X}_{p'+p''+l+k+2,q'+q''+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}_1,K_1,K_2,K_P,K_j:C \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(s)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(u)}}} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{af_k+g_k} & \cdot \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\varsigma_j^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\varsigma_j^{(l)}}} & \mathrm{B}_1,L_1,L_j,L_Q,:D \end{array}\right) \tag{3.17}$$

We obtain the A-function of $s+u+k+l$ variables.

Where $$P_1=(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{h}(af_j+g_j)^{\sigma_j}\right\} \tag{3.18}$$

$$B_{u,v}=(b-a)^{\sum_{i=1}^{v}(\mu_i+\mu'_i+\rho_i+\rho'_i)\eta_{G_i,g_i}}\left\{\prod_{j=1}^{h}(af_j+g_j)^{-\sum_{i=1}^{v}(\lambda_i+\lambda'_i)\eta_{g_i,h_i}}\right\}G_v \tag{3.19}$$

where $G_v=\phi_1\dfrac{\prod_{i=1}^{v}\phi_i u_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon_{G_i}^{(i)}g_i!}$

ϕ_1,ϕ_i for $i=1,\cdots,v$ are defined respectively by (1.2) and (1.3)

$$B_u=\frac{(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!} \tag{3.20}$$

Provided that

(A) $m', n', p', m'_i, n'_i, p'_i, c'_i \in \mathbb{N}^*; i = 1, \cdots, s$ **;** $a'_j, b'_j, c'^{(i)}_j, d'^{(i)}_j, A'^{(i)}_j, B'^{(i)}_j, C'^{(i)}_j, D'^{(i)}_j \in \mathbb{C}$

$m'', n'', p'', m''_i, n''_i, p''_i, c''_i \in \mathbb{N}^*; i = 1, \cdots, u$ **;** $a''_j, b''_j, c''^{(i)}_j, d''^{(i)}_j, A''^{(i)}_j, B''^{(i)}_j, C''^{(i)}_j, D''^{(i)}_j \in \mathbb{C}$

(B) $a, b \in \mathbb{R}(a < b); \mu_i, \rho_i, \mu'_j, \rho'_j \lambda^{(i)}_v; \lambda'^{(j)}_v \in \mathbb{R}^+$, $f_i, g_j, \tau_v, \sigma_j \in \mathbb{C}$ $(i = 1, \cdots, r; j = 1, \cdots; s; v = 1, \cdots, k)$

$(C)\ \max_{1 \leqslant j \leqslant k} \left\{ \left| \frac{(b-a)f_i}{af_i + g_i} \right| \right\} < 1$

(D) $Re\big[\alpha + \sum_{i=1}^{s} \mu_i \min_{1 \leqslant j \leqslant m'_i} \frac{d'^{(i)}_j}{D'^{(i)}_j} + \sum_{i=1}^{u} \mu'_i \min_{1 \leqslant j \leqslant m''_i} \frac{d''^{(i)}_j}{D''^{(i)}_j}\big] > 0$

$Re\big[\beta + \sum_{i=1}^{s} \rho_i \min_{1 \leqslant j \leqslant m'_i} \frac{d'^{(i)}_j}{D'^{(i)}_j} + \sum_{i=1}^{u} \rho'_i \min_{1 \leqslant j \leqslant m''_i} \frac{d''^{(i)}_j}{D''^{(i)}_j}\big] > 0$

(E) $\xi^*_i = Im \left(\sum_{j=1}^{p'} A'^{(i)}_j - \sum_{j=1}^{q'} B'^{(i)}_j + \sum_{j=1}^{q'_i} D'^{(i)}_j - \sum_{j=1}^{p'_i} C'^{(i)}_j \right) = 0; i = 1, \cdots, s$

$\xi'^*_i = Im \left(\sum_{j=1}^{p''} A''^{(i)}_j - \sum_{j=1}^{q''} B''^{(i)}_j + \sum_{j=1}^{q''_i} D''^{(i)}_j - \sum_{j=1}^{p''_i} C''^{(i)}_j \right) = 0; i = 1, \cdots, u$

(F) $|arg(\Omega_i) z_k| < \frac{1}{2} \eta_k \pi, \xi^* = 0, \eta_i > 0$

$$Re\left(\sum_{j=1}^{n} A'^{(i)}_j - \sum_{j=n+1}^{p} A'^{(i)}_j + \sum_{j=1}^{m} B'^{(i)}_j - \sum_{j=m+1}^{q} B'^{(i)}_j + \sum_{j=1}^{m_i} D'^{(i)}_j - \sum_{j=m_i+1}^{q_i} D'^{(i)}_j + \sum_{j=1}^{n_i} C'^{(i)}_j - \sum_{j=n_i+1}^{p_i} C'^{(i)}_j \right)$$

$$-\mu'_i - \rho'_i - \sum_{l=1}^{k} \lambda'^{(i)}_j > 0\,;\ i = 1, \cdots, s$$

$|arg(\Omega'_i) z'_k| < \frac{1}{2} \eta'_k \pi, \xi'^* = 0, \eta'_i > 0$

$$Re\left(\sum_{j=1}^{n''} A''^{(i)}_j - \sum_{j=n''+1}^{p''} A''^{(i)}_j + \sum_{j=1}^{m''} B''^{(i)}_j - \sum_{j=m''+1}^{q''} B''^{(i)}_j + \sum_{j=1}^{m''_i} D''^{(i)}_j - \sum_{j=m''_i+1}^{q''_i} D''^{(i)}_j + \sum_{j=1}^{n''_i} C''^{(i)}_j - \sum_{j=n''_i+1}^{p''_i} C''^{(i)}_j \right)$$

$$-\mu'_i - \rho'_i - \sum_{l=1}^{k} \lambda'^{(i)}_j > 0\,;\ i = 1, \cdots, u$$

(G) $\left| arg \left(z_i \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda^{(i)}_j} \right) \right| < \frac{1}{2} \eta_i \pi \ \ (a \leqslant t \leqslant b; i = 1, \cdots, s)$

$$\left|arg\left(z'_i\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(i)}_j}\right)\right| < \frac{1}{2}\eta'_i\pi \quad (a \leqslant t \leqslant b; i=1,\cdots,u)$$

(H) $P \leqslant Q+1$. The equality holds, when , in addition,

either $P > Q$ and $\left|z''_i\left(\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta^{(i)}_j}\right)\right|^{\frac{1}{Q-P}} <1 \qquad (a \leqslant t \leqslant b)$

or $P \leqslant Q$ and $\max_{1\leqslant i\leqslant k}\left[\left|\left(z''_i\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta^{(i)}_j}\right)\right|\right] < 1 \qquad (a \leqslant t \leqslant b)$

(I) The multiple series occuring on the right-hand side of (3.17) is absolutely and uniformly convergent.

Proof

First expressing the multivariable A-function in serie with the help of (1.6) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the A-function of s-variables and u-variables defined by Gautam et al [1] by the Mellin-Barnes contour integral with the help of the equation (1.8) and (1.16) respectively, the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1). Now collect the power of (f_jt+g_j) with $j=1,\cdots,k$ and use the equations (2.1) and (2.2) and we obtain $k-$Mellin-Barnes contour integral and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Interpreting $(r+s+k+l)-$Mellin-barnes contour integral in multivariable A-function defined by Gautam et al [1], we obtain the desired result.

4. Multivariable H-function

If $A'^{(i)}_j, B'^{(i)}_j, C'^{(i)}_j, D'^{(i)}_j \in \mathbb{R}$,$m'=0$ and $A''^{(i)}_j, B''^{(i)}_j, C''^{(i)}_j, D''^{(i)}_j \in \mathbb{R}$ and $m''=0$, the multivariable A-functions reduces to multivariable H-functions defined by Srivastava et al [6]. We obtain the following formula.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j} A\left(\begin{array}{c} \text{z"'}_1(t-a)^{\mu_1+\mu'_1}(b-t)^{\rho_1+\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda^{(1)}_j-\lambda'^{(1)}_j} \\ \cdot \\ \cdot \\ \cdot \\ \text{z"'}_v(t-a)^{\mu_v+\mu'_v}(b-t)^{\rho_v+\rho'_v}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda^{(v)}_j-\lambda'^{(v)}_j} \end{array}\right)$$

$$H\left(\begin{array}{c} \mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda^{(1)}_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_s(t-a)^{\mu_s}(b-t)^{\rho_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda^{(s)}_j} \end{array}\right)$$

$$H\left(\begin{array}{c} z'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ . \\ . \\ . \\ z'_u(t-a)^{\mu'_u}(b-t)^{\rho'_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(u)}} \end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l} z''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right]\mathrm{d}t=(b-a)^{\alpha+\beta-1}$$

$$=P_1\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\sum_{g_1,\cdots,g_v=0}^{\infty}\sum_{M_1=0}^{\alpha^{(1)}}\cdots\sum_{M_v=0}^{\alpha^{(v)}}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{i=1}^{v}z_i'''^{\eta_{h_i,k_i}}\prod_{k=1}^{u}z''^{R_k}B_uB_{u,v}$$

$$H^{0,n'+n''+l+k+2;X}_{p'+p''+l+k+2,q'+q''+l+k+1;Y}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}_1,K_1,K_2,K_P,K_j:C \\ \cdots & . \\ \cdots & . \\ \frac{z_s(b-a)^{\mu_s+\rho_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(s)}}} & . \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z'_u(b-a)^{\mu'_u+\rho'_u}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(u)}}} & . \\ \frac{(b-a)f_1}{af_1+g_1} & . \\ \cdots & . \\ \cdots & . \\ \frac{(b-a)f_k}{af_k+g_k} & . \\ \frac{z''_1(b-a)^{\tau_1+\upsilon_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(1)}}} & . \\ \cdots & . \\ \cdots & . \\ \frac{z''_l(b-a)^{\tau_l+\upsilon_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \mathrm{B}_1,L_1,L_j,L_Q,:D \end{array}\right) \quad (4.1)$$

under the same conditions and notations that (3.17) with $A'^{(i)}_j, B'^{(i)}_j, C'^{(i)}_j, D'^{(i)}_j \in \mathbb{R}$,$m' = 0$ and $A''^{(i)}_j, B''^{(i)}_j, C''^{(i)}_j, D''^{(i)}_j \in \mathbb{R}$ and $m'' = 0$

Remark

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable A-functions defined by Gautam et al [1].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable A-functions, defined by Gautam et al [1], a expansion of multivariable A-function and a generalized hypergeometric function with general arguments. The formula established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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