Dom-Color Number of a Graph

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Abstract: *Let G be a graph with $\chi(G) = k$.then G is called k-chromatic.In a coloring of G, the set of all vertices with a given color is called a color class. Let $C = \{V_1, V_2, \ldots, V_k\}$ be a k-coloring of G. Let d_C denote the number of color classes in C which are dominating sets of G. Then $d_\chi = \max\ d_C$, where the maximum is taken over all k colorings of G , which we call the dom-color number of G.*

A partition of V into independent dominating sets of G is called an independent domatic partition of G or indomatic partition of G. A graph G which admits an independent domatic partition is called indominable. The maximum order of an independent domatic partition of G is called the indomatic number of G and is denoted by $d_i(G)$.

The chromatic bondage number $\rho(G)$ is the minimum number of edges between two color classes in a k- coloring of G, where the minimum is taken over all k-colorings of G. We present several interesting results on dom-color number and chromatic bondage number .

Keywords: *dom-color number , Chromatic bondage number, indomatic number*

I.INTRODUCTION

The chromatic number is well studied parameter whose history dates back to the famous four-color problem and the early work of Kempe [9] in 1879 and Heawood [8] in 1890.Fink et al.,[3] studied the concept of the bondage number.

Throughout this paper, we assume that G = (V,E) is a finite, simple connected graph with at least two vertices.

II.MAIN RESULTS

Example 1.1.

(i) $d_\chi = \chi(C_n) = 2$, if n is even. If n is odd, then $\chi(C_n) = 3$ and $d_\chi(C_n) = 2$.

(ii) $d_\chi(K_n) = \chi(K_n) = n$

(iii) For any bipartite graph G, $d_\chi(G) = \chi(G) = 2$.

Theorem 1.2. *In a k-chromatic graph G,any k-coloring of G yields another k-coloring of G containing a color class which is a dominating set of G.*

Remark 1.3. It follows from Theorem 1.2 that $d_\chi(G) \geq 1$ and hence $1 \leq d_\chi(G) \leq \chi(G)$. Further these bounds are sharp. For the complete graph K_n, we have $d_\chi(K_n) = \chi(K_n) = n$. Also if H is any connected graph of order at least 3, then the corona $G = H \circ K1$ does not have two disjoint independent dominating sets and hence $d_\chi(G) = 1$. In fact we have the following theorem.

Theorem 1.4. *Given integers a and b with $1 \leq a \leq b$, there exists a graph G such that $d_\chi(G) = a$ and $\chi(G) = b$.*

Proof. If a = 1, then for the corona $G = K_b \circ K_1$, we have $d_\chi(G) = 1$ and $\chi(G) = b$. If $a \geq 2$, let G be the graph obtained from K_b by attaching a copy of K_a at a vertices of K_b. Then $d_\chi(G) = a$ and $\chi(G) = b$.

For any graph G , $d_\chi(G) = \chi(G)$ if and only if G admits a k-coloring in which every color class is a dominating set of G and we proceed to study graphs with this property.

Theorem 1.5. *For any uniquely colorable graph G, $d_\chi(G) = \chi(G)$.*

Proof. Let $C = \{V_1, V_2, \ldots, V_k\}$ be the *k*-coloring of G, where $\chi(G) = k$. Suppose there exists a color class in C, say V_1 which is not a dominating set of G. Then there exists a vertex $v \in V_i$ where i > 1, such that v is not adjacent any of the vertices of V_1. Now, $C_1 = \{V_1 \cup \{v\}, \ldots, V_i - \{v\}, V_{i+1} \ldots V_k\}$ is a *k*-coloring of G, which is a contradiction. Thus each V_i is a dominating set of G. Hence $d_\chi(G) = \chi(G)$.

Corollary 1.6. *If G is a uniquely colorable graph, then $d_i(G) = \chi(G)$.*

Remark 1.7. *The converse of Theorem 1.4 is not true.*

Consider the graph G given in Figure 1.1. Clearly $\chi(G) = 3$. Also $C_1 = \{\{v_2,v_5\},\{v_3,v_6\},\{v_1,v_4\}\}$ and $C_2 = \{\{v_2,v_4\},\{v_1,v_3\}, \{v_1,v_5\}\}$ are two different 3-colorings of G and hence G is not uniquely colorable. Further, in any 3-coloring of G each color class contains exactly one vertex from $\{v_1,v_2,v_6\}$ and exactly one vertex from $\{v_3,v_4,v_5\}$. Hence every color class forms a dominating set of G, so that $d_\chi(G) = 3$.

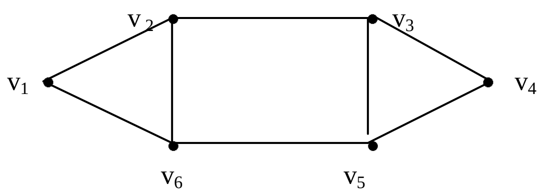

Figure 1.1

Remark 1.8. If $d_\chi(G) = \chi(G)$, then G is indominable and $\chi \leq d_i$. This inequality can be strict. For the graph G give in Figure 1.2, $\chi(G) = 3$. Also $\{\{1,6\},\{3,7\},\{4,8\},\{2,5\}\}$ is an indomatic partition of G and hence $d_i(G) = 4$

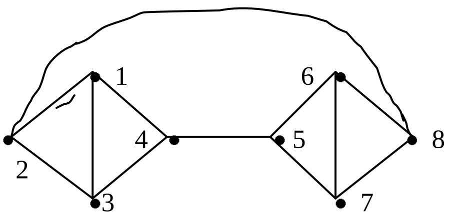

Figure 1.2

Remark 1.9. Let G be an indominable graph. Since $d_i(G) \geq \chi(G)$ and $\chi(G) \geq d_\chi(G)$, we have $d_i(G) \geq d_\chi(G)$.

Theorem 1.10. *Let G be a k-chromatic graph. If* $= \frac{m}{\binom{k}{2}}$ *, then every k-coloring of G is an independent domatic partition of G.*

Proof. Suppose $= \frac{m}{\binom{k}{2}}$. Then the number of edges between any two color classes in any k-coloring of G is $\frac{m}{\binom{k}{2}}$. Let $\{V_1,V_2,...,V_k\}$ be any k-coloring of G. We claim that each V_i is a dominating set of G. Suppose on the contrary that V_1 is not a dominating set of G. Then there exists a vertex v in a color class other than V_1, say V_2, such that v is not adjacent to any of the vertices of V_1. Hence $\{V_1 \cup \{v\}, V_2 - \{v\}, V_3,...,V_k\}$ is a k-coloring of G and the number of edges between the color class $V_2 - \{v\}$ and some color class is less than $\frac{m}{\binom{k}{2}}$, which is a contradiction. Thus each V_i is a dominating set of G.

Corollary 1,11. *Let G be a k-chromatic graph with* $\rho = \frac{m}{\binom{k}{2}}$. *Then* $d_\chi(G) = \chi(G)$.

Corollary 1.12. *For any k-chromatic graph G with* $\rho = \frac{m}{\binom{k}{2}}$, $i(G) \leq \frac{n}{k}$ *and this bound is sharp.*

Proof. Let $\{V_1,V_2,...,V_k\}$ be a k-coloring of G. Since each V_i is a dominating set of G, $|V_i| \geq i(G)$ for all $i = 1,2,...,k$ and hence $i(G) \leq \frac{n}{k}$. This bound is sharp, since for the complete k-partite graph $G = K_{\lambda,\lambda,...,\lambda}$, we have $i(G) = \frac{n}{k}$.

Remark 1.13. The converse of Theorem 1.9 is not true. For the 3-chromatic graph G given in Figure 1.3,

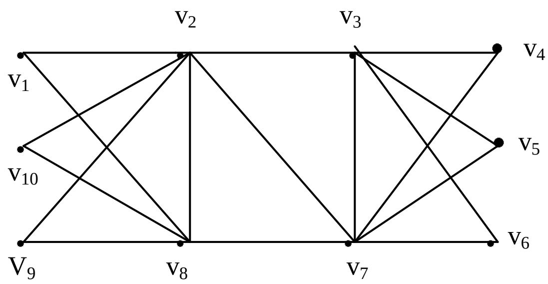

Figure 1.3

$\{\{v_2,v_4,v_5,v_6\},\{v_3,v_8\},\{v_1,v_7,v_9,v_{10}\}\}$ is the unique 3-coloring of G in which each color class is a dominating set of G. However $\rho = 5 \neq \frac{m}{\binom{k}{2}}$.

Problem 1.14. *Characterize k-chromatic graphs G with* $\rho(G) = \frac{m}{\binom{k}{2}}$ *for which* $i(G) = \frac{n}{k}$.

We observe that a k-chromatic graph G with $\rho(G) = \frac{m}{\binom{k}{2}}$ for which $i(G) = \frac{n}{k}$ must be a k-partite graph with a partition $\{V_1, V_2, \ldots, V_k\}$ such that each V_i is an $i(G)$-set of G and the number of edges in $\langle V_i \cup V_j \rangle$ is same for all $i \neq j$.

Theorem 1.9 leads to the following problem.

Problem 1.15. *Characterize k-chromatic graphs G in which every k-coloring of G is an independent domatic partition of G.*

Theorem 1.16. *For any uniquely colorable graph G,* $\rho(G) \geq 2i(G)-1$. *Further, equality holds if and only if the k-coloring of G contains two color classes* V_1 *and* V_2 *such that* $|V_1| = |V_2| = i(G)$ *and* $\langle V_1 \cup V_2 \rangle$ *is a tree.*

Proof. Let $\{V_1, V_2, \ldots, V_k\}$ be the k-coloring of G. Since G is uniquely colorable, it follows from Theorem 1.4 that each V_i is a dominating set of G and hence $|V_i| \geq i(G)$. Now, it follows from Theorem 1.45 that the induced subgraph $\langle V_i \cup V_j \rangle$ is connected for all $i \neq j$ so that the number of edges in $\langle V_i \cup V_j \rangle$ is at least $2i(G)-1$. Thus $\rho(G) \geq 2i(G)-1$. Obviously, equality holds if and only if the k-coloring of G contains two color classes V_1 and V_2 such that $|V_1| = |V_2| = i(G)$ and $\langle V_1 \cup V_2 \rangle$ is a tree.

Remark 1.17. *The above bound is sharp. For the complete graph* K_n, *we have* $\rho(K_n) = 1 = 2i(K_n)-1$.

The following are some interesting problems for further investigation.

Problem 1.18.

(i) *Characterize uniquely colorable graphs G for which* $\rho(G) = 2i(G)-1$.

(ii) *Characterize graphs G for which* $\rho(G) = \frac{m}{\binom{k}{2}}$

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