A Review on Total and Paired Domination of Cartesian product Graphs

[1] S.Divya

[1]Assistant Professor, Department of Mathematics, Subramanya Arts and Science College, Palani.

ABSTRACT – A dominating set D for a graph G is a subset of V(G) such that any vertex not in D has at least one neighbor in D. The domination number γ(G) is the size of a minimum dominating set in G. Vizing's conjecture from 1968 states that for the Cartesian product of graphs G and H ,γ(G)γ(H) ≤ γ(G□H), and Clark and Suen (2000) proved that γ(G)γ(H)≤2γ(G□H). In this paper, we modify the approach of Clark and Suen to prove a variety of similar bounds related to total and paired domination, and also extend these bounds to then n-Cartesian product of graphs A^1throughA^n.

Keywords: Domination, Total domination, Paired domination, Vizing's conjecture.

1. INTRODUCTION

A graph is nothing but a representation of any physical situation involving discrete objects and a relationship among them. A dominating set D for a graph G is a subset of V(G) such that any vertex not in D has at least one neighbor in D. The domination number γ(G) is the size of a minimum dominating set in G. Vizing's conjecture from 1968 states that for the Cartesian product of graphs G and H ,γ(G)γ(H) ≤ γ(G□H), and Clark and Suen (2000) proved that γ(G)γ(H)≤2γ(G□H). In this paper, we modify the approach of Clark and Suen to prove a variety of similar bounds related to total and paired domination, and also extend these bounds to then n-Cartesian product of graphs A^1 throughA^n.

2. TOTAL AND PAIRED DOMINATION OF CARTESIAN PRODUCT GRAPHS

We begin by introducing some notation which will be utilized throughout the proofs in this section. Given S⊆V(G□H), the **projection of S onto graphs G and H** is defined as $\Phi_G(S)$ = { g ∈ V(G) | ∃ h ∈ V(H) with gh ∈ S }, $\Phi_H(S)$ = { h ∈ V(H) | ∃ g ∈ V(G) with gh ∈ S}.In the case of the **n-product graph** A^1□….□A^n, we project a set of vertices in V(A^1□….□A^n) down to a particular graph A^i. Therefore, given S ⊆ V(A^1□….□A^n), we define $\Phi_A{}^i(S)$ = { a ∈ V(A^i) | ∃ $u^1 \cdots u^n$ ∈ S with a = u^i }.

For gh ∈ V(G□H), the **G-neighborhood** and **H-neighborhood** of gh are defined as follows:

$N_{\underline{G}\square H}$ (gh) = {g′h ∈ V(G□H) | g′∈N_G(g)},

$N_{G\square \underline{H}}$ (gh) = {gh′ ∈ V(G□H) | h′∈N_H(h)}.

Thus, $N_{\underline{G}\square H}$ (gh) and $N_{G\square \underline{H}}$ (gh) are both subsets of V(G□H). Additionally, E(G□H) can be partitioned into two sets, **G-edges** and **H-edges**, where G-edges = {(gh, g′h) ∈ E(G□H) | h ∈ V(H) and (g, g′) ∈ E(G)}, H-edges = {(gh, gh′) ∈ E(G□H) | g ∈ V(G) and (h, h′) ∈ E(H)}.

In the case of the n-product graph A^1□….□A^n, we identify the **i-neighborhood** of a particular vertex, and partition the set of edges E(A^1□….□A^n) into n sets. Thus, we define E_i to be E_i = { ($u^1 \ldots u^n$,$v^1 \ldots v^n$) | (u^i,v^i) ∈ E(A^i), and u_j = v_j, for all other indices j ≠ i}, And for a vertex u ∈ V(A^1□….□A^n), we define $N_{\square A^i}$(u) = { v ∈ V(A^1□….□A^n) | v and u are connected by E_i-edge}.

Finally, we need two elementary propositions about matrices that will be utilized throughout the proofs.

PROPOSITION I.

Let M be a binary matrix. Then either

(a) each column contains a 1, or

(b) each row contains a 0.

Prop.1 refers only to $d_1 \times d_2$ binary matrices.

PROOF:

Let M be the matrix containing only 0/1 entries.

For a proof by contradiction, Assume there exists a row (say i) which does not contain a 0, and a column (say j) which does not contain a 1.

Then the entry M[i,j] is neither 0 nor 1.

This is a contradiction.

Prop.II is a generalization of prop.I for $d_1 \times d_2 \times \ldots \times d_n$ n-ary matrices.

PROPOSITION II.

Let M be a $d_1 \times d_2 \times \ldots \times d_n$, n-ary matrix (n-ary in this case signifies that M contains entries only in the range {1….n}). Then there exists a j ∈ {1….n} (not necessarily unique), such that each of the $d_1 \times d_2 \times \ldots \times d_{j-1} \times 1 \times d_{j+1} \times \ldots \times d_n$ submatrices of M contains an entry with value j. Such a matrix M is called a j-matrix. Note that, given any $d_1 \times d_2 \times \ldots \times d_n$ matrix, there are d_j submatrices of the form $d_1 \times d_2 \times \ldots \times d_{j-1} \times 1 \times d_{j+1} \times \ldots \times d_n$. We will denote such a submatrix as M[:, i_j,:] with$1 \leq i_j \leq d_j$.

PROOF:

Let M be a $d_1 \times d_2 \times \ldots \times d_n$ n-ary matrix, there exists at least one $1 \times d_2 \times \ldots \times d_n$ submatrix that does not contain a 1.without loss of generality, let M[i_1,:] with $1 \leq i_1 \leq d_1$ be such matrix. Next, consider j=2. Since M is also not a 2-matrix, let M[:, i_2,:] with $1 \leq i_2 \leq d_2$ be a $d_1 \times 2 \times d_3 \times \ldots \times d_n$ submatrix that does not contain a 2.Therefore,M[i_1, i_2,:] is a $1 \times 1 \times d_3 \times \ldots \times d_n$ submatrix that contains neither a 1 nor a 2.we continue this pattern for $1 \leq j \leq n-1$.since M is not a j-matrix for $1 \leq j \leq n-1$. Let M[$i_1, i_2, \ldots i_{n-1},:$] be the $1 \times \ldots 1 \times d_n$ submatrix containing no elements in the set {1,…,n-1}.Therefore, for all $1 \leq x \leq d_n$, M[$i_1, i_2, \ldots i_{n-1}, x$] $= n$, and all of the $d_1 \times \ldots \times d_{n-1} \times 1$ submatrices of M contains an entry with value n. Thus, M is an n-matrix.

THEOREM 3.9

Given graphs G and H containing no isolated vertices, $\max\{\gamma(G)\gamma_t(G), \gamma_t(G)\gamma(G)\} \leq 2\gamma(G\Box H)$.

PROOF:

Let $\{u_1 \ldots\ldots u_{\gamma_t(G)}\}$ be a γ_t-set of G. Partition V(G) into sets, $D_1, \ldots\ldots, D_{\gamma_t(G)}$ such that $D_i \subseteq N_G(u_i)$. Let $\{\bar{u}_1 \ldots\ldots \bar{u}_{\gamma(H)}\}$ be a γ-set of H. Partition V(H) into sets $\bar{D}_1, \ldots\ldots \bar{D}_{\gamma(H)}$ such that $\bar{u}_j \in \bar{D}_j$ and $\bar{D}_j \subseteq N_H[\bar{u}_j]$. We note that $\{D_1, \ldots. D_{\gamma_t(G)}\} \times \{\bar{D}_1, \ldots\ldots \bar{D}_{\gamma(H)}\}$ is a partition of V(G□H). Let D be a γ-set of G□H. Then, for each gh ∉ D, either $N_{\underline{G}\Box H}(gh) \cap D$ or $N_{G\Box\underline{H}}(gh) \cap D$ is nonempty. Based on this observation, we define the binary |V(G)| × |V(H)| matrix F such that:

$$F(g, h) = \begin{cases} 1 \text{ if } gh \in D \text{ or } N_{G\Box\underline{H}}(gh) \cap D \neq \phi, \\ \quad 0 \text{ otherwise}. \end{cases}$$

Since F is a |V(G)|×|V(H)| matrix, each of the $D_i \times \bar{D}_j$subsets of V(G□H) determines a submatrix of F.

For $i = 1, \ldots \gamma_t(G)$,. Let $Z_i = D \cap (D_i \times V(H))$, and let $S_i = \{\bar{D}_x$ | the submatrix of F determined by $D_i \times \bar{D}_x$ satisfies Prop. Ia, with x ∈ {1, . . . ,γ(H)}} For j = 1, . . . , ,γ(H), let $\bar{Z}_j = D \cap (V(G) \times \bar{D}_j)$, and let $\bar{S}_j = \{D_x$ | the submatrix of F determined by $D_x \times \bar{D}_j$ satisfies Prop. Ib, with x ∈ {1, . . . , $\gamma_t(G)$}}

Let $d_H = \sum_{i=1}^{\gamma_t(G)} |S_i|$ and $d_G = \sum_{j=1}^{\gamma(H)} |\bar{S}_j|$. Since the partition of V(G□H) composed of elements $D_i \times \bar{D}_j$ contains, $\gamma_t(G)\gamma(H)$ components, and since every $D_i \times \bar{D}_j$ submatrix of F satisfies either conditions (a) or (b) of Prop. I (possibly both), $\gamma_t(G)\gamma(H) \leq d_H + d_G$. We will now prove two subclaims which will allow us to bound the size of our various sets.

CLAIM – 1:

If the submatrix of F determined by $D_i \times \bar{D}_j$ satisfies Prop. Ia, then $\bar{D}_j$ is dominated by $\Phi_H(Z_i)$.

CLAIM – 2:

If the submatrix of F determined by $D_i \times \bar{D}_j$ satisfies Prop. Ib, then D_i is dominated by $\phi_G(\bar{Z}_j)$. Additionally, ∀g ∈ $D_i \cap \phi_G(\bar{Z}_j)$, there exists a vertex g′ ∈ $\phi_G(\bar{Z}_j)$. Such that (g, g′) ∈ E(G).

We note that this claim does not imply that $\phi_G(\bar{Z}_j)$ is a total dominating set, but the claim is a slightly stronger condition on domination. When applying this condition, we will say that the set D_i is non-self dominated by $\phi_G(\bar{Z}_j)$.

.CLAIM -3:

For $i = 1, \ldots, \gamma_t(G)$, $|S_i| \leq |Z_i|$. Similarly, for j = 1, . . . , γ(H), $|\bar{S}_j| \leq |\bar{Z}_j|$.

To conclude the proof, we observe that

$$d_H = \sum_{i=1}^{\gamma_t(G)} |S_i| \leq \sum_{i=1}^{\gamma_t(G)} |Z_i| \leq |D|,$$

$$d_G = \sum_{j=1}^{\gamma(H)} |\bar{S}_j| \leq \sum_{j=1}^{\gamma(H)} |\bar{Z}_j| \leq |D|.$$

Hence, $\gamma_t(G)\gamma(G) \leq d_H + d_G \leq 2|D| \leq 2(G\Box H)$.

Moreover, we can similarly prove that $\gamma(G)\gamma_t(G) \leq 2(G\Box H)$.

Therefore, $\max\{\gamma(G)\gamma_t(G), \gamma_t(G)\gamma(G)\} \leq 2\gamma(G\Box H)$.

3. PAIRED DOMINATION OF CARTESIAN PRODUCT GRAPHS

Let γ_{pr}(G) denote the paired domination number and G□H denote the Cartesian product of graphs G and H. In this paper we show that for all graphs G and H without isolated vertex, γ_{pr}(G)γ_{pr}(H) ≤ 7γ_{pr}(G□H).

THEOREM : For any graphs G and H,

$$\gamma(G)\,\gamma(H) \leq 2\,\gamma(G\Box H).$$

PROOF:

Let D be a dominating set of G□H. It is sufficient to show that,

$$\gamma(G)\,\gamma(H) \;\leq\; 2|D|. \qquad \ldots\ldots\ldots (1)$$

Let $\{u_1, u_2, \ldots . u_{\gamma(G)}\}$ be a dominating set of G. Form a partition $\{\Pi_1, \Pi_2, \ldots . \Pi_{\gamma(G)}\}$ of V(G) so that for all i:

(i) $u_i \in \Pi_i$, and (ii) u ∈ Π_i implies u = u_i or u is adjacent to u_i. This partition of V(G) induces a partition $\{D_1, D_2, \ldots . D_{\gamma(G)}\}$ of D where D_i = (Π_i × V(H)) ∩ D.

Let P_i be the projection of D_i onto H. That is, P_i = {v | (u,v) ∈ D_i for some u ∈ Π_i}. Observe that for any i, P_i ∪ (V(H) - $N_H[P_i]$) is a dominating set of H, and hence the number of vertices in V(H) not dominated by P_i satisfies the inequality

$$|V(H) - N_H[P_i]| \geq \gamma(H) - |P_i|. \qquad \ldots\ldots\ldots (2)$$

For v ∈ V(H), let Q_v = D ∩ (V(G) × {v}) = {(u,v) ∈ D | u ∈ V(G)}. And C be the subset of {1,2,…γ(G)} × V(H) given by C = $\{(i, v) \mid \Pi_i \times \{v\} \subseteq N_{G\Box H}[Q_v]\}$. Let N = $|C|$. By counting in two different ways we shall find upper and lower bounds for N.

Let

$$L_i = \{(i,v) \in C \mid v \in V(H)\}, \text{ and}$$
$$R_v = \{(i,v) \in C \mid 1 \leq i \leq \gamma(G)\}.$$

Clearly

$$N = \sum_{i=1}^{\gamma(G)} |L_i| = \sum_{v\in V(H)} |R_v|.$$

Note that if v ∈ V(H) - $N_H[P_i]$, then the vertices in Π_i × {v} must be dominated by vertices in Q_v and therefore (i,v) ∈ L_i. This implies that $|L_i| \geq |V(H) - N_H[P_i]|$. Hence,

$$N \geq \sum_{i=1}^{\gamma(G)} |V(H) - N_H[P_i]|$$

And it follows from (2) that

$$N \geq \gamma(G)\,\gamma(H) - \sum_{i=1}^{\gamma(G)} |P_i|$$
$$\geq \gamma(G)\,\gamma(H) - \sum_{i=1}^{\gamma(G)} |D_i|.$$

So we obtain the following lower bound for N.

$$N \geq \gamma(G)\,\gamma(H) - |D|. \qquad \ldots\ldots\ldots (3)$$

For each v ∈ V(H), $|R_v| \leq |Q_v|$. If not,

$$\{u \mid (u,v) \in Q_v\} \cup \{u_j \mid (j,v) \notin R_v\}$$

is a dominating set of G with cardinality$|Q_v|$ + (γ(G) - $|R_v|$) = γ(G) – ($|R_v|$ - | Q_v|) < γ(G), And we have a contradiction. This observation shows that

$$N = \sum_{v\in V(H)} |R_v| \leq \sum_{v\in V(H)} |Q_v| = |D|. \ldots\ldots (4)$$

It follows from (3) and (4) that

$$\gamma(G)\,\gamma(H) - |D| \leq N \leq |D|,$$
$$\gamma(G)\,\gamma(H) \leq 2\,\gamma(G\Box H).$$

Hence the theorem is proved.

4. APPLICATIONS OF DOMINATION IN GRAPHS

Domination in graphs has applications to several fields. Domination arises in facility location problems, where the number of facilities (e.g., hospitals, fire stations) is fixed and one attempts to minimize the distance that a person needs to travel to get to the closest facility. A similar problem occurs when the maximum distance to a facility is fixed and one attempts to minimize the number of facilities necessary so that everyone is serviced. Concepts from domination also appear in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveying (e.g., minimizing the number of places a surveyor must stand in order to take height measurements for an entire region).

(i) School Bus Routing

Most school in the country provide school buses for transporting children to and from school Most also operate under certain rules, one of which usually states that no child shall have to walk farther than, say one quarter km to a bus pickup point. Thus, they must construct a route for each bus that gets within one quarter km of every child in its assigned area. No bus ride can take more than some specified number of minutes, and Limits on the number of children that a bus can carry at any one time. Let us say that the following figure represents a street map of part of a city, where each edge represents one pick up block. The school is located at the large vertex. Let us assume that the school has decided that no child shall have to walk more than two blocks in order to be picked up by a school bus. Construct a route for a school bus that leaves the school, gets within two blocks of every child and returns to the school.

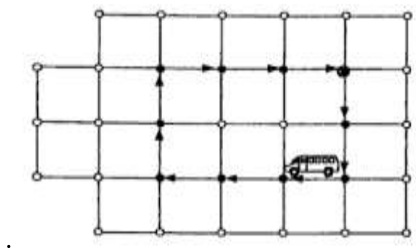

Figure-1: School Bus Routing

(ii) Computer Communication Networks

Consider a computer network modeled by a graph G = (V, E), for which vertices represents computers and edges represent direct links between pairs of computers. Let the vertices in following figure represent an array, or network, of 16 computers, or processors. Each processor to which it is directly connected. Assume that from time to time we need to collect information from all processors. We do this by having each processor route its information to one of a small set of collecting processors (a dominating set). Since this must be done relatively fast, we cannot route this information over too long a path. Thus we identify a small set of processors which are close to all other processors. Let us say that we will tolerate at most a two unit delay between the time a processor sends its information and the time it arrives at a nearby collector. In this case we seek a distance-2 dominating set among the set of all processors. The two shaded vertices form a distance- dominating set in the hypercube network in following figure

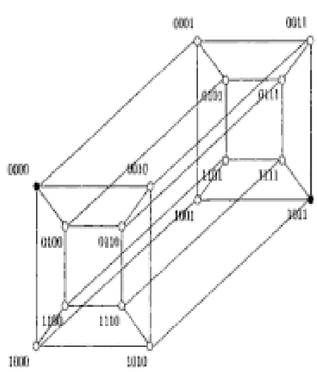

Figure -2 : Computer Communication Networks

(iii) Radio Stations

Suppose that we have a collection of small villages in a remote part of the world. We would like to locate radio stations in some of these villages so that messages can be broadcast to all of the villages in the region. Since each radio station has a limited broadcasting range, we must use several stations to reach all villages. But since radio stations are costly, we want to locate as few as possible which can reach all other villages.

Let each village be represented by a vertex. An edge between two villages is labeled with the distance, say in kilometers, between the two villages

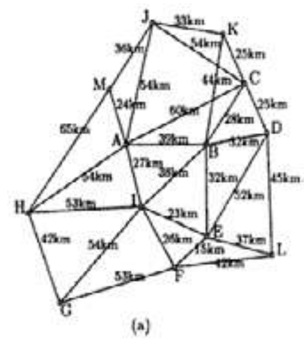

Figure-3: Radio Station

Let us assume that a radio station has a broadcast range of fifty kilometers. What is the least number of stations in a set which dominates (within distance 50) all other vertices in this graph? A set (B, F, H,J} of cardinality four is indicated in the following figure(b).

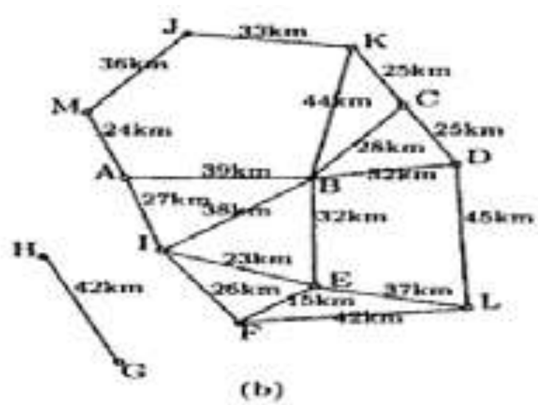

Figure-4: Radio Station

Here we have assumed that a radio station has a broadcast range of only fifty kilometers, we can essentially remove all edges in the graph, which represent a distance of more than fifty kilometers. We need only to find a dominating set in this graph. Notice that if we could afford radio stations which have a broadcast range of seventy kilometers, three radio stations would sufficient.

(iv) Locating Radar Stations Problem

The problem was discussed by Berge. A number of strategic locations are to be kept under surveillance. The goal is to locate a radar for the surveillance at as few of these locations as possible. How a set of locations in which the radar stations are to be placed can be determined.

(v) Nuclear Power Plants Problem

A similar known problem is a nuclear power plants problem. There are various locations and an arc can be drawn from location x to location y if it is possible for a watchman stationed at x to observe a warning light located at y. How many guards are needed to observe all of the warning lights, and where should they be located? At present, domination is considered to be one of the fundamental concepts in graph theory and its various applications to ad hoc networks, biological networks, distributed computing, social networks and web graphs partly explain the increased interest. Such applications usually aim to select a subset of nodes that will provide some definite service such that every node in the network is 'close' to some node in the subset. The following examples show when the concept of domination can be applied in modeling real-life problems.

(vi) Modeling Biological Networks

Using graph theory as a modeling tool in biological networks allows the utilization of the most graphical invariants in such a way that it is possible to identify secondary RNA (Ribonucleic acid) motifs numerically. Those graphical invariants are variations of the domination number of a graph. The results of the research carried out in show that the variations of the domination number can be used for correctly distinguishing among the trees that represent native structures and those that are not likely candidates to represent RNA.

(vii) Modeling Social Networks

Dominating sets can be used in modeling social networks and studying the dynamics of relations among numerous individuals in different domains. A social network is a social structure made of individuals (or groups of individuals), which are connected by one or more specific types of interdependency. The choice of initial sets of target individuals is an important problem in the theory of social networks. In the work of Kelleher and Cozzens, social networks are modeled in terms of graph theory and it was shown that some of these sets can be found by using the properties of dominating sets in graphs.

(viii) Facility Location Problems

The dominating sets in graphs are natural models for facility location problems in operational research. Facility location problems are concerned with the location of one or more facilities in a way that optimizes a certain objective such as minimizing transportation cost, providing equitable service to customers and capturing the largest market share.

5. CONCLUSION

This paper "TOTAL AND PAIRED DOMINATION OF CARTESIAN PRODUCT GRAPHS", can make an in-depth study in Total domination and Paired domination using Cartesian product graphs. We discussed the various applications of graph theory and dealt with the total and paired domination of Cartesian product graphs. We also dealt with the paired domination of Cartesian product graphs and the real life applications of domination in graphs.

REFERENCES

[1] B.Stjan, B.Brešar, P.Dorbec, W.Goddard, B.Hartnell, M.Henning, S.Klavžar, and D.Rall. Vizing's conjecture: A survey and recent results, 2009.

[2] W.Clark and S.Suen. An inequality realated to Vizing's conjecture. Electronic journal of combinatorics, 7(Note 4), 2000.

[3] P.T.Ho. A note on the total domination number. Utilitas Mathematica, 77:97-100, 2008.

[4] X.M.Hou and F.Jiang. Paired domination of Cartesian products of graphs, Journal of Mathematical Research and Exposition, 30(1): 181-185, 2010.

[5] B.Brešar, M.Henning, D.Rall. Paired-domination of Cartesian products of graphs[J]. Util. Math., 2007, 73: 255-265.

[6] Cockayne E J, Dawes R M, Hedetniemi S T. Total domination in graphs[J]. Networks, 1980. 10(3): 211-219.

[7] J.A. Pondy and U.S.R. Murthy. Graph theory with Applications., chapter-I to V.

[8] Bert Hartnell and Douglas F. Rall, Domination in Cartesian Products: Vizing's Conjecture, in Domination in Graphs-Advanced Topics edited by Haynes, et al, Marcel Dekker, Inc, New York, 1998, 163-189.

[9] V.G. Vizing, The Cartesian Product of Graphs, Vycisal, Sistemy 9,1963, 30-43.

[10] HOU Xinmin. Total domination of Cartesian products of graphs [J]. Discuss, Math,Graph theory, 2007, 27(1): 175-178.

[11] HAYNES T W, SLATER P.J. Paired domination in Graphs [J], Networks, 1998, 32(3): 199-206.
[12] NOWAKOWSKI R J, RALL D F. Associative graph products and their independence, domination and colouring numbers [J]. Discuss. Math. Graph theory, 1996, 16(1): 53-79.
[13] VIZING V.G. Some unsolved problems in graph theory [J]. Uspehi Mat. Nauk, 1968, 6(144): 117-134. (in Russian).
[14] PELEG D, ULLMAN J D. An optimal synchronizer for the hypercube [J]. SIAM J. comput., 1989, 18(4): 740-747.
[15] Domination in Graph with its Applications, PREETI GUPTA, Prestige institute of Engineering and Science, Indore.