# Strongly Multiplicative Labeling of Some Snake Related Graphs

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**Abstract:** A graph $G$ with $p$ vertices is said to be strongly multiplicative if the vertices of $G$ can be labeled with $p$ consecutive positive integers $1, 2, ..., p$ such that label induced on the edges by the product of labels of end vertices are all distinct. In this paper we investigate strongly multiplicative labeling of some snake related graphs. We prove that alternate triangular snake and alternate quadrilateral snake are strongly multiplicative. We also prove that double alternate triangular snake and double alternate quadrilateral snake are strongly multiplicative. Strongly multiplicative labeling of double quadrilateral snake, braid graph and triangular ladder have also been discussed.

***Index Terms*—Strongly Multiplicative Labeling; Alternate Triangular Snake, Alternate Quadrilateral Snake; Double Alternate Triangular Snake, Double Alternate Quadrilateral Snake, Braid Graph, Triangular Ladder.**

***MSC 2010 Codes* – 05C78**

## I. Introduction

In this research article, by a graph we mean finite, connected, undirected, simple graph $G = (V(G), E(G))$ of order $|V(G)| = p$ and size $|E(G)| = q$.

**Definition 1.1**: A *graph labeling* is an assignment of numbers to the vertices or edges or both subject to certain condition(s). A latest survey of all the graph labeling techniques can be found in Gallian Survey[2].

**Definition 1.2**: A graph $G = (V(G), E(G))$ with $p$ vertices is said to be *multiplicative* if the vertices of $G$ can be labeled with $p$ distinct positive integers such that label induced on the edges by the product of labels of end vertices are all distinct.

Multiplicative labeling was introduced by Beineke and Hegde[1]. In the same paper they proved that every graph $G$ admits multiplicative labeling and defined strongly multiplicative labeling as follows.

**Definition 1.3**: A graph $G = (V(G), E(G))$ with $p$ vertices is said to be *strongly multiplicative* if the vertices of $G$ can be labeled with $p$ consecutive positive integers $1, 2, ..., p$ such that label induced on the edges by the product of labels of end vertices are all distinct.

Beineke and Hegde[1] proved the following results.

- Every cycle $C_n$ is strongly multiplicative.
- Every wheel $W_n$ is strongly multiplicative.
- The complete graph $K_n$ is strongly multiplicative $\Leftrightarrow n \leq 5$.
- The complete bipartite graph $K_{n,n}$ is strongly multiplicative $\Leftrightarrow n \leq 4$.
- Every spanning subgraph of a strongly multiplicative graph is strongly multiplicative.
- Every graph is an induced subgraph of a strongly multiplicative graph.

**Definition 1.4**: The *alternate triangular snake* $A(T_n)$ is the graph obtained from a path $P_n$ with vertices $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to a new vertex $v_i$ ,where $1 \leq i \leq n-1$ for even $n$ and $1 \leq i \leq n-2$ for odd $n$. In other words every alternate edge of a path $P_n$ is replaced by cycle $C_3$.

**Definition 1.5**: The *alternate quadrilateral snake* $A(QS_n)$ is obtained from a path $P_n$ with vertices $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to new vertices $v_i, w_i$ respectively and then joining $v_i$ and $w_i$ ,where $1 \leq i \leq n-1$ for even $n$ and $1 \leq i \leq n-2$ for odd $n$. In other words every alternate edge of path $P_n$ is replaced by cycle $C_4$

**Definition 1.6**: The *double quadrilateral snake* $D(QS_n)$ is obtained from a path $P_n$ with vertices $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ to new vertices $v_i, v_j^{'}$ and $w_i, w_i^{'}$ respectively and adding the edges $v_i w_i$ and $v_i^{'} w_i^{'}$. In other words $D(QS_n)$ consists of two quadrilateral snakes that have a common path.

**Definition 1.7**: The $Z - P_n$ *graph* is obtained from the pair of paths $P_n'$ and $P_n''$ by joining $i^{th}$ vertex of path $P_n'$ with $(i+1)^{th}$ vertex of path $P_n''$ with new edges, for all $1 \leq i \leq (n-1)$.

**Definition 1.8**: The *braid graph* $B_n$ is obtained from pair of paths $P_n'$ and $P_n''$ by joining $i^{th}$ vertex of path $P_n'$ with $(i+1)^{th}$ vertex of path $P_n''$ and $i^{th}$ vertex of path $P_n''$ with $(i+2)^{th}$ vertex of path $P_n'$ with new edges, for all $1 \leq i \leq (n-2)$.

**Definition 1.9**: The *triangular ladder* $TL_n$ is obtained from the ladder $L_n = P_n \times P_2; (n \geq 2)$ by adding the edges $u_i v_{i+1}$ for all $1 \leq i \leq (n-2)$, where the consecutive vertices of two copies of paths are $v_1, v_2, ..., v_n$ and $u_1, u_2, ..., u_n$ and the edges are $u_i v_i$.

**Definition 1.10**: The *double alternate triangular snake* $DA(T_n)$ is obtained from a path path $P_n$ with vertices $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to new vertices $v_i$ and $w_i$. In other words $DA(T_n)$ consists of two alternate triangular snakes that have a common path.

**Definition 1.11**: The *double alternate quadrilateral snake* $DA(QS_n)$ is obtained from a path $P_n$ with vertices $u_1, u_2, ..., u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to new vertices $v_i, v_i^{'}$ and $w_i, w_i^{'}$ respectively and adding the edges $v_i w_i$ and $v_i^{'} w_i^{'}$. In other words $DA(QS_n)$ consists of two alternate quadrilateral snakes that have a common path.

For any undefined term in graph theory we rely upon D B West[6].

## II. Main Results

**Theorem 2.1** The alternate triangular snake $A(T_n)$ is strongly multiplicative.

**Proof:** Let $G = A(T_n)$ be alternate triangular snake.
We have $V(G) = \{u_i, v_j \ / \ 1 \leq i \leq n \text{ and } 1 \leq j \leq \lfloor n/2 \rfloor\}$.
We note that

$$|V(G)| = \begin{cases} \frac{3n}{2}; & n \equiv 0 (mod\ 2) \\ \frac{3n-1}{2}; & n \equiv 1 (mod\ 2) \end{cases}$$

and

$$|E(G)| = \begin{cases} 2n-1; & n \equiv 0 (mod\ 2) \\ 2n-2; & n \equiv 1 (mod\ 2) \end{cases}$$

We define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., |V(G)|\}$ as follows:
$f(v_i) = 3i - 1;$ $\quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$
$f(u_{(2i-1)}) = 3i - 2;$ $\quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$
$f(u_{2i}) = 3i;$ $\quad 1 \leq i \leq \lceil \frac{n}{2} \rceil$

The labeling pattern defined above encompasses all possible layout's of vertices. In each possibility the graph $G$ under consideration admits strongly multiplicative labeling. That is, the alternate triangular snake $A(T_n)$ is strongly multiplicative.

**Illustration 2.2**: The alternate triangular snake $A(T_7)$ and its strongly multiplicative labeling is shown in the *Figure 1*.

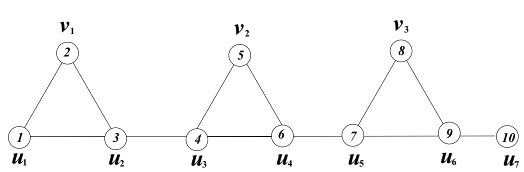


Figure 1: Strongly multiplicative labeling of the alternate triangular snake $A(T_7)$.

**Theorem 2.3** The double quadrilateral snake $D(QS_n)$ is strongly multiplicative.

**Proof:** Let $G = D(QS_n)$ be double quadrilateral snake.
We have $V(G) = \{u_i, v_j \ / \ 1 \leq i \leq n \text{ and } 1 \leq j \leq \lfloor n/2 \rfloor\}$.
We note that $|V(G)| = 5n - 4$ and $|E(G)| = 7(n-1)$.

We define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., |V(G)|\}$ as follows:
$f(v_i) = 5i - 4;$ $\quad 1 \leq i \leq n$
$f(u_i) = 5i - 3;$ $\quad 1 \leq i \leq (n-1)$
$f(u_i') = 5i - 2;$ $\quad 1 \leq i \leq (n-1)$
$f(w_i) = 5i;$ $\quad 1 \leq i \leq (n-1)$
$f(w_i') = 5i + 1;$ $\quad 1 \leq i \leq (n-1)$

The labeling pattern defined above encompasses all possible layout's of vertices. In each possibility the graph $G$ under consideration admits strongly multiplicative labeling. That is, the double quadrilateral snake $D(QS_n)$ is strongly multiplicative.

**Illustration 2.4**: The double quadrilateral snake $D(QS_6)$ and its strongly multiplicative labeling is shown in the *Figure 2*.

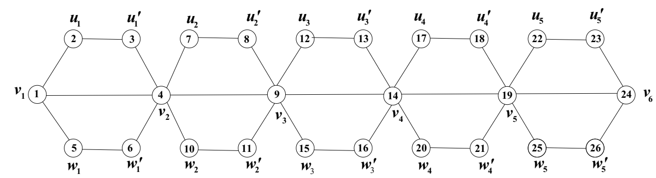


Figure 2: Strongly multiplicative labeling of the double quadrilateral snake $D(QS_6)$.

**Theorem 2.5** The $Z - P_n$ *graph* is strongly multiplicative.

**Proof:** Let $G = Z - P_n$ be the graph obtained from the pair of paths $P_n'$ and $P_n''$. Let $v_1', v_2', ..., v_n'$ and $v_1'', v_2'', ..., v_n''$ be the vertices of the path $P_n'$ and $P_n''$ respectively. To obtain $Z - P_n$ join $i^{th}$ vertex $v_i'$ of path $P_n'$ with $(i+1)^{th}$ vertex $v_{i+1}''$ of path $P_n''$ for all $1 \leq i \leq (n-1)$. We note that $|V(G)| = 2n$ and $|E(G)| = 3n - 3$.

We define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., 2n\}$ as follows:
$f(v_i') = 2i;$ $\quad 1 \leq i \leq n.$
$f(v_i'') = 2i - 1;$ $\quad 1 \leq i \leq n.$

The labeling pattern defined above encompasses all possible layout's of vertices. In each possibility the graph $G$ under consideration admits strongly multiplicative labeling. That is, the $Z - P_n$ *graph* is strongly multiplicative.

**Illustration 2.6**: The $Z - P_8$ graph and its strongly multiplicative labeling is shown in the *Figure 3*.

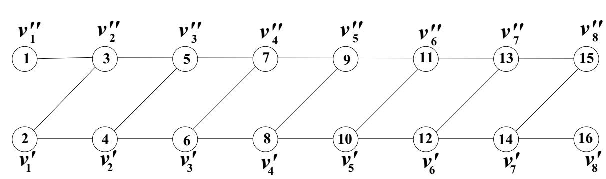


Figure 3: Strongly multiplicative labeling of the $Z - P_8$.

**Theorem 2.7** The *braid graph* $B_n$ is strongly multiplicative.

**Proof:** Let $G = B_n$ be the graph obtained from the pair of paths $P'_n$ and $P''_n$. Let $v'_1, v'_2, ..., v'_n$ and $v''_1, v''_2, ..., v''_n$ be the vertices of the paths $P'_n$ and $P''_n$ respectively. To find $B_n$ join $i^{th}$ vertex $v'_i$ of path $P'_n$ with $(i+1)^{th}$ vertex $v''_{i+1}$ of path $P''_n$ and $i^{th}$ vertex $v''_i$ of path $P''_n$ with $(i+2)^{th}$ vertex $v'_{i+2}$ of path $P'_n$, for all $1 \leq i \leq (n-2)$. We note that $|V(G)| = 2n$ and $|E(G)| = 4n - 5$.

We define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., 2n\}$ as follows:
$f(v'_i) = 2i - 1; \qquad 1 \leq i \leq n.$
$f(v''_i) = 2i; \qquad 1 \leq i \leq n.$

The labeling pattern defined above encompasses all possible layout's of vertices. In each possibility the graph $G$ under consideration admits strongly multiplicative labeling. That is, the braid graph $B_n$ is strongly multiplicative.

**Illustration 2.8**: The braid graph $B_6$ and its strongly multiplicative labeling is shown in the *Figure 4*.

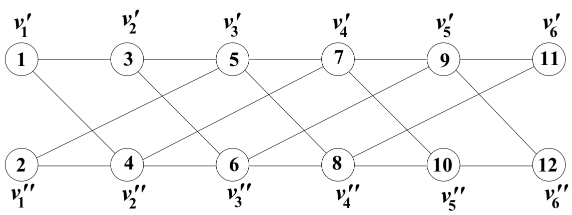


Figure 4: Strongly multiplicative labeling of the braid graph $B_6$.

**Theorem 2.9** The *triangular ladder* $(TL_n)$ is strongly multiplicative.

**Proof:** Let $G = TL_n$ be the triangular ladder obtained from the ladder $L_n = P_n \times P_2; (n \geq 2)$ by adding the edges $u_i v_{i+1}$ for all $1 \leq i \leq (n-2)$, where the consecutive vertices of two copies of paths are $v_1, v_2, ..., v_n$ and $u_1, u_2, ..., u_n$ and the edges are $u_i v_i$. We note that $|V(G)| = 2n$ and $|E(G)| = 4n - 3$.

We define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., 2n\}$ as follows:
$f(v_i) = 2i; \qquad 1 \leq i \leq n.$
$f(u_i) = 2i - 1; \qquad 1 \leq i \leq n.$

The labeling pattern defined above encompasses all possible layout's of vertices. In each possibility the graph $G$ under consideration admits strongly multiplicative labeling. That is, the triangular ladder $TL_n$ is strongly multiplicative.

**Illustration 2.10**: The $TL_8$ graph and its strongly multiplicative labeling is shown in the *Figure 5*.

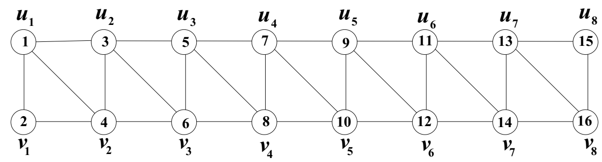


Figure 5: Strongly multiplicative labeling of the triangular ladder $TL_8$.

**Theorem 2.11** The alternate quadrilateral snake $A(QS_n)$ is strongly multiplicative.

**Proof:** Let $G = A(QS_n)$ be an alternate quadrilateral snake We have $V(G) = \{u_i, v_j, w_j \;/\; 1 \leq i \leq n \text{ and } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\}$. We note that

$$|V(G)| = \begin{cases} 2n; & n \equiv 0(mod\ 2) \\ 2n - 1; & n \equiv 1(mod\ 2) \end{cases}$$

and

$$|E(G)| = \begin{cases} \frac{5n-2}{2}; & n \equiv 0(mod\ 2) \\ \frac{5n-5}{2}; & n \equiv 1(mod\ 2) \end{cases}$$

We define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., |V(G)|\}$ as follows:
$f(u_{(2i-1)}) = 4i - 3; \qquad 1 \leq i \leq \lceil \frac{n}{2} \rceil$
$f(u_{2i}) = 4i; \qquad 1 \leq i \leq \lceil \frac{n}{2} \rceil$
$f(v_j) = 4j - 2; \qquad 1 \leq j \leq \lfloor \frac{n}{2} \rfloor$
$f(w_j) = 4j - 1; \qquad 1 \leq j \leq \lfloor \frac{n}{2} \rfloor$

The labeling pattern defined above encompasses all possible layout's of vertices. In each possibility the graph $G$ under consideration admits strongly multiplicative labeling. That is, the alternate quadrilateral snake $A(QS_n)$ is strongly multiplicative.

**Illustration 2.12**: The alternate quadrilateral snake $A(QS_7)$ and its strongly multiplicative labeling is shown in the *Figure 6*.

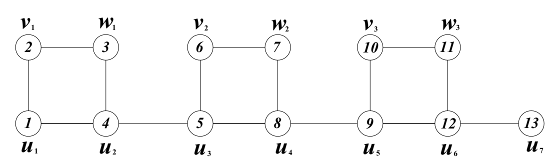


Figure 6: The strongly multiplicative labeling of the alternate quadrilateral snake $A(QS_7)$.

**Theorem 2.13** The double alternate triangular snake $DA(TS_n)$ is strongly multiplicative.

**Proof:** Let $G = DA(TS_n)$ be the double alternate triangular snake with the vertex set $V(G) = \{u_i, v_j, w_j \;/\; 1 \leq i \leq n,\ 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\}$. We note that

$$|V(G)| = \begin{cases} 2n; & n \equiv 0(mod\ 2) \\ 2n - 1; & n \equiv 1(mod\ 2) \end{cases}$$

and

$$|E(G)| = \begin{cases} 3n - 1; & n \equiv 0(mod\ 2) \\ 3n - 3; & n \equiv 1(mod\ 2) \end{cases}$$

We define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., |V(G)|\}$ as follows:
$f(u_{(2i-1)}) = 4i - 3; \qquad 1 \leq i \leq \lceil \frac{n}{2} \rceil$

$f(u_{2i}) = 4i - 1; \qquad 1 \le i \le \lceil \frac{n}{2} \rceil$
$f(v_j) = 4j - 2; \qquad 1 \le j \le \lfloor \frac{n}{2} \rfloor$
$f(w_j) = 4j; \qquad 1 \le j \le \lfloor \frac{n}{2} \rfloor$

The labeling pattern defined above encompasses all possible layout's of vertices. In each possibility the graph $G$ under consideration admits strongly multiplicative labeling. That is, the double alternate triangular snake $DA(TS_n)$ is strongly multiplicative.

**Illustration 2.14**: The double alternate triangular snake $DA(TS_8)$ and its strongly multiplicative labeling is shown in the *Figure 7*.

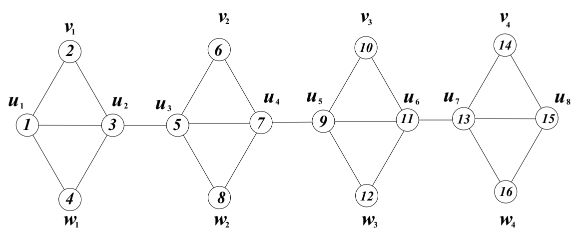

Figure 7: Strongly multiplicative labeling of the double alternate triangular snake $DA(TS_8)$.

**Theorem 2.15** The double alternate quadrilateral snake $DA(QS_n)$ is strongly multiplicative.

**Proof:** Let $G = DA(QS_n)$ be the double alternate quadrilateral snake with the vertex set $\{u_i, v_j, w_j, v_i^{'}, w_j^{'}: 1 \le i \le n, 1 \le j \le \lfloor \frac{n}{2} \rfloor\}$. We note that

$$|V(G)| = \begin{cases} 3n; & n \equiv 0(mod\ 2) \\ 3n-2; & n \equiv 1(mod\ 2) \end{cases}$$

and

$$|E(G)| = \begin{cases} 4n-1; & n \equiv 0(mod\ 2) \\ 4n-4; & n \equiv 1(mod\ 2) \end{cases}$$

We define vertex labeling $f : V(G) \to \{1, 2, ..., |V(G)|\}$ as follows:

$f(u_{(2i-1)}) = 6i - 4; \qquad 1 \le i \le \lceil \frac{n}{2} \rceil$
$f(u_{2i}) = 6i - 1; \qquad 1 \le i \le \lceil \frac{n}{2} \rceil$
$f(v_j) = 6j - 5; \qquad 1 \le j \le \lfloor \frac{n}{2} \rfloor$
$f(w_j) = 6j - 2; \qquad 1 \le j \le \lfloor \frac{n}{2} \rfloor$
$f(v_j^{'}) = 6j - 3; \qquad 1 \le j \le \lfloor \frac{n}{2} \rfloor$
$f(w_j^{'}) = 6j; \qquad 1 \le j \le \lfloor \frac{n}{2} \rfloor$

The labeling pattern defined above encompasses all possible layout's of vertices. In each possibility the graph $G$ under consideration admits strongly multiplicative labeling. That is, the double alternate quadrilateral snake $DA(QS_n)$ is strongly multiplicative.

**Illustration 2.16**: The double alternate quadrilateral snake $DA(QS_7)$ and its strongly multiplicative labeling is shown in the *Figure 8*.

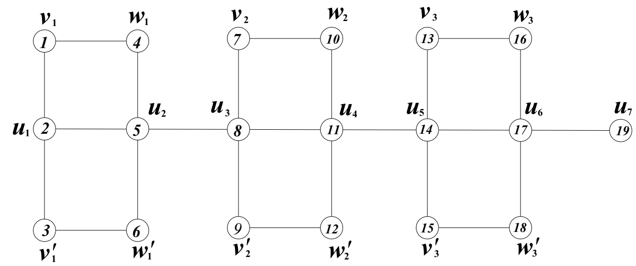

Figure 8: The strongly multiplicative labeling of the double alternate quadrilateral snake $DA(QS_7)$.

## III. Concluding Remark

To explore some new strongly multiplicative graphs is an open problem.

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