Convergence of sequences of functions in D_{L^p} and $D_{L^p,\eta}, 1 \leq p < \infty$ through their analytic representations

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Abstract. *In this paper we prove that the uniform convergence of a sequence* $\hat{f}_n$ *of analytic representations for functions* f_n *in* D_{L^p} *on suitable sets implies convergence of the sequence* f_n *in* D_{L^p}*. Additionally, the boundary function* $\hat{f}$ *of the sequence of the analytic representations* $\hat{f}_n$ *is analytic representation for the boundary function* f *of the sequence* f_n *. Similar results are given for the new distribution space* $D'_{L^p,\eta}$ *which is the weighted version of the space* D'_{L^p} *and generalized it.*

Keywords- D_{L^p} *and* $D_{L^p,\eta}$ *spaces, distribution, analytic representation*

I. INTRODUCTION

We use the standard notation from the Schwartz distribution theory.

The boundary value representation has been studied for a long time and from different points of view.

One of the first result is that if $f \in L^1$, then the function

$$\hat{f}(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} > \text{, for } \operatorname{Im} z \neq 0$$

is the Cauchy representation of f i.e.

$$\lim_{y \to 0^+} < \hat{f}(x+iy) - \hat{f}(x-iy), \varphi(x) > = < f, \varphi >,$$

for every $\varphi \in D$.

D_{L^p}, $1 \leq p < \infty$ denotes the space of all infinitely differentiable functions φ for which $\varphi^{(\beta)} \in L^p$ for each n-tuple β of nonnegative integers.

$B = D_{L^\infty}$ is the space of all infinitely differentiable functions which are bounded on $\mathbb{R}^n$.

$\dot{B}$ is the subspace of B that consists of all functions $\varphi \in B$ which vanish at infinity together with each of their derivatives.

The topology of D_{L^p} is given in terms of the norms

$$\|\varphi\|_{m,\,p} = \left(\int_{\mathbb{R}^n} \left|\varphi^{(\beta)}(x)\right|^p dx \right)^{1/p},$$

$$|\beta| \le m, \quad m = 0,1,2,3,\ldots.$$

A sequence of functions (φ_λ) of D_{L^p} converges to a function φ in the topology of D_{L^p}, $1 \le p \le \infty$ as $\lambda \to \lambda_0$ if each $\varphi_\lambda \in D_{L^p}$, $\varphi \in D_{L^p}$, and

$$\lim_{\lambda\to\lambda_0} \left\|\varphi_\lambda{}^{(\beta)} - \varphi^{(\beta)}\right\|_{L^p} = \lim_{\lambda\to\lambda_0} \left(\int_{\mathbb{R}^n} \left|\varphi_\lambda{}^{(\beta)}(x) - \varphi^{(\beta)}(x)\right|^p dx \right)^{1/p} = 0$$

, for every β.

A sequence of functions (φ_λ) converges to the function φ in $\dot{B}$ as $\lambda \to \lambda_0$ if each $\varphi_\lambda \in \dot{B}$, $\varphi \in \dot{B}$, and

$$\lim_{\lambda\to\lambda_0} \left\|\varphi_\lambda{}^{(\beta)} - \varphi^{(\beta)}\right\|_{L^\infty} = 0.$$

D is dense in D_{L^p}, $1 \le p < \infty$ and in $\dot{B}$, but not in $B = D_{L^\infty}$. Also D_{L^p} is dense in L^p. If $\varphi \in D_{L^p}$ for $1 \le p < \infty$ then φ is bounded and converges to 0 at infinity with the same being true for all derivatives of φ. We have $D \subset D_{L^p} \subset D_{L^q} \subset \dot{B}$ if $1 \le p \le q < \infty$.

D'_{L^p}, $1 < p \le \infty$ is the space of all continuous linear functionals on D_{L^q} where $\frac{1}{p} + \frac{1}{q} = 1$. D'_{L^1} is the space of all continuous linear functionals on $\dot{B}$.

The space D'_{L^p} is a subspace of D'. Indeed, if $\Lambda \in D'_{L^p}$ then, since D is subspace of D_{L^q}, we have that $< \Lambda, \varphi >$ is well defined for all $\varphi \in D$. Clearly, Λ is linear on D. Since convergence in D implies convergence in D_{L^q} then $< \Lambda, \varphi_\lambda > \to 0$ when $\varphi_\lambda \to 0$ in D as $\lambda \to \lambda_0$. Thus $\Lambda \in D'$. The uniqueness of the linear functional Λ follows from the fact that D is dense in D_{L^q}. Similar reasoning yields that $D'_{L^1} \subset D'$ since $D \subset \dot{B}$.

The following theorem gives the structures of D'_{L^p}.

Structure Theorem. A distribution Λ belongs to D'_{L^p}, $1 \le p \le \infty$ if and only if Λ is a finite sum of distributional derivatives of functions in L^p, i.e. there is an integer $m \ge 0$ depending only on Λ such that

$$\Lambda = \sum_{|\beta|\le m} f_\beta{}^{\beta}, \text{ where } f_\beta \in L^p \text{ for each } \beta,\ |\beta| \le m.$$

II. MAIN RESULTS

Theorem 1. Let f_n be a bounded sequence in D_{L^p} for $1 \le p < \infty$ and let $(\hat{f}_n(z))$ be a sequence of analytic representation for every term of f_n, respectively.

Then the following conditions hold:

a) If the sequence $(\hat{f}_n(z))$ of analytic representations converges uniformly to the function $\hat{f}(z)$ on the sets of the form $\mathbb{R} \times (0,T)$ or $\mathbb{R} \times (-T,0)$, for $T > 0$ as $n \to \infty$, then there exists a function $f \in D_{L^p}$ such that $f_n \to f$ as $n \to \infty$ in D_{L^p}.

b) The boundary function $\hat{f}(z)$ is analytic representation for the boundary function $f \in D_{L^p}$.

Proof. We will first show that if $f \in D'_{L^p}$ then f has Cauchy representation $\hat{f}(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} >$, for $\operatorname{Im} z \neq 0$, i.e. we will prove that

$$\lim_{y\to 0^+} < \hat{f}(x+iy) - \hat{f}(x-iy), \varphi(x) > = < f, \varphi >,$$

where $\varphi \in D \subset D_{L^q}, \frac{1}{p} + \frac{1}{q} = 1$.

Note that $\hat{f}^{k} \in L^p$ for any positive integer k.

By the structure theorem $f = \sum_{k=1}^{m} f_k^{(k)}$, where $f_i \in L^p$.

Let $\varphi \in D$ be arbitrarily chosen. Since φ has compact support, there exists some $a > 0$ such that $\operatorname{supp} \varphi \subset [-a, a]$.

We have

$$I(y) = \int_{\mathbb{R}} (\hat{f}(x+iy) - \hat{f}(x-iy))\varphi(x)dx =$$
$$\int_{\mathbb{R}} \left(\frac{1}{2\pi i} < f(t), \frac{1}{t-z} > - \frac{1}{2\pi i} < f(t), \frac{1}{t-\bar{z}} >\right)\varphi(x)dx$$
$$= \int_{\mathbb{R}} \frac{1}{2\pi i} \sum_{k=1}^{m} \left(\int_{\mathbb{R}} \frac{f_k^{(k)}(t)dt}{t-z} - \int_{\mathbb{R}} \frac{f_k^{(k)}(t)dt}{t-\bar{z}}\right)\varphi(x)dx.$$

Integrating by parts m-times in the last two integrals in the brackets, we get

$$I(y) = \int_{\mathbb{R}} \frac{1}{2\pi i} \sum_{k=1}^{m} k!\left(\int_{\mathbb{R}} \frac{f_k(t)dt}{(t-z)^{k+1}} - \int_{\mathbb{R}} \frac{f_k(t)dt}{(t-\bar{z})^{k+1}}\right)\varphi(x)dx.$$

Now we apply Fubini's theorem and get

$$I(y) = \int_{\mathbb{R}} \sum_{k=1}^{m} k! f_k(t)dt \frac{1}{2\pi i}\left(\int_{\mathbb{R}} \frac{\varphi(x)dx}{(t-z)^{k+1}} - \int_{\mathbb{R}} \frac{\varphi(x)dx}{(t-\bar{z})^{k+1}}\right).$$

Again, we integrate by parts m-times in the last two integrals and get that

$$I(y) =$$
$$= \int_{\mathbb{R}} \sum_{k=1}^{m} k! \frac{(-1)^k}{k!} f_k(t)dt \frac{1}{2\pi i}\left(\int_{\mathbb{R}} \frac{\varphi^{(k)}(x)dx}{t-z} - \int_{\mathbb{R}} \frac{\varphi^{(k)}(x)dx}{t-\bar{z}}\right)$$
$$= \int_{\mathbb{R}} \sum_{k=1}^{m} (-1)^k f_k(t)dt \frac{y}{\pi} \int_{\mathbb{R}} \frac{\phi(x)dx}{|t-z|^2},$$

where $\phi = \varphi^{(k)} \in D.$

From Lemma 4 in [1] we have that $\frac{y}{\pi} \int_{\mathbb{R}} \frac{\phi(x)dx}{|t-z|^2} = \phi^*(t+iy)$ and that $\phi^*(t+iy) \to \phi(t)$ uniformly on compact subset of the complex plane as $y \to 0^+$.

Therefore

$$I(y) = \int_{\mathbb{R}} \sum_{k=1}^{m} (-1)^k f_k(t)\phi^*(t+iy)dt.$$

By the Lebesgue dominated convergence theorem, we get

$$\lim_{y\to 0^+}\int_{\mathbb{R}}(\hat{f}(x+iy)-\hat{f}(x-iy))\varphi(x)dx =$$
$$\lim_{y\to 0^+} I(y) = \lim_{y\to 0^+}\int_{\mathbb{R}}\sum_{k=1}^{m}(-1)^k f_k(t)\phi^*(t+iy)dt =$$
$$\int_{\mathbb{R}}\sum_{k=1}^{m}(-1)^k f_k(t)\phi(t)dt =$$
$$\int_{\mathbb{R}}\sum_{k=1}^{m}(-1)^k f_k(t)\varphi^{(k)}(t)dt =$$
$$\int_{\mathbb{R}}\sum_{k=1}^{m}(-1)^k(-1)^k f_k^{(k)}(t)\varphi(t)dt =$$
$$\int_{\mathbb{R}} f(t)\varphi(t)dt = < f,\varphi > .$$

Now we will show a). First we prove that the sequence (f_n) is a Cauchy sequence.

Let $\varphi \in D$ and let $n,m \geq n_0$ for some $n_0 \in \mathbb{N}$. We consider the difference

$$\begin{aligned}
&< f_n,\varphi > - < f_m,\varphi > \\
&= < f_n,\varphi > - \int_{\mathbb{R}}(\hat{f}_n(z)-\hat{f}_n(\bar{z}))\varphi(x)dx \\
&+ \int_{\mathbb{R}}(\hat{f}_n(z)-\hat{f}_n(\bar{z}))\varphi(x)dx \\
&- \int_{\mathbb{R}}(\hat{f}_m(z)-\hat{f}_m(\bar{z}))\varphi(x)dx \\
&+ \int_{\mathbb{R}}(\hat{f}_m(z)-\hat{f}_m(\bar{z}))\varphi(x)dx \\
&- < f_m,\varphi > = I_1 + I_2 + I_3,
\end{aligned}$$

where

$$I_1 = < f_n,\varphi > - \int_{\mathbb{R}}(\hat{f}_n(z)-\hat{f}_n(\bar{z}))\varphi(x)dx,$$
$$I_2 = \int_{\mathbb{R}}(\hat{f}_n(z)-\hat{f}_n(\bar{z}))\varphi(x)dx - \int_{\mathbb{R}}(\hat{f}_m(z)-\hat{f}_m(\bar{z}))\varphi(x)dx,$$
$$I_3 = \int_{\mathbb{R}}(\hat{f}_m(z)-\hat{f}_m(\bar{z}))\varphi(x)dx - < f_m,\varphi > .$$

Since $\hat{f}_n(z)$ is analytic representation for (f_n), respectfully, for the integrals I_1 and I_3 we have that, for arbitrary $\varepsilon > 0$ there exist y_0 such that for $y_0 < y$, $|I_1| < \frac{\varepsilon}{3}$ and $|I_3| < \frac{\varepsilon}{3}$.

Now we consider the second integral and get that

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}}(\hat{f}_n(z)-\hat{f}_m(z))\varphi(x)dx + \\
&\int_{\mathbb{R}}(\hat{f}_m(\bar{z})-\hat{f}_n(\bar{z}))\varphi(x)dx \\
&= I_2' + I_2'' .
\end{aligned}$$

Since the sequence $(\hat{f}_n(z))$ is convergent, it implies that it is a Cauchy sequence. Therefore, for an arbitrary $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for $n,m \geq n_0$, $|I_2'| < \frac{\varepsilon}{6}$ and $|I_2''| < \frac{\varepsilon}{6}$.

From the above estimates, we get that the sequence (f_n) is a Cauchy sequence in D'. Since D' is complete, there exists $\Lambda \in D'$ such that

$$\lim_{n\to\infty} f_n = \Lambda \quad \text{in } D', \text{ i.e.}$$

$$\lim_{n\to\infty} < f_n, \varphi > = < \Lambda, \varphi > \text{ for } \varphi \in D.$$

Since the space D is dense in D_{L^q}, the linear functional Λ can be extend to continuous linear functional on D_{L^q} by the Banach-Steinhaus Theorem.

Thus, there exists $f \in D_{L^p}$ such that

$$f_n \to f \quad \text{in } D_{L^p}.$$

The proof that the function $\hat{f}(z)$ is analytic representation for f is similar to the one in the first part, so we will omitted it.

In the sequel, we will consider the spaces $D_{L^p,m}$ and $D'_{L^p,m}$.

First we give some facts about these spaces. With D_E we denote the space of all infinitely differentiable functions φ such that

$$\|\varphi\|_{E,N} = \max \ \|\varphi^{(\alpha)}\|_E : |\alpha| \le N \ ,$$

where E is a reflexive space. Note that the spaces L^p for $1 < p < \infty$ are reflexive.

Let m be a polynomially bounded measurable weighted function from $\mathbb{R}^n$ into $(0,\infty)$ that fulfills the requirement

$$m(x+h) \le M(1+|h|)^{\tau} m(x), \text{ for some } M, \tau > 0.$$

With L^p_η we denote the space of all functions g such that $\|g\|_{p,m} = \|gm\|_p < \infty.$

If $E = L^p_\eta$, then the corresponding dual space is denoted by $E'_* = E' = (L^p_\eta)' = L^q_\eta$ for $1 < p, q < \infty$.

With $D_{L^p,\eta}$ we denote the space of all functions g such that $(g\,\eta)^{(\alpha)} \in L^p$ or

$$\|(g\,\eta)^{(\alpha)}\|_{L^p} = (\int_{\mathbb{R}} |(g\,\eta)^{(\alpha)}|^p d\mu)^{1/p} < \infty, \ 1 \le p < \infty.$$

The dual space of $D_{L^p,\eta}$ is the space $D'_{L^p,\eta}$ of all continuous linear functionals on $D_{L^q,\eta^{-1}}$ or

$$D'_{L^p,\eta} = (D_{L^q,\eta^{-1}})', \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

Structure theorem for a distribution of $D'_{L^p,\eta}$.

$\Lambda \in D'_{L^p,\eta}$ if and only if $\Lambda = \sum_{|\beta| \le m} (\eta(x) f_\beta(x))^{\beta}$, where $\eta(x) f_\beta(x) \in L^p, |\beta| \le m$ and $1 < p < \infty$.

Theorem 2. Let (f_n) be a bounded sequence of functions of $D_{L^p,\eta}$ for $1 \le p < \infty$ and let $(\hat{f}_n(z))$ be a sequence of analytic representation for every (f_n), respectively. Then the following conditions holds:

a) If the sequence $(\hat{f}_n(z))$ of analytic representations converges uniformly on the set of the form $\mathbb{R} \times (0,T)$ or $\mathbb{R} \times (-T,0)$ then there exists a function $f \in D_{L^p,\eta}$ such that $f_n \to f$ in $D_{L^p,\eta}$.

b) The boundary function $\hat{f}(z)$ is analytic representation for the boundary function f.

Proof. First we will show that if $f \in D'_{L^p,\eta}$ then f has Cauchy representation.

By the structure theorem, we have $f = \sum_{|\alpha| \le m} (\eta(x) f_\alpha(x))^{\alpha}$

, where $\eta(x) f_\alpha(x) \in L^p$.

Let $\hat{f}(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} >$, for $\operatorname{Im} z \neq 0$.

Then

$$\int_{\mathbb{R}} (\hat{f}(x+iy) - \hat{f}(x-iy))\varphi(x)dx = \int_{\mathbb{R}} \frac{1}{2\pi i}\left(\int_{\mathbb{R}} \frac{f(t)dt}{t-z} - \int_{\mathbb{R}} \frac{f(t)dt}{t-\bar{z}}\right)\varphi(x)dx = I_1 + I_2 .$$

Let us consider the first integral

$$I_1 = \int_{\mathbb{R}} \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)dt}{t-z}\varphi(x)dx = \int_{\mathbb{R}} \frac{1}{2\pi i}\int_{\mathbb{R}} \sum_{k=1}^{m} \frac{(f_k(t)\eta(t))^{(k)}dt}{t-z}\varphi(x)dx = \int_{\mathbb{R}} \sum_{k=1}^{m} \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{(f_k(t)\eta(t))^{(k)}dt}{t-z}\varphi(x)dx.$$

With partial integration m-times we get that

$$\int_{\mathbb{R}} \sum_{k=1}^{m} k!\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f_k(t)\eta(t)dt}{(t-z)^{k+1}}\varphi(x)dx$$

Since $f_k(t)\eta(t) \in L^p$ and $\frac{1}{(t-z)^{k+1}} \in L^q$, the last integral might be written as follows

$$\int_{\mathbb{R}} \sum_{k=1}^{m} k!\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f_k(t)\eta(t)dt}{(t-z)^{k+1}}\varphi(x)dx = \sum_{k=1}^{m} k!\frac{1}{2\pi i}\int_{\mathbb{R}} f_k(t)\eta(t)dt\int_{\mathbb{R}} \frac{\varphi(x)dx}{(t-z)^{k+1}}.$$

Another m-times partial integration in the last integral gives

$$I_1 = \sum_{k=1}^{m} k!\frac{1}{2\pi i}\int_{\mathbb{R}} f_k(t)\eta(t)dt\int_{\mathbb{R}} \frac{\varphi(x)dx}{(t-z)^{k+1}} = \sum_{k=1}^{m} k!\frac{-1^{k}}{k!}\frac{1}{2\pi i}\int_{\mathbb{R}} f_k(t)\eta(t)dt\int_{\mathbb{R}} \frac{\varphi^{(k)}(x)}{t-z}dx, = \sum_{k=1}^{m} (-1)^k \frac{1}{2\pi i}\int_{\mathbb{R}} f_k(t)\eta(t)dt\int_{\mathbb{R}} \frac{\phi(x)}{t-z}dx,$$

where $\phi(x) = \varphi^{(k)}(x)$.

Analogously, for the second integral we have that

$$I_2 = \sum_{k=1}^{m} (-1)^k \frac{1}{2\pi i}\int_{\mathbb{R}} f_k(t)\eta(t)dt\int_{\mathbb{R}} \frac{\phi(x)}{t-\bar{z}}dx .$$

Thus, $I_1 + I_2 = \sum_{k=1}^{m} (-1)^k \int_{\mathbb{R}} f_k(t)\eta(t)dt \frac{y}{\pi}\int_{\mathbb{R}} \frac{\phi(x)dx}{|t-z|^2}$

and then we use Lemma 4 in $[1]$ i.e. that $\frac{y}{\pi}\int_{\mathbb{R}} \frac{\phi(x)dx}{|t-z|^2} = \phi^*(t+iy)$ and converges uniformly to $\phi(t)$ on compact subset of the plane.

Finally with the use of the Lebesgue dominated convergence theorem, we get that

$$\begin{aligned}
&\lim_{y\to 0^+}\int_{\mathbb{R}}(\hat{f}(x+iy)-\hat{f}(x-iy))\varphi(x)dx \\
&=\lim_{y\to 0^+}\sum_{k=1}^{m}(-1)^k\int_{\mathbb{R}} f_k(t)\eta(t)\phi^*(t+iy)dt \\
&=\sum_{k=1}^{m}(-1)^k\int_{\mathbb{R}} f_k(t)\eta(t)\phi(t)dt \\
&=\sum_{k=1}^{m}(-1)^k\int_{\mathbb{R}} f_k(t)\eta(t)\varphi^{(k)}(t)dt \\
&=\sum_{k=1}^{m}(-1)^k(-1)^k\int_{\mathbb{R}} (f_k(t)\eta(t))^{(k)}\varphi(t)dt \\
&=\int_{\mathbb{R}}\sum_{k=1}^{m}(f_k(t)\eta(t))^{(k)}\varphi(t)dt =< f,\varphi > .
\end{aligned}$$

The proof of the rest of theorem is similar to the one of theorem 1, so we will omitted it.

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