Stability of $L_{4,5}$ in the R3BP Under the Combined Effects of Stokes Drag and Finite Straight Segment

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Abstract: *We have investigated the existence and stability of non-collinear libration points in the restricted three body problem (R3BP) with dissipative force Stokes drag. The bigger primary is taken as spherical and smaller one as a finite straight segment. There exist five libration points, out of which two are non-collinear with the primaries and three are almost collinear with the primaries. The location of libration points for different values of parameters* μ, α *and* l *are evaluated numerically and shown graphically. The linear stability of non-collinear libration points is also discussed.*

Key Words: *Restricted three body problem, Libration points, Straight segment, Linear stability, Dissipative force, Stokes drag.*

I. Introduction

In the restricted three body problem the mass of the one of the bodies is considered to be negligibly small, so that the two other bodies (called primaries) can be described by the two body problem and the body with negligible mass moves in the given field of two bodies with given Keplerian motion such that it does not influence the motion of the primaries but is influenced by them. The problem possesses five libration points out of which three are collinear and two are non-collinear. The collinear libration points are unstable for all values of mass parameter μ, while non-collinear libration points are stable for the mass parameter $\mu \leq 0.0385\ldots$, Szebehely [14]. It is further assumed that the motion of all the three bodies is co-planar (planar restricted three body problem) and /or that the orbit of the two massive bodies is circular (circular restricted three body problem). The five Lagrange solutions denoted by L_1, L_2, L_3, L_4 and L_5 exist in the R3BP. In the circular restricted three body problem the above five configurations remains constant in the reference system co-rotating with the two massive bodies and are known as libration points (equilibrium points or stationary solutions). There are numerous applications of the R3BP in cosmology and stellar dynamics. In fact, the equations of motion of so many dynamical systems are similar to those of the R3BP. There are many examples of R3BP in space dynamics. One of them is the classical three body problem, namely, the “Sun-Earth-Moon” combination and describing the motion of the Moon. Another example, motion of a Trojan asteroid attracted by the Sun and Jupiter, which has a similarity with the R3BP. A lot of work has been done on R3BP till today. The location and stability of equilibrium points in the planar circular restricted three body problem when the third body is acted on by a variety of drag forces has been investigated by Muray [10]. He found that L_4 and L_5 are asymptotically stable. Liou et al. [8] have examined the effects of radiation pressure, Poynting-Robertson drag and Solar wind drag on dust grains trapped in mean motion resonances with the Sun and Jupiter in the R3BP. They concluded that all dust grain orbits are unstable in time when P-R and solar wind drag are included in the Sun-Jupiter-dust system. Shu et al. [13] determined a criterion of linear stability of libration points in the perturbed restricted three body problem and investigated the effect of drag in the linear stability of triangular libration points. Mishra et al. [9] examined the stability of triangular equilibrium points in photogravitational elliptic restricted three body problem with Poynting-Robertson drag by considering the smaller primary as an oblate spheroid and bigger primary as radiating. They concluded that the triangular equilibrium points remain unstable. Jain and Aggarwal [3] determined the existence and stability of libration points in the restricted problem under the effect of Poynting Robertson Light Drag and conclude that both the non-collinear libration points are unstable. Jain and Aggarwal [4] have performed an analysis in the R3BP with Stokes drag effect by taking both primaries as the point masses and found that non-collinear stationary solutions are linearly unstable. Aggarwal et al. [1] discussed the non-linear stability of the triangular libration point L_4 of the R3BP under the presence of the third and fourth order resonances by taking bigger primary as an oblate body and the smaller one as a triaxial body and both are source of radiation. They found that L_4 is always unstable.

Furthermore, the motion of a particle under the gravitational field of a massive straight segment has been studied by Riaguas et al. [11]. This model is used as an approximation to the gravitational field of irregular shaped bodies such as asteroids, nuclei and planets moons. Riaguas et al. [12] have studied the non-linear stability of the equilibria in the gravity field of the finite straight segment and determined the orbital stability of the equilibria, for all values of the parameter k. Jain et al. [6] investigated the stationary solutions in the R3BP when the smaller primary is a finite straight segment. They determined that the collinear stationary solutions are unstable for all values of mass parameter μ, whereas the non-collinear stationary solutions are stable for a critical value $\mu_c=\mu_0-0.0073562l^2$. The equilibrium solutions and linear stability of m_3 and m_4 considering one of the primaries as an oblate spheroid have been examined by Aggarwal and Kaur [2]. They concluded that there are no non-collinear equilibrium solutions of the system. By considering smaller primary as an oblate spheroid, the existence and stability of the non-collinear libration points with Stokes drag effect have been examined by Jain and Aggarwal [5]. They found that the non-collinear libration points are unstable. Kumar et al. [7] studied existence and stability of libration points in the R3BP under the combined effects of finite straight segment and oblateness. They found that, there exist five libration points, out of which three are collinear and two are non-collinear with the primaries. The collinear libration points are unstable for all values of mass parameter μ, and the non-collinear libration points are stable if $\mu<\mu_c$, where $\mu_c=0.038521-0.007356\, l^2-0.285002\, A$.

In the present paper, we discuss the linear stability of non-collinear libration points L_4 and L_5 in the R3BP by considering the combined effects of Stokes drag and finite straight segment. There are six sections in this paper. In Section II, the equations of motion of the infinitesimal mass m_3 are determined. Section III comprises location of the libration points. In Section IV, we have investigated the stability of the non-collinear libration points. In Section V, the conclusion of this problem is drawn. And in last section, appendix is given.

II. Equations of Motion

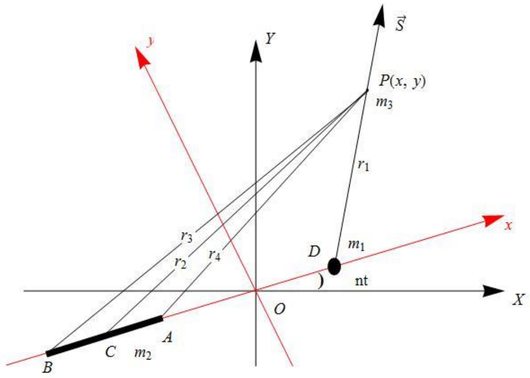

Fig.1. The configuration of R3BP with Stokes drag when m_1 is a point mass and m_2 is a finite straight segment

Let m_1 be a point mass and m_2 be the mass of a finite straight segment (called primaries), are moving with angular velocity n (say) in circular orbits about their common centre of mass O. There is an infinitesimal mass m_3 which is moving in the plane of motion of m_1 and m_2 ($m_1 \geq m_2$). $O(XYZ)$ and $O(xyz)$ are inertial and synodic coordinate system respectively. The line joining m_1 and m_2 is taken as X-axis and O their centre of mass as origin and the line passing through O and perpendicular to OX and lying in the plane of motion of m_1 and m_2 is taken as Y-axis. $O(xyz)$ initially coincident with the inertial coordinate system $O(XYZ)$. The synodic axes are rotating with angular velocity n (say) about Z-axis (the z-axis is coincident with Z-axis) (Fig. 1).

The equations of motion of m_3 in the dimensionless synodic coordinate system are

$$\begin{aligned} &\ddot{x}-2n\dot{y}=\Omega_x-k(\dot{x}-y+\alpha S'_y), \\ &\ddot{y}+2n\dot{x}=\Omega_y-k(\dot{y}+x-\alpha S'_x), \end{aligned} \tag{1}$$

where

$$\Omega=\frac{1}{2}n^2\left(x^2+y^2\right)+\frac{(1-\mu)}{r_1}+\frac{\mu}{2l}\log\frac{(r_3+r_4+2l)}{(r_3+r_4-2l)},$$

$$\Omega_x=n^2x-\frac{(1-\mu)(x-\mu)}{r_1^3}-\frac{2\mu(x-\mu+1)}{r_3r_4(r_3+r_4)},$$

$$\Omega_y=n^2y-\frac{(1-\mu)y}{r_1^3}-\frac{2\mu y(r_3+r_4)}{r_3r_4\left\{(r_3+r_4)^2-4l^2\right\}},$$

$$\mu=\frac{m_2}{m_1+m_2}\leq\frac{1}{2},\Rightarrow m_2=\mu, m_1=(1-\mu),$$

$$r_1^2=(x-\mu)^2+y^2, r_2^2=(x-\mu+1)^2+y^2,$$

$$r_3^2=\{x-(\mu-1-l)\}^2+y^2,$$

$$r_4^2=\{x-(\mu-1+l)\}^2+y^2,$$

$n^2=(1+l^2), l<<1,$
$2l=$dimensionless length of the straight segment AB,
$\vec{S}=$Stokes drag Force acting on m_3 due to m_1 along m_1m_3.
The components of Stokes drag along the synodic axes are
$S_x=k(\dot{x}-y+\alpha S'_y)$ and $S_y=k(\dot{y}+x-\alpha S'_x)$,
where $k\in(0,1)$ is the dissipative constant depending on several physical parameters like the viscosity of the gas, the radius and mass of the particle. $S'=S'(r)=r^{-3/2}$, is the Keplerian angular velocity at distance $r=\sqrt{x^2+y^2}$ from the origin O and $\alpha\in(0,1)$ is the ratio between the gas and Keplerian velocities. $\vec{r}=\overline{OP}=x\hat{i}+y\hat{j}, \vec{\omega}=nK=$angular velocity of the axes $O(x, y)$=constant. The Stokes drag effect is of the order of $k=10^{-5}, \alpha=0.05$ (generally $k\in(0,1)$ and $\alpha\in(0,1)$ as stated above).

III. Libration points

The libration points are the solution of the equations

$$n^2x-\frac{(1-\mu)(x-\mu)}{r_1^3}-\frac{2\mu\{x-(\mu-1)\}}{r_3r_4(r_3+r_4)}+k\left[y+\frac{3}{2}\alpha\left(x^2+y^2\right)^{-7/4}y\right]=0, \qquad (2)$$

$$n^2y-\frac{(1-\mu)y}{r_1^3}-\frac{2\mu y(r_3+r_4)}{r_3r_4\{(r_3+r_4)^2-4l^2\}}-k\left[x+\frac{3}{2}\alpha\left(x^2+y^2\right)^{-7/4}x\right]=0. \qquad (3)$$

If we take $k=0$, then it is conformity with Jain et al. [6]. The non-collinear libration points when smaller primary is a finite straight segment are given by Jain et al. [6]

$$x_0=\left(\mu-\frac{1}{2}\right)+\frac{(\mu+3)}{24(\mu-1)}l^2,$$
$$y_0=\pm\left(\frac{\sqrt{3}}{2}+\frac{(19-23\mu)}{24\sqrt{3}(\mu-1)}l^2\right). \qquad (4)$$

Now, we assume that the solution of the Eqs. (2) and (3) when $k\neq0$ and $y\neq0$

$$x'=x_0+\alpha_1,\ y'=y_0+\alpha_2,\ \alpha_1,\alpha_2<<1. \qquad (5)$$

Putting these values of x', y' in Eqs. (2) and (3) and applying Taylor's series, we get

$$\alpha_1\left[1+(1-\mu)a_1+\mu\left\{\frac{2(x_0-\mu+1)^2}{a_2r'_3r'_4}a_3-\frac{2}{a_2r'_3r'_4}\right\}\right]+\alpha_2\left[\frac{3(x_0-\mu)(1-\mu)y_0}{r'^{5/2}_1}+\mu\left\{\frac{2(x_0-\mu+1)y_0}{a_2r'_3r'_4}a_3\right\}\right]+k\left[y_0+\frac{3}{2}\alpha\left(x_0^2+y_0^2\right)^{-7/4}y_0\right]=0, \qquad (6)$$

and

$$\alpha_1\left[(1-\mu)\left\{\frac{3y_0(x_0-\mu)}{r'^{5/2}_1}\right\}+\mu\left\{\frac{-2y_0a_4}{r'_3r'_4a_5}+\frac{4y_0a_2^2}{r'_3r'_4a_5}a_4+\frac{2y_0\{x_0-(\mu-1+l)\}a_2}{r'_3(r'_4)^{3/2}a_5}+\frac{2y_0\{x_0-(\mu-1-l)\}a_2}{(r'_3)^{3/2}r'_4a_5}\right\}\right]$$
$$+\alpha_2\left[1+(1-\mu)a_6+\mu\left\{\frac{2y_0^2a_2}{(r'_3)^{3/2}r'_4a_5}+\frac{2y_0^2a_2}{r'_3(r'_4)^{3/2}a_5}-\frac{2y_0a_7}{r'_3r'_4a_5}-\frac{2a_2}{r'_3r'_4a_5}+\frac{4y_0a_2^2a_7}{r'_3r'_4a_5}\right\}\right]-k\left[x_0+\frac{3}{2}\alpha\left(x_0^2+y_0^2\right)^{-7/4}x_0\right]=0, \qquad (7)$$

where

$$r'_1=\sqrt{(x_0-\mu)^2+y_0^2}, r'_3=\sqrt{\{x_0-(\mu-1-l)\}^2+y_0^2},$$
$$r'_4=\sqrt{\{x_0-(\mu-1+l)\}^2+y_0^2}.$$
$$a_1=\frac{3(x_0-\mu)^2}{r'^{5/2}_1}-\frac{1}{r'^{3/2}_1},\ a_2=r'_3+r'_4,$$
$$a_3=\frac{1}{r'_3r'_4}+\frac{1}{r'^{1/2}_4}+\frac{1}{r'^{1/2}_1},$$
$$a_4=\frac{x_0-(\mu-1+l)}{r'_4}+\frac{x_0-(\mu-1-l)}{r'_3},$$
$$a_5=a_2^2-4l^2,\ a_6=\frac{3y_0^2}{r'^{5/2}_1}-\frac{1}{r'^{3/2}_1},$$
$$a_7=\frac{y_0}{r'_4}+\frac{y_0}{r'3}.$$

Substituting the values of x_0 and y_0 in Eqs. (6) and (7), we get

$$\alpha_1=\frac{1}{48\sqrt{3}}(1-60\alpha)k+\frac{1}{144\sqrt{3}}(189\alpha-16)kl^2,$$
$$\alpha_2=\left(\frac{1}{144}-\frac{5}{12}\alpha\right)k+\left(-\frac{1}{27}-\frac{11}{16}\alpha\right)kl^2.$$

Thus, the location of non-collinear libration points $L_4(x', y')$ and $L_5(x', y')$ in our case are

$$x' = \left(\mu - \frac{1}{2}\right) + \frac{(\mu+3)}{24(\mu-1)} l^2 + \frac{1}{48\sqrt{3}}(1-60\alpha)k + \frac{1}{144\sqrt{3}}(189\alpha - 16)k\, l^2,$$

$$y' = \pm\left(\frac{\sqrt{3}}{2} + \frac{(19-23\mu)}{24\sqrt{3}(\mu-1)} l^2\right) + \left(\frac{1}{144} - \frac{5}{12}\alpha\right)k + \left(-\frac{1}{27} - \frac{11}{16}\alpha\right)k\, l^2.$$

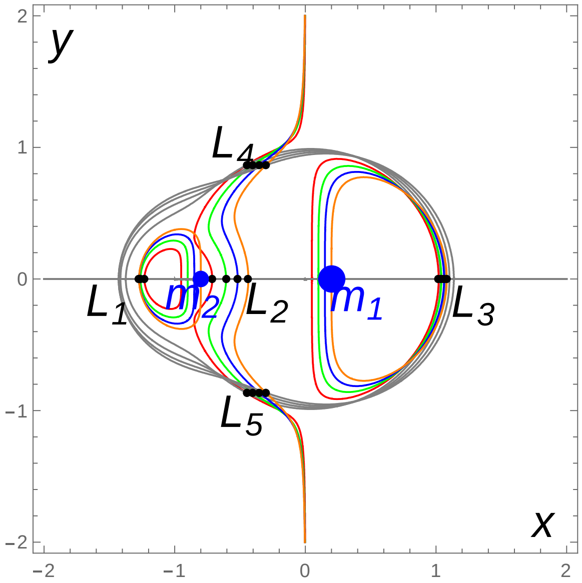

Fig. 2(a) The location of libration points for $\alpha = 0.05$, $k = 10^{-5}, l = 0.05$ and $\mu = 0.05$ (red, gray), $\mu = 0.10$ (green, gray), $\mu = 0.15$ (blue, gray), $\mu = 0.20$ (orange, gray),

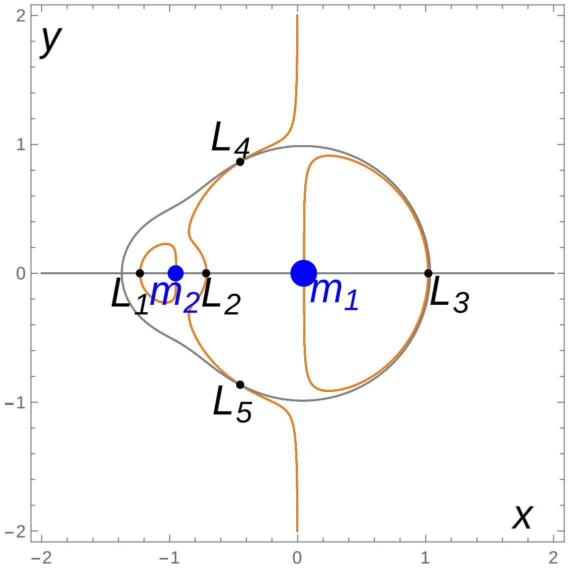

Fig. 2(b) The location of libration points for $\mu = 0.05$, $k = 10^{-5}, l = 0.05$ and $\alpha = 0.02, 0.05, 0.10, 0.15$

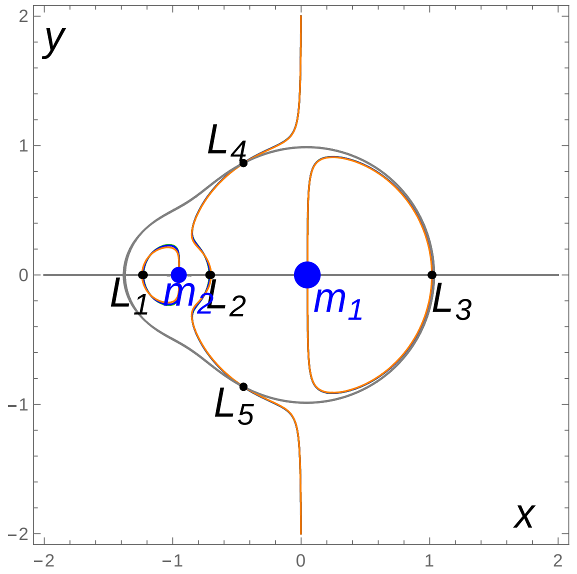

Fig. 2(c) The location of libration points for $\mu = 0.05$, $k = 10^{-5}, \alpha = 0.05$ and $l = 0.001, 0.01, 0.05, 0.10$.

On solving Eqs. (2) and (3) numerically, we get five libration points L_1, L_2, L_3, L_4 and L_5, out of which L_4 and L_5 are non-collinear with the primaries and L_1, L_2, L_3 are almost collinear with the primaries. For fixed values of parameters $\alpha = 0.05$, $k = 10^{-5}$, $l = 0.05$, and $\mu = 0.05$, we get five libration points.

In Figure 2(a), the location of libration points for different values of mass parameter μ are shown. We have taken $\alpha = 0.05, k = 10^{-5}, l = 0.05$ and $\mu = 0.05$, 0.10, 0.15, 0.20. The numerical values of libration points for different values of μ are given in Table 1. It is observed that as μ increases, L_1 moves away from m_2 along the x-axis. L_2 moves towards the centre of mass O along the x-axis. L_3 moves away from m_1 along the x-axis. L_4 and L_5 moves parallel to x-axis and towards the y-axis.

The location of libration points for different values of α are shown in Fig. 2(b). We have taken $\mu = 0.05, k = 10^{-5}, l = 0.05$ and $\alpha = 0.02$, 0.05, 0.10, 0.15. The numerical values of libration points for different values of α are given in Table 2. The location of libration points for different values of l are shown in Fig. 2(c). We have taken $\mu = 0.05, k = 10^{-5}, \alpha = 0.05$ and $l = 0.001$, 0.01, 0.05, 0.10. The numerical values of libration points for different values of l are given in Table 3. It is observed from Fig. 2(b) and 2(c), the effects of α and l on the location of libration points are very negligible.

μ	L_1	L_2	L_3	$L_{4,5}$
0.05	(-1.23082, 6.86556 $\times 10^{-6}$)	(-0.71192, 1.70634 $\times 10^{-6}$)	(1.02000, -2.42891 $\times 10^{-4}$)	(-0.45042, $\pm$ 0.86485)
0.10	(-1.26166, 8.40468 $\times 10^{-6}$)	(-0.60647, 1.51175 $\times 10^{-6}$)	(1.04080, -1.20483 $\times 10^{-4}$)	(-0.40040, $\pm$ 0.86489)
0.15	(-1.27191, 9.74215 $\times 10^{-6}$)	(-0.51755, 1.47960 $\times 10^{-6}$)	(1.06150, -7.96326 $\times 10^{-5}$)	(-0.35042, $\pm$ 0.86491)
0.20	(-1.27236, 11.0466 $\times 10^{-6}$)	(-0.43614, 1.60978 $\times 10^{-6}$)	(1.08205, -5.91649 $\times 10^{-5}$)	(-0.30044, $\pm$ 0.86493)

Table 1. Libration points, when $\alpha = 0.05, k = 10^{-5}, l = 0.05.$

α	L_1	L_2	L_3	$L_{4,5}$
0.02	(-1.23082, 6.72144 $\times 10^{-6}$)	(-0.71192, 1.50397 $\times 10^{-6}$)	(1.2000, -2.33360 $\times 10^{-4}$)	(-0.45042, $\pm$ 0.86485)
0.05	(-1.23082, 6.86556 $\times 10^{-6}$)	(-0.71192, 1.70634 $\times 10^{-6}$)	(1.2000, -2.42891 $\times 10^{-4}$)	(-0.45042, $\pm$ 0.86485)
0.10	(-1.23082, 7.10577 $\times 10^{-6}$)	(-0.71192, 2.04361 $\times 10^{-6}$)	(1.2000, -2.58776 $\times 10^{-4}$)	(-0.45042, $\pm$ 0.86485)
0.15	(-1.23082, 7.34598 $\times 10^{-6}$)	(-0.71192, 2.38089 $\times 10^{-6}$)	(1.2000, -2.74662 $\times 10^{-4}$)	(-0.45043, $\pm$ 0.86484)

Table 2. Libration points, when $\mu = 0.05, k = 10^{-5}, l = 0.05.$

l	L_1	L_2	L_3	$L_{4,5}$
0.001	(-1.22809, 7.15184 $\times 10^{-6}$)	(-0.71522, 1.78288 $\times 10^{-6}$)	(1.02083, -2.43901 $\times 10^{-4}$)	(-0.45008, $\pm$ 0.86598)
0.010	(-1.22820, 7.14019 $\times 10^{-6}$)	(-0.71509, 1.77978 $\times 10^{-6}$)	(1.02079, -2.43861 $\times 10^{-4}$)	(-0.45010, $\pm$ 0.86594)
0.050	(-1.23082, 6.86556 $\times 10^{-6}$)	(-0.71192, 1.70634 $\times 10^{-6}$)	(1.02000, -2.42891 $\times 10^{-4}$)	(-0.45042, $\pm$ 0.86485)
0.100	(-1.23906, 6.09515 $\times 10^{-6}$)	(-0.70201, 1.49801 $\times 10^{-6}$)	(1.01755, -2.39893 $\times 10^{-4}$)	(-0.45142, $\pm$ 0.86147)

Table 3. Libration points, when $\mu = 0.05, k = 10^{-5}, \alpha = 0.05.$

IV. Stability of non-collinear libration points

Following the procedure of Jain and Aggarwal [5], we get the characteristic equation

$$\lambda^4 - \left(k_{x',\dot{x}'} + k_{y',\dot{y}'}\right)\lambda^3 + [2(e-n^2) - f - h - k_{x',x'} + 2n - k_{y',y'} + p + i]\lambda^2 + \left[\left(n^2 - e + f\right)k_{x',\dot{x}'} + (h-e-i)k_{y',\dot{y}'} + 2n\left(k_{x',y'} + k_{y',x'}\right) + n^2 k_{y',\dot{y}'} - pk_{x',\dot{x}'} + j\left(2n - k_{y',\dot{x}'}\right) + m\left(2n + k_{x',\dot{y}'}\right)\right]\lambda + \left[\left(e-h-n^2\right)\left(e-f-n^2\right) - g^2 + \left(n^2 - e + f\right)k_{x',x'} + \left(n^2 - e + h\right)k_{y',y'} - g\left(k_{x',y'} + k_{y',x'}\right) + p\left(e-h+i-n^2\right) + i\left(e-f-n^2\right) - pk_{x',x'} - i\,k_{y',y'} + m\left(g + j + k_{x',y'}\right) + j\left(k_{y',x'} - g\right)\right] = 0,$$

where the values of *e, f, g, h, i, j, m, p* and k_i*'s* are given in appendix.

In general form, the above equation can be written as

$$\lambda^4 + \sigma_3\lambda^3 + (\sigma_{20} + \sigma_2)\lambda^2 + \sigma_1\lambda + (\sigma_{00} + \sigma_0) = 0,$$

where $\sigma_{00} = \left(e-h-n^2\right)\left(e-f-n^2\right) - g^2,$

$$\sigma_0 = \left(n^2 - e + f\right)k_{x',x'} + \left(n^2 - e + h\right)k_{y',y'} - g\left(k_{x',y'} + k_{y',x'}\right) + p\left(e-h+i-n^2\right) + i\left(e-f-n^2\right) - pk_{x',x'} - i\,k_{y',y'} + m\left(g + j + k_{x',y'}\right) + j\left(k_{y',x'} - g\right),$$

$$\sigma_1 = \left(n^2 - e + f\right)k_{x',\dot{x}'} + (h-e-i)k_{y',\dot{y}'} + 2n\left(k_{x',y'} + k_{y',x'}\right) + n^2 k_{y',\dot{y}'} - pk_{x',\dot{x}'} + j\left(2n - k_{y',\dot{x}'}\right) + m\left(2n + k_{x',\dot{y}'}\right),$$

$$\sigma_{20} = 2(e-n^2) - f - h,$$

$$\sigma_2 = -k_{x',x'} + 2n - k_{y',y'} + p + i,$$

$$\sigma_3 = -\left(k_{x',\dot{x}'} + k_{y',\dot{y}'}\right),$$

Here σ_{00}, σ_{20} and $\sigma_i (i = 0,1,2,3)$ can be derived by evaluating *e, f, g, h, i, j, m* and *p* defined in the appendix. The value of the coefficient in the zero drag case is denoted by adding additional subscript 0. Now, we have

$$\sigma_{00} = 3\mu + \frac{15}{2}\mu l^2 + \left(\frac{-7}{96\sqrt{3}} - \frac{173\alpha}{16\sqrt{3}}\right)l^2 k + \left(\frac{43}{288\sqrt{3}} + \frac{1271\alpha}{48\sqrt{3}}\right)\mu l^2 k,$$

$$\sigma_{20} = -3 + \mu + \left(-3 + \frac{3}{2}\mu\right)l^2 + \left(\frac{-7}{288\sqrt{3}} - \frac{173\alpha}{48\sqrt{3}}\right)l^2 k + \left(\frac{5}{288\sqrt{3}} + \frac{193\alpha}{48\sqrt{3}}\right)\mu l^2 k,$$

$$\sigma_0=\frac{33}{16}\mu+\frac{-9}{8}\mu l+\frac{63}{16}\mu l^2+\left(\frac{-3\sqrt{3}}{2}-\frac{63\sqrt{3}\alpha}{32}\right)k$$
$$+\left(\frac{313\sqrt{3}}{128}+\frac{1113\sqrt{3}\alpha}{128}\right)\mu k+\left(\frac{-169}{128\sqrt{3}}-\frac{13\sqrt{3}\alpha}{8}\right)\mu lk$$
$$+\left(\frac{-25}{4\sqrt{3}}-\frac{469\sqrt{3}\alpha}{128}\right)l^2k+\left(\frac{25945}{2304\sqrt{3}}+\frac{88763\alpha}{1537\sqrt{3}}\right)\mu l^2k,$$
$$\sigma_1=\sqrt{3}\mu+\sqrt{3}\mu l+\frac{5}{12\sqrt{3}}\mu l^2+\left(-1+\frac{21\alpha}{4}\right)k$$
$$+\left(\frac{115}{144}+\frac{1337\alpha}{48}\right)\mu k+\left(\frac{-19}{144}+\frac{95\alpha}{12}\right)\mu lk$$
$$+\left(1+\frac{21}{8}\alpha\right)l^2k+\left(\frac{-35}{36}+\frac{22913\alpha}{576}\right)\mu l^2k,$$
$$\sigma_2=-\frac{1}{4}\mu-\frac{1}{6}\mu l^2+\frac{21\sqrt{3}\alpha}{8}k+\left(\frac{1}{48\sqrt{3}}+\frac{149\alpha}{32\sqrt{3}}\right)\mu k$$
$$+\frac{49}{4\sqrt{3}}\alpha l^2k+\left(\frac{-341}{3456\sqrt{3}}+\frac{30469\alpha}{1152\sqrt{3}}\right)\mu l^2k,$$
$$\sigma_3=2k.$$

To check the stability of non-collinear libration points, we have followed the procedure of Jain and Aggarwal [5]. We have found that in our case $\sigma_1=-k$ and $\sigma_3=-2k$ and therefore, $\sigma_1>\sigma_3$ and hence, $L_{4,5}$ is not asymptotically stable. Further, one of the roots of λ has positive real root. Hence $L_{4,5}$ is not stable. Hence we conclude that $L_{4,5}$ are linearly unstable.

V. Conclusion

The existence of the libration points in the restricted three body problem under the combined effects of Stokes drag and straight segment has been investigated. There exist five libration points, out of which two are non-collinear with the primaries and three are almost collinear with the primaries. Using the terminology of Murray [10], we have examined the stability of non-collinear libration points and found that in our case $\sigma_1=-k$ and $\sigma_3=-2k$ which proves that $L_{4,5}$ are linearly unstable.

The location of the libration points for various values of parameters μ,α and l are shown in Fig. 2. From Fig. 2(a), we observed that as mass parameter μ increases, L_1 moves away from the smaller primary, L_2 moves towards the centre of mass O, L_3 moves away from the bigger primary m_1, and $L_{4,5}$ moves parallel to x-axis and towards the y-axis. It is observed that the effects of α and l on the libration points are very negligible. The numerical values of libration points are given in Tables 1, 2 and 3. If we take $k=0$ and $l=0$, our results confirm with the classical restricted three body problem (Szebehely [14]). If $l=0$ and $k\neq 0$, then the results confirm with Jain and Aggarwal [4]. If we consider $k=0$ and $l\neq 0$, then the results show the conformity with Jain et al. [6].

VI. Appendix

$$e=\frac{1-\mu}{r'^3_1},\ f=\frac{3(1-\mu)}{r'^5_1}y'^2,$$
$$g=\frac{3(1-\mu)(x'-\mu)}{r'^5_1}y',\ h=\frac{3(1-\mu)(x'-\mu)^2}{r'^5_1},$$
$$i=\frac{2\mu}{r'_3r'_4a_2}\left\{1-\frac{a_8}{a_2}-a_9-a_{10}\right\},$$
$$j=\frac{2\mu(x'-\mu+1)y'}{r'_3r'_4a_2}\left\{\frac{1}{r'^2_3}+\frac{1}{r'^2_4}+\frac{1}{r'_3r'_4}\right\},$$
$$m=\frac{2\mu y'}{r'_3r'_4a_5}\left(\frac{a_{11}}{r'_4}+\frac{a_{12}}{r'_3}\right)\left(1-\frac{2a_2^2}{a_5}\right)$$
$$-\frac{2\mu y'a_2}{r'_3r'_4a_5}\left(\frac{a_{11}}{r'^2_4}+\frac{a_{12}}{r'^2_3}\right),$$
$$p=\frac{2\mu a_2}{r'_3r'_4a_5}-\frac{2\mu y'^2a_2}{r'_3r'_4a_5}\left(\frac{1}{r'^2_3}+\frac{1}{r'^2_4}\right)$$
$$+\frac{2\mu y'}{r'_3r'_4a_5}\left(\frac{y'}{r'_3}+\frac{y'}{r'_4}\right)\left\{1-\frac{2a_2}{a_5}\right\},$$
$$a_8=\frac{(x'-\mu+1)\{x'-(\mu-1+l)\}}{r'_4}+\frac{x'-(\mu-1-l)}{r'_3},$$
$$a_9=\frac{(x'-\mu+1)\{x'-(\mu-1+l)\}}{r'^2_4},$$
$$a_{10}=\frac{(x'-\mu+1)\{x'-(\mu-1-l)\}}{r'^2_3},$$
$$a_{11}=\{x'-(\mu-1+l)\},\ a_{12}=\{x'-(\mu-1-l)\},$$
$$k_{x',x'}=-\frac{21}{4}\alpha\left(x'^2+y'^2\right)^{-11/4}x'y'k,\ k_{x',\dot{x}'}=-k,$$
$$k_{x',y'}=k-\frac{21}{4}\alpha\left(x'^2+y'^2\right)^{-11/4}y'^2k,\ k_{x',\dot{y}'}=0,$$
$$k_{y',x'}=-k-\frac{21}{4}\alpha\left(x'^2+y'^2\right)^{-11/4}x'^2k,\ k_{y',\dot{x}'}=0,$$
$$k_{y',y'}=-\frac{21}{4}\alpha\left(x'^2+y'^2\right)^{-11/4}x'y'k,\ k_{y',\dot{y}'}=-k.$$

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