IDEMPOTENTS OF $M_3(R[x])$

Meenu Khatkar

Department of Mathematics, Indian Institute of Technology Delhi,
New Delhi, 110016, India

Abstract

The aim of this paper is to study idempotents in the matrix ring $M_3(R[x])$.

Keywords: Idempotents, Units, Polynomial rings, Matrix rings.
Mathematics Subject Classification (2010): 16S50, 13F20,16U60.

1 Introduction

Idempotents in rings play a critical role in the study of rings. Several classes of elements are defined using idempotents and units, for example, clean elements (the elements that can be expressed as sum of an idempotent and a unit, cf. [2], [8]), strongly clean elements (the elements that can be expressed as a sum of an idempotent and a unit that commute, cf. [9]), unit regular elements (the elements that can be written as eu for some idempotent e and unit u, cf. [9]), Lie regular elements (the elements that can be written as $eu - ue$ where e is an idempotent and u is a unit, cf. [10]), etc. Due to their importance, the idempotents generated interest among several researchers and efforts have been made to compute idempotents of rings.

The problem of obtaining structure and presentation of unit groups of rings have also drawn attention of several researchers. Important contributions have been made in some special cases (for example see [4], [5], [7], [10], [11]). These studies, however, are far from complete and a lot more needs to be done. In the case of polynomial rings, Kanwar, Leroy and Matczuk showed that for an abelian ring (a ring in which all idempotents are central) R, idempotents in the polynomial ring $R[x]$ over R are precisely idempotents in R ([1], Lemma 1). In fact, a ring is reduced if and only if the unit group of $R[x]$ is same as the unit group of R. Not much, however, is known in the case of polynomial rings over matrix rings (equivalently, matrix rings over polynomial rings).

In this article, we study idempotents in matrix ring $M_3(R[x])$. Throughout, a ring will mean an associative ring with unity. For any ring R, $E(R)$ will denote the set of all idempotents in R. For any positive integer n, $M_n(R)$ will denote the ring of $n \times n$ matrices over a ring R.

We will use standard definitions for determinant and trace of matrices over commutative rings (cf. [7]). Recall that the determinant of product of two matrices over a commutative ring is the product of the determinant of two matrices.

2 Idempotents of $M_3(R[x])$

We now give some results that will be useful in our study. We begin with the following proposition that may also be of independent interest.

Proposition 2.1. *[3] Let R be any ring with unity and $a = \sum_{i=0}^{n} a_i x^i$ is an element in $R[x]$ such that $a^2 - a \in R$. If any of the following conditions hold:*

1. *R has no non-zero nilpotent elements,*
2. *$a_0 a_i = a_i a_0$ for $1 \leq i \leq n$ and $2a_0 - 1$ is a unit in R,*

then $a \in R$.

Proof. If R has no non-zero nilpotent elements and $a^2 - a \in R$, then it is easy to see that $a_i = 0$ for $1 \leq i \leq n$. The proof, in the second case, is similar to the proof of Lemma 1 in [1]. We give a brief outline for the sake of completeness. If $a \notin R$ and a_i $(i > 0)$ is the first non-zero coefficient in a, then $a^2 - a \in R$ gives $2a_0 a_i - a_i = 0$. But then $a_i = 0$ as $2a_0 - 1$ is a unit in R, a contradiction. Thus $a \in R$. □

In particular, we have the following corollary.

Corollary 2.2. *[1, Lemma 1] If R is a commutative ring, then $E(R[x]) = E(R)$.*

Corollary 2.3. *[3] If R is a ring with no non-zero nilpotent elements, then $E(R[x]) = E(R)$.*

Theorem 2.4. *[3] Let R be a commutative ring. Then the trace of every non-trivial idempotent in $M_2(R)$ with determinant 0 is an idempotent.*

However the result does not holds for $\mathbb{M}_3(R)$ in general.
For example matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in $\mathbb{M}_3(\mathbb{Z}_6[x])$ is a non-trivial idempotent having determinant 0, but the trace is not an idempotent in $\mathbb{Z}_6[x]$.

Theorem 2.5. *Let R be a commutative ring. Then the trace of every non-trivial idempotent in $\mathbb{M}_3(R)$ with determinant 0 is an idempotent if and only if $2(bd + cg + fh = ae + ei + ai)$.*

Proof. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be a non-trivial idempotent in $\mathbb{M}_3(R)$ and let the determinant of A is 0. Thus $aei + bfg + cdh = afh + bdi + cge$. Since A is an idempotent, $\begin{pmatrix} a^2 + bd + cg & ab + be + ch & ac + bf + ci \\ ad + ed + fh & bd + e^2 + fh & cd + ef + fi \\ ag + hg + gi & bg + eh + hi & cg + fh + i^2 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.
Thus $a^2 + bd + cg = a$, $bd + e^2 + fh = d$, and $cg + fh + i^2 = a + d$.
Therefore $a^2 + e^2 + i^2 + 2bd + 2cg + 2fh = a + e + i$.
We can see that $a+e+i$ is an idempotent in R if and only if $2(bd+cg+fh = ae+ei+ai)$. □

Since we are taking R to be a coomutative ring. Therefore the idempotents in ring $R[x]$ are same as that of the ring R, so the above result holds for $\mathbb{M}_3(R[x])$.

References

[1] P. Kanwar, A. Leroy, and J. Matczuk, *Idempotents in ring extensions*, J. Algebra 389 (2013), 128–136.

[2] P. Kanwar, A. Leroy, and J. Matczuk, *Clean elements in polynomial rings*, Noncommutative Rings and their Applications, Contemporary Mathematics, Amer. Math. Soc. 634 (2015), 197–204.

[3] P. Kanwar, M. Khatkar, and R. K. Sharma, *Idempotents and Units of Matrix Rings over Polynomial Rings*, Int. Electron. J. Algebra (To appear).

[4] P. Kanwar, R. K. Sharma, and P. Yadav, *Lie regular generators of general linear groups II*, Int. Electron. J. Algebra Volume 13 (2013), 91–108.

[5] C. Lanski, *Some remarks on rings with solvable units*, Ring Theory, (Proc. Conf., Park City, Utah, 1971), Academic Press, New York (1972), 235-240.

[6] B. R. McDonald, *Elementary Number Theory*, Marcel Dekker, New York. (1984).

[7] M. Mirowicz, *Units in group rings of infinite dihedral group*, Canad. Math. Bull. Volume 34(1)(1991), 83–89.

[8] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. Volume 229 (1977), 269–278.

[9] W. K. Nicholson, *Strongly clean rings and Fitting's lemma*. Comm. Algebra Volume 27 (1999), 3583–3592.

[10] R. K. Sharma, P. Yadav, and P. Kanwar, *Lie regular generators of general linear groups*, Comm. Algebra Volume 40 (2012), 1304–1315.

[11] L. Wiechecki, *Group algebra units and tree actions*, J. Algebra 311 (2007), 781–799.