Idempotents of $M_2(\mathbb{Z}_6[x])$

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Abstract

The aim of this paper is to study idempotents in the matrix ring $M_2(\mathbb{Z}_6[x])$.

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1 Introduction

Idempotents in rings play a critical role in the study of rings. Several classes of elements are defined using idempotents and units, for example, clean elements (the elements that can be expressed as sum of an idempotent and a unit, cf. [8], [13]), strongly clean elements (the elements that can be expressed as a sum of an idempotent and a unit that commute, cf. [14]), unit regular elements (the elements that can be written as eu for some idempotent e and unit u, cf. [6], [14]), Lie regular elements (the elements that can be written as $eu - ue$ where e is an idempotent and u is a unit, cf. [15]), etc. Due to their importance, the idempotents generated interest among several researchers and efforts have been made to compute idempotents of rings.

The problem of obtaining structure and presentation of unit groups of rings have also drawn attention of several researchers. Important contributions have been made in some special cases (for example see [1], [2], [3], [5], [9], [10], [12], [15], [16]). These studies, however, are far from complete and a lot more needs to be done. In the case of polynomial rings, Kanwar, Leroy and Matczuk showed that for an abelian ring (a ring in which all idempotents are central) R, idempotents in the polynomial ring $R[x]$ over R are precisely idempotents in R ([7], Lemma 1). In fact, a ring is reduced if and only if the unit group of $R[x]$ is same as the unit group of R. Not much, however, is known in the case of polynomial rings over matrix rings (equivalently, matrix rings over polynomial rings).

In this article, we study idempotents in matrix ring $M_2(\mathbb{Z}_6[x])$. Throughout, a ring will mean an associative ring with unity and for any positive integer n, $\mathbb{Z}_n$ will denote the ring of integers modulo n. For any ring R, $E(R)$ will denote the set of all idempotents in R. For any positive integer n, $M_n(R)$ will denote the ring of $n \times n$ matrices over a ring R and $GL(n, R)$ will denote the general linear group (the group of all $n \times n$ invertible matrices over ring R).

We will use standard definitions for determinant and trace of matrices over commutative rings (cf. [12]). More precisely for a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over a commutative ring R, determinant of A is $ad - bc$ and trace of A is $a + d$. Recall that the determinant of product of two matrices over a commutative ring is the product of the determinant of two matrices.

2 Idempotents of $M_2(\mathbb{Z}_6[x])$

We now give some results that will be useful in our study. We begin with the following proposition that may also be of independent interest.

Proposition 2.1. *Let R be any ring with unity and $a = \sum_{i=0}^{n} a_i x^i$ is an element in $R[x]$ such that $a^2 - a \in R$. If any of the following conditions hold:*

1. *R has no non-zero nilpotent elements,*
2. *$a_0 a_i = a_i a_0$ for $1 \leq i \leq n$ and $2a_0 - 1$ is a unit in R,*

then $a \in R$.

Proof. If R has no non-zero nilpotent elements and $a^2 - a \in R$, then it is easy to see that $a_i = 0$ for $1 \leq i \leq n$. The proof, in the second case, is similar to the proof of Lemma 1 in [7]. We give a brief outline for the sake of completeness. If $a \notin R$ and a_i $(i > 0)$ is the first non-zero coefficient in a, then $a^2 - a \in R$ gives $2a_0 a_i - a_i = 0$. But then $a_i = 0$ as $2a_0 - 1$ is a unit in R, a contradiction. Thus $a \in R$. $\square$

In particular, we have the following corollary.

Corollary 2.2. *[7, Lemma 1] If R is a commutative ring, then $E(R[x]) = E(R)$.*

Corollary 2.3. *If R is a ring with no non-zero nilpotent elements, then $E(R[x]) = E(R)$.*

Theorem 2.4. *Any non-trivial idempotent in $M_2(\mathbb{Z}_6[x])$ is of one of the following forms:*

1. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$
2. $\begin{pmatrix} a(x) & b(x) \\ c(x) & 1 - a(x) \end{pmatrix}$, *where* $a(x)\{1 - a(x)\} - b(x)c(x) = 0$

3. $\begin{pmatrix} 3a(x) & 3b(x) \\ 3c(x) & 3(1-a(x)) \end{pmatrix}$, *where* $a(x)\{1-a(x)\} - b(x)c(x) = 2f(x)$

4. $\begin{pmatrix} 2a(x) & 2b(x) \\ 2c(x) & 4-2a(x) \end{pmatrix}$, *where* $a(x)\{1-2a(x)\} - 2b(x)c(x) = 3g(x)$

5. $\begin{pmatrix} 3+2a(x) & 2b(x) \\ 2c(x) & 1-2a(x) \end{pmatrix}$, *where* $a(x)\{1-2a(x)\} - 2b(x)c(x) = 3h(x)$

6. $\begin{pmatrix} 1+3a(x) & 3b(x) \\ 3c(x) & 4-pa(x) \end{pmatrix}$, *where* $a(x)\{1-a(x)\} - b(x)c(x) = 2\phi(x)$,

where $a(x)$, $b(x)$, $c(x)$, $f(x)$, $g(x)$, $h(x)$, *and* $\phi(x)$ *are polynomials in* $\mathbb{Z}_6[x]$, *not necessarily non-zero.*

Proof. Since the idempotents in $\mathbb{Z}_6[x]$ are precisely the idempotents in $\mathbb{Z}_6$. Therefore the idempotents in $\mathbb{Z}_6[x]$ are 0, 1, 3, and 4. Now let $A = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$ be a non-trivial idempotent of $M_2(\mathbb{Z}_6[x])$. For convenience, we will write a, b, c, d for $a(x)$, $b(x)$, $c(x)$, $d(x)$ respectively. Since A is an idempotent, we have $a^2+bc = a$, $b(a+d) = b$, $c(a+d) = c$, and $bc + d^2 = d$. Also since determinant of A is an idempotent in $\mathbb{Z}_6$, so the determinant of A is 0 or 1 or 3 or 4.
If determinant of A is 1 then $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a trivial idempotent in $M_2(\mathbb{Z}_6[x])$. Hence, the determinant of A is 0 or 3 or 4. Also, trace of A is in $\mathbb{Z}_6$, that is, $a+d \in \mathbb{Z}_6$.

Case 1: Determinant of A is 0. This means $ad - bc = 0$. Since A is an idempotent, therefore, $a^2+bc+bc+d^2 = a^2+2bc+d^2 = a^2+2ad+d^2 = a+d$. It means $a+d$ is an idempotent in $\mathbb{Z}_6[x]$. Thus $a+d$ is either 0 or 1 or 3 or 4.
If $a+d = 0$ then we get A to be a zero matrix, which is a trivial idempotent in $M_2(\mathbb{Z}_6[x])$.
If $a+d = 1$ then $d = 1-a$ and hence $ad-bc = 0$ gives $a^2+bc = a$. Also $(a+d)b = b$, $(a+d)c = c$, and $bc+d^2 = 1-a$. Thus, $A^2 = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$. Thus, in this case, $A = \begin{pmatrix} a(x) & b(x) \\ c(x) & 1-a(x) \end{pmatrix}$, where $a(x),\ b(x),\ c(x) \in \mathbb{Z}_6[x]$ such that $a(x)\{1-a(x)\} = b(x)c(x)$.
If $a+d = 3$ then $d = 3-a$ and hence $ad-bc = 0$ gives $a^2+bc = 3a$. Thus $A^2 = \begin{pmatrix} 3a & 3b \\ 3c & 3-3a \end{pmatrix}$. Since A is an idempotent, $2a = 0$, $2b = 0$, and $2c = 0$. Therefore, $a = 3a'(x)$, $b = 3b'(x)$, and $c = 3c'(x)$, where $a'(x)$, $b'(x)$ and $c'(x)$ are polynomials in $\mathbb{Z}_6[x]$. Now since $ad-bc = 0$, we get $3a'(x)\{1-a'(x)\} = 3b'(x)c'(x)$, which is equivalent to $a'(x)\{1-a'(x)\} - b'(x)c'(x) = 2f(x)$ for some polynomial $f(x) \in \mathbb{Z}_6[x]$. Hence, $A = \begin{pmatrix} 3a(x) & 3b(x) \\ 3c(x) & 3(1-a(x)) \end{pmatrix}$, where $a(x),\ b(x),\ c(x) \in \mathbb{Z}_6[x]$ such that $a(x)\{1-a(x)\} - b(x)c(x) = 2f(x)$ for some $f(x) \in \mathbb{Z}_6[x]$.
If $a+d = 4$ then $d = 4-a$ and hence $ad-bc = 0$ gives $a^2+bc = 4a$. Thus,

$A^2 = \begin{pmatrix} 4a & 4b \\ 4c & 3a+4-a \end{pmatrix}$. Since A is an idempotent, $3a = 0$, $3b = 0$, and $3c = 0$. Hence, as in the previous case, $A = \begin{pmatrix} 2a(x) & 2b(x) \\ 2c(x) & 4-2a(x) \end{pmatrix}$, where $a(x),\ b(x),\ c(x) \in \mathbb{Z}_6[x]$ such that $a(x)\{1-2a(x)\} - 2b(x)c(x) = 3g(x)$ for some $g(x) \in \mathbb{Z}_6[x]$.
Next, we consider the case where determinant of A is 3.

Case 2: Determinant of A is 3. This means $ad - bc = 3$, that is, $2ad - 2bc = 0$. Since A is an idempotent, therefore, $a^2 + bc + bc + d^2 = a^2 + 2bc + d^2 = a^2 + 2ad + d^2 = a + d$. It means $a + d$ is an idempotent in $\mathbb{Z}_6[x]$. Thus $a + d$ is either 0 or 1 or 3 or 4.
If $a + d = 1$ then $ad - bc = 3$ gives $a^2 + bc = 3 + a(mod\ 6)$ and hence
$A^2 = \begin{pmatrix} 3+a & b \\ c & 4-a \end{pmatrix}$. Since A is an idempotent, we get $3 + a = a$, a contradiction.
If $a + d = 3$ then $ad - bc = 3$ gives $a^2 + bc = 3a - 3$ and hence
$A^2 = \begin{pmatrix} 3a-3 & 3b \\ 3c & 3a \end{pmatrix}$. Since A is an idempotent, we get $2a = 3$. Thus $2a_0 = 3(mod\ 6)$ where a_0 is the term without x in a. This is not possible as $\gcd(2, 6) = 2$ and 2 does not divide 6.
It is easy to see that the determinant of $(A + 3I)$ is 0 and the trace of $(A + 3I)$ is trace of A. Therefore, by the previous case, $A + 3I$ is either the zero matrix in $M_2(\mathbb{Z}_6[x])$ or $\begin{pmatrix} 2a(x) & 2b(x) \\ 2c(x) & 4-2a(x) \end{pmatrix}$, where $a(x),\ b(x),\ c(x) \in \mathbb{Z}_6[x]$ such that $a(x)\{1-2a(x)\} - 2b(x)c(x) = 3g(x)$. Hence A is $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ or $\begin{pmatrix} 3+2a(x) & 2b(x) \\ 2c(x) & 1-2a(x) \end{pmatrix}$, where $a(x),\ b(x),\ c(x) \in \mathbb{Z}_6[x]$ such that $a(x)\{1-2a(x)\} - 2b(x)c(x) = 3h(x)$ for some $h(x) \in \mathbb{Z}_6[x]$.
Finally, we consider the case where determinant of A is 4.

Case 3: Determinant of A is 4. In this case, $ad - bc = 4$, $a^2 + bc = a$, and $bc + d^2 = d$ give $(a + d)^2 = a + d + 2$. Since $a + d \in \mathbb{Z}_6$, we get $a + d$ is either 2 or 5.
If $a + d = 2$ then $d = 2 - a$ and hence $ad - bc = 4$ gives $a^2 + bc = 2a - 2$. Thus $A^2 = \begin{pmatrix} 2a+2 & 2b \\ 2c & 4a \end{pmatrix}$. Since A is an idempotent, we get $a = 4$, $b = c = 0$. Thus $A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.
If $a + d = 5$ then $d = 5 - a$ and hence $ad - bc = 4$ gives $a^2 + bc = 5a + 2$. Thus $A^2 = \begin{pmatrix} 5a+2 & 5b \\ 5c & 3+a \end{pmatrix}$. Since A is an idempotent, we have $4a = 4$, $4b = 0$, and $4c = 0$.
Hence, as earlier, $A = \begin{pmatrix} 1+3a(x) & 3b(x) \\ 3c(x) & 4-3a(x) \end{pmatrix}$, where $a(x),\ b(x),\ c(x) \in \mathbb{Z}_6[x]$ such that $a(x)\{1-a(x)\} - b(x)c(x) = 2\phi(x)$ for some $\phi(x) \in \mathbb{Z}_6[x]$. $\square$

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